Hints[1] for exercise sheet 7

Exercise 1

To show sublinearity of q: Try starting like this: Fix $x, x' \in X$ and $\varepsilon > 0$. By the definition of infimum, there exist $\{A_1, ..., A_n\} \subseteq G$ and $\{B_1, ..., B_m\} \subseteq G$ such that

$$\tfrac{1}{n} p(A_1 x + ... + A_n x) < q(x) + \epsilon \quad \text{and} \quad \tfrac{1}{m} p(B_1 x' + ... + B_m x') < q(x') + \epsilon.$$

The strategy is then to show that $2\varepsilon + q(x) + q(x') \geq q(x + x')$.

Using the property that $p(Ax) \leq p(x)$ for all $x \in X$ and all $A \in G$,

$$\begin{aligned} \tfrac{1}{n} p(A_1 x + ... + A_n x) = \tfrac{m}{nm} p(A_1 x + ... + A_n x) &= \sum_{j=1}^{m} \tfrac{1}{nm} p(A_1 x + ... + A_n x) \\ &\geq \sum_{j=1}^{m} \tfrac{1}{nm} p(B_j A_1 x + ... + B_j A_n x), \end{aligned}$$

and similarly...

Once you have applied Hahn-Banach to obtain a linear functional F, to show that $F(Ax) = F(x)$, consider $|F(Ax) - F(x)| = |F(Ax - x)|$ and consider $\{A^n, A^{n-1}, ..., A, \mathrm{id}\} \subseteq G$ and let n vary...

Exercise 2

A possible strategy: show the following two facts, and then combine them with the hypotheses of the exercise to produce a solution.

Fact 1: Let X, Y be vector spaces and $f : X \to Y$ a linear map. Then

$$(f^{**} \circ J_X)(x) = (J_Y \circ f)(x) \quad \forall x \in X.$$

Fact 2: Let X, Y be normed vector spaces and $f : X \to Y$ a continuous linear map. If f is injective, then $f^* : Y^* \to X^*$ is surjective.

To show the second fact, use Hahn-Banach (the image $f(X) \subseteq Y^*$ is a linear subspace on which there is a functional that one might want to extend to a functional on all of Y^*...)

[1] Try by yourself first!

Exercise 3

For $y \in H$, consider $f_y : H \to \mathbb{K}, f_y(x) := \langle Ax, y\rangle$. Apply the Banach-Steinhaus theorem to the family of maps

$$\mathcal{F} = \{f_y \mid \|y\| = 1\}.$$

Cauchy-Schwarz might also be helpful.

Exercise 4

A weak formulation of the boundary value problem is

$$-\int_U \Delta v(x)\Delta u(x)dx = \int_U f(x)v(x)dx$$

for all $v \in H_0^{2,2}(U)$ and $u \in H_0^{2,2}(U)$. Explain why!! The idea is that a solution of the original biharmonic equation should give a solution of the "weak formulation" of the problem (in the special case that the assumptions of the original problem are satisfied).

For the question of existence and uniqueness of solutions, consult the relevant section in the lecture notes on Hilbert space methods. In particular, one can use the Lax-Milgram theorem.

Exercise 5

a. Define $B_y := B(\cdot, y) : X \to \mathbb{K}$, and $B_x := B(x, \cdot) : X \to \mathbb{K}$. Consider the family

$$\mathcal{F} := \{B_y : y \in X, \|y\| \leq 1\}$$

Then you know which theorem to apply!

b. To show that B is not continuous in $\mathcal{P} \times \mathcal{P}$, show that for any $C > 0$ there exists p for which $|B(p, p)| > C\|p\|^2$.