Hints[1] for exercise sheet 5

Exercise 1

! Warning ! There was a typo (sign error) in the statement of the exercise: it should be $\widehat{\nabla f}(k) = ik\hat{f}(k)$ and not $\widehat{\nabla f}(k) = -ik\hat{f}(k)$.

The solutions to this exercise can be found in the book of Lieb and Loss. If you are stuck, take a look at the solutions there... they are pretty complete, but there are still some details to fill in... Notice that we use quite a bit of Fourier theory!

Exercise 2

One possibility: use Riesz's theorem (Theorem 3.3.1 in the lecture notes) and show that the conditions stated there are equivalent to the conditions of the exercise. To do this, it is enough to show that, under the assumptions

(i) A is bounded

(ii) $\sup_{f\in A} \int_{\mathbb{R}^n \setminus B_R(0)} |f(x)|^2 dx \to 0$ as $R \to \infty$

we have

(a) $\sup_{f\in A} \|f(h+\cdot) - f\|_2 \to 0$ as $|h| \to 0$ if and only if

(b) $\sup_{f\in A} \int_{\mathbb{R}^n \setminus B_R(0)} |\hat{f}(k)|^2 dk \to 0$ as $R \to \infty$.

I would show $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$ separately. Among other things, use the Fourier transform, and in particular Plancherel's theorem and properties of convolutions. Note that for $f_h(x) := f(x+h)$ one has $\hat{f}_h(x) = e^{ih\cdot k}\hat{f}(k)$.

For the part $(a) \Rightarrow (b)$, perhaps start like this: Choose a function $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\chi \subset B_1(0)$, $0 \le \chi(x) \le 1\ \forall x \in \mathbb{R}^n$ and $\chi(x) = 1\ \forall |x| \le 1/2$. Use then (justify why!) that

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_R(0)} |\hat{f}(k)|^2 dk =& \|\hat{f}\|_2^2 - \int_{B_R(0)} |\hat{f}(k)|^2 dk \\
\le& \|f\|_2^2 - \int_{\mathbb{R}^n} \chi(k/R) |\hat{f}(k)|^2 dk \\
=& \int_{\mathbb{R}^n} |f(x)|^2 dx - \int_{\mathbb{R}^n \times \mathbb{R}^n} R^n \hat{\chi}(R(x-y)) \overline{f(x)} f(y) dx dy
\end{aligned}$$

[1] Try by yourself first!

Exercise 3

b. For the uniqueness, one can use that $H = H_0^{1,2} \oplus (H_0^{1,2})^\perp$, since $H_0^{1,2}$ is closed; in other words, any $u \in H$ has a unique decomposition of the form $u = u' + u''$ with $u' \in H_0^{1,2}$ and $u'' \in (H_0^{1,2})^\perp$.

c. Integrate by parts!

Exercise 4

Use the Arzela-Ascoli theorem for both parts.