Hints[1] for exercise sheet 2

Exercise 1:

c. For the triangle inequality, use Cauchy-Schwarz. To show non-completeness, consider for example the sequence of functions

$$f_n(x) = \begin{cases} (2x)^n, & x \in [0, 1/2) \\ 1, & x \in [1/2, 1] \end{cases}$$

Exercise 2:

d. When showing that $M^\perp \subseteq \overline{spanM}^\perp$, perhaps show first that $M^\perp \subseteq (spanM)^\perp$.

Exercise 3:

a. To show that $M^\perp \subseteq A$, that is equivalent to show that $A^c \subseteq (M^\perp)^c$, i.e. that for every $g \in A^c$ there exists $f \in M$ such that $(f, g) \neq 0$. Given g, one can construct such an f "by hand"...

b. Let $A \subseteq B \subseteq C$ be subsets of a topological space. Note that if A is dense in B (meaning that $\overline{(A)} \supseteq B$) and B is dense in C, then A is dense in C.

Here one can for example show that $c_c(\mathbb{K})$ is dense in $l^2(\mathbb{K})$, and that M is dense in $c_c(\mathbb{K})$...

Exercise 4:

b. By far, the difficult part of this exercise is to show that, given a norm which satisfies the parallelogram identity, there exists an inner product which induces that norm. Here is a sketch of an outline of a proof:

The basic idea is to define a function $V \times V \to \mathbb{K}$ which should be the desired inner product, and then prove that this function is indeed an inner product and does indeed induce the norm. Our strategy will be to show how to reduce that case when $\mathbb{K} = \mathbb{C}$ to the case when $\mathbb{K} = \mathbb{R}$, and then solve the latter case.

For $\mathbb{K} = \mathbb{C}$, from part **a.** we know that we should make the following definition as our ansatz:

$$(x, y) := \frac{1}{4}\{(\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)\} \quad \forall x, y \in V.$$

[1]Try by yourself first!

Step 1. Reduce to the case $\mathbb{K} = \mathbb{R}$. Let $V_{\mathbb{R}}$ be the real vector space obtained by considering the same set of vectors V, but with scalar multiplication defined only for real numbers. Consider a real form of $V_{\mathbb{R}}$, i.e. a subspace U of $V_{\mathbb{R}}$ such that $V_{\mathbb{R}} = U \oplus iU$. The norm on V induces a norm on $V_{\mathbb{R}}$. Obtain a real inner product on $V_{\mathbb{R}}$ and use it to build a (hermitian) inner product on V.

Step 2. Now we only need to consider the case that $(V, \|\cdot\|)$ is a real vector space with norm satisfying the parallelogram identity. The putative inner product we want should be defined now via the real polarization identity:

$$(x, y) := \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2).$$

Show that this function is symmetric and positive definite. We are then reduced to the task of showing that this function is bilinear.

Step 3. Show that $(x, y + z) = (x, y) + (x, z)$ for all $x, y, z \in V$. Note that

$$(x, y + z) = \frac{1}{4}(\|x + y + z\|^2 - \|x - y - z\|^2)$$

Use the parallelogram identity to expand the two terms on the right-hand side. Then consider the difference of the obtained equations...

Step 4. Show that $(\lambda x, y) = \lambda(x, y) \qquad \forall \lambda \in \mathbb{Z}$. Make use of Step 3 and induction....

Step 5. Show that $(\lambda x, y) = \lambda(x, y)\ \forall \lambda \in \mathbb{Q}$. Represent rationals as ratios of integers...

Step 6. Show that $(\lambda x, y) = \lambda(x, y)\ \forall \lambda \in \mathbb{R}$. For this, consider the function $\mathbb{R}\backslash\{0\} \to \mathbb{R}, \lambda \to \frac{1}{\lambda}(\lambda x, y)$ and use continuity...

Exercise 4:

b. Use real and imaginary parts...