Hints[1] for exercise sheet 8

Exercise 1

a. To show that $(X, \mathcal{A}, \mu)$ is not σ-finite, I would use a proof by contradiction. If a set has finite measure, what does that mean here?

b. Consider $\Psi(f) = \sum_{x\in[0,1]} f(x)$. Is this well-defined? Does it define a linear functional?

Exercise 2

One strategy of proof is to use the Lax-Milgram theorem from the lecture notes. For this: show that

$$g_f(x) := \int_X L(x,y)f(y)dy$$

is an $L^2(X)$ function (use Hölder and Fubini), show that the integral equation of the exercise is equivalent to

$$\langle cf - g_f, \phi\rangle = \langle -u, \phi\rangle \qquad \text{for all } \phi \in L^2(X) \tag{1}$$

where where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2(X)$. Try setting

$$a(f,\phi) := \langle cf - g_f, \phi\rangle$$

and checking the hypotheses of Lax-Milgram...

Exercise 3

b. Let $\{x_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers; define $E_n = \cup_{k=1}^{\infty}\left(x_k - \frac{1}{2^k n}, x_k + \frac{1}{2^k n}\right)$ and $E = \cap_{n=1}^{\infty} E_n$....

Exercise 4

a. Don't forget the special case when $A = \emptyset$!
b. Note: for this exercise, assume that $\mathcal{F}$ is non-empty.

Exercise 5

Use the triangle inequality.

[1] Try by yourself first!