Hints[1] for exercise sheet 6

Exercise 1

a. Try the following: write $u(x) = (2\pi)^{n/2} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi} d\xi$ (this is possible, since u is in L^2). Derive an estimate of the form

$$\|u\|_{L^\infty} \leq (2\pi)^{n/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-m} d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^m |\hat{u}(\xi)|^2 d\xi \right)^{1/2},$$

(Cauchy-Schwarz!). Then show that the first term is bounded by a constant depending only on n and m, and that the second term is bounded by something involving the H^m norm.

b. Try this: Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $u(x) = \chi(x) \ln|\ln|x||$, where $\chi \in C_c^\infty(\mathbb{R}^2)$ and $0 \leq \chi(x) \leq 1\ \forall x \in \mathbb{R}^2$, $\chi(x) = 1$ for $|x| \leq 1/4$ and $\chi(x) = 0$ for $|x| \geq 1/2$.

Show that $u \notin L^\infty(\mathbb{R}^2)$ but that $u \in H^1(\mathbb{R}^2)$.

For the latter, use polar coordinates to get

$$\|u\|_{L^2(\mathbb{R}^2)} \leq 2\pi \int_0^{1/2} (\ln|\ln r|)^2\, r dr$$

Since $\ln(x) \leq 1 - x$ one has for $0 \leq r \leq 1/2$

$$|\ln|\ln r|| \leq |1 - |\ln r|| \leq |\ln r| = -\ln r;$$

this can be used to estimate.

Also one needs to show that the weak derivative of u is in L^2. Show that the weak derivative is given by the function

$$f = \frac{-x}{x^2 \ln|x|} \chi(x) + \ln|\ln|x|| \nabla\chi(x);$$

for this proceed by approximation as we have been doing often the last weeks!

Exercise 2

For E_1 and E_2, compute $\|x^{-\alpha}\|_{L^1}$.

Hint for E_3: it is bounded but not pre-compact. To show that latter, one can try a proof by contradiction using the following fact:

[1] Try by yourself first!

Lemma: Let $f : X \to Y$ be a continuous function between metric spaces. Assume that the metric space (X, d_X) is complete. Then:

$$A \subseteq X \text{ precompact} \Rightarrow f(A) \subseteq Y \text{ precompact.}$$

To apply the lemma, try constructing a continous linear function $F : L^1((0,1)) \to Y$, where Y is some other metric space, and such that $F(E_3) \subseteq Y$ is not precompact. For example, try defining F to be something similar to the Fourier transform...

For E_4 apply Arzela-Ascoli.

Exercise 3

a.

Consider $(f_n)_{n\in\mathbb{N}}$ dense in X^*. Following the hint, for every fixed $n \in \mathbb{N}$ there exists $x_n \in X$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \|f\|/2$. Then

$$M = \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$$

is separable. To solve the exercise, show that $M = X$. For this, assume $X \neq M$ and let $x_0 \in X \setminus M$. By Corollary 4.2.9. there exists $f \in X^*$ such that

$$f(x_0) = d > 0 \text{ and } f\big|_M = 0...$$

b. Try a proof by contradiction. Let $x \in \Omega$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Omega$, and consider the family $f_s = \chi_{B_{s\varepsilon}(x)}$ for $0 < s \leq 1$ and a dense subset $(g_k)_{k\in\mathbb{N}} \subset L^\infty(\Omega)$.

Show that there exists at most one $s = s(k)$ such that

$$\|f_s - g_k\|_{L^\infty(\Omega)} < 1/2, \tag{1}$$

and conclude that one could thus construct a surjective map $\mathbb{N} \ni k \mapsto s(k) \to\in (0,1]$.

Exercise 4

Define $p(f) := \sup_{x\in E} f_+(x)$, this is a sublinear functional on $B(E,\mathbb{R})$. Use Hahn-Banach!

With $f_+ := \max(f,0)$ and $f_- := \max(-f,0)$, recall that $f = f_+ - f_-$ and $|f| = f_+ + f_-$...

Also: note that $\overline{T}(f) \geq 0$ iff $-\overline{T}(f) = \overline{T}(-f) \leq 0$...

Exercise 5

You may use the following fact without proof: if $f : X \to Y$ is a bijective map between Banach spaces such that f is continuous, then f^{-1} is also continuous. (This is called the inverse mapping theorem).

A possible strategy: assuming X is reflexive, show that Y is reflexive by showing that f^{**} is surjective and that $f^{**} \circ J_X = J_Y \circ f$...

For the opposite direction (Y reflexive $\Rightarrow$ X reflexive), make use Theorem 4.3.4 from the lecture!