Random walk on a discrete torus and random interlacements

David Windisch

Department of Mathematics
ETH Zurich

July 2008 / Summer School Disentis 2008

- Discrete torus $E = (\mathbb{Z}/N\mathbb{Z})^d,\ d \geq d_0,\ u > 0$ (small).
- $(X_n)_{n \geq 0}$: simple random walk on $(\mathbb{Z}/N\mathbb{Z})^d$
- Study the vacant set $E \setminus X_{[0,uN^d]}$ ($X_{[0,uN^d]} = \{X_0, \ldots, X_{[uN^d]}\}$).

1. Logarithmic components of the vacant set
2. Local pictures converge to random interlacements

1. Logarithmic components of the vacant set

2. Local pictures converge to random interlacements

Theorem (Benjamini, Sznitman, 2008)

For any $d \geq d_0$ sufficiently large, there is a constant $c_0(d) > 0$ such that for any $\beta, \gamma \in (0,1)$,

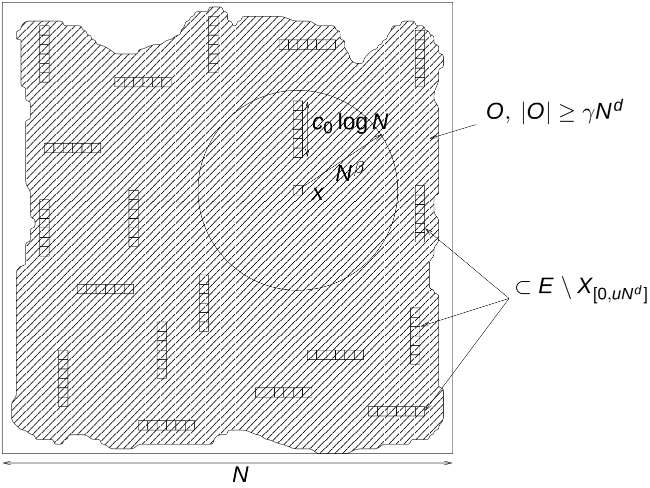
$$\lim_N P\left[\mathcal{G}_{\beta,uN^d} \cap \{|O| \geq \gamma N^d\}\right] = 1, \text{ for small } u > 0,$$

where

$$\begin{aligned}\mathcal{G}_{\beta,uN^d} = \{ & \text{ for all } x \in E, \text{ there is a segment} \\ & [x', x' + [c_0 \log N]e_i] \subset B(x, N^\beta) \setminus X_{[0,uN^d]} \text{ for some } 1 \leq i \leq d, \\ & \text{ AND all vacant segments of this length belong to a unique} \\ & \text{component } O \text{ of } E \setminus X_{[0,uN^d]}.\}\end{aligned}$$

O: "giant component"

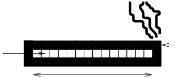
$E = (\mathbb{Z}/N\mathbb{Z})^d$



Theorem

For any $d \geq d_0$, there is a constant $c_1(d) > 0$ such that for any sufficiently small $u > 0$, $(\mathbb{Z}/N\mathbb{Z})^d \setminus X_{[0,uN^d]}$ contains a component consisting only of a segment of length $[c_1 \log N]$ with probability tending to 1 as $N \to \infty$.

Proof:

$E \setminus X_{[0,uN^d]} \supset$  $\subset X_{[0,uN^d]}$ "cigar"

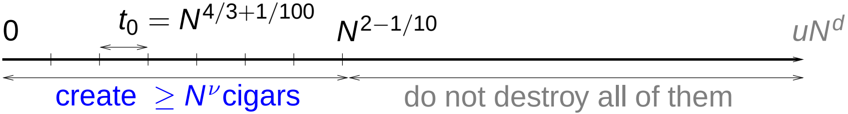
$c_1 \log N$

Show that the following occurs with high probability for some $\nu(d) > 0$:

1. Until time $N^{2-1/10}$, the random walk has produced at least $N^{\nu(d)}$ cigars,
2. some of which remain unvisited up until time uN^d.

0 — $N^{2-1/10}$ — uN^d

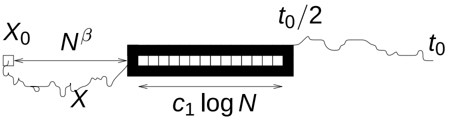
create $\geq N^{\nu}$ cigars | do not destroy all of them



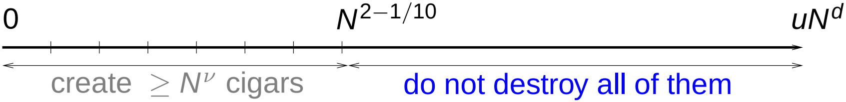
Use

Lemma ($d \geq 5$)

$$\lim_N P\left[\begin{array}{c} X_I \cap X_J = \emptyset \text{ for all subintervals } I, J \text{ of } [0, N^{2-1/10}] \\ \text{with mutual distance at least } N^{4/3}. \end{array}\right] = 1.$$

$P\Big[$  $\Big]$

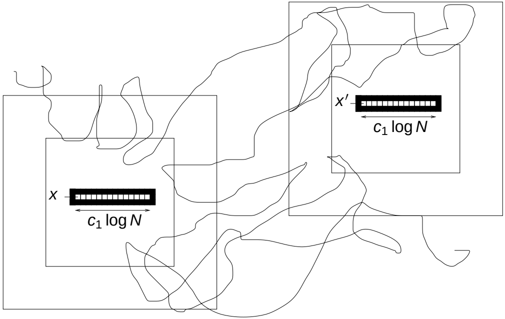
$$\geq c(\log N)N^{-\beta(d-2)} \times e^{-cc_1 \log N} \times c \geq cN^{-\epsilon}$$



For $x \in (\mathbb{Z}/N\mathbb{Z})^d$, show that

$$P\left[[x, x + [c_1 \log N]e_1] \cap X_{[0,uN^d]} = \emptyset\right] \geq \exp\{-cuc_1 \log N\} \gg N^{-\nu},$$

for $u > 0$ small. Then apply technique for bounding the covariance between such events.



1 Logarithmic components of the vacant set

2 Local pictures converge to random interlacements

For $x \in (\mathbb{Z}/N\mathbb{Z})^d$, define the vacant configuration $\omega_{x,uN^d} : \mathbb{Z}^d \to \{0,1\}$ in the neighborhood of x by

$$\omega_{x,uN^d}(.) = \mathbf{1}\{X_m \neq \pi(.) + x, \text{ for all } 0 \leq m \leq uN^d\},$$

where π denotes the projection $\mathbb{Z}^d \to (\mathbb{Z}/N\mathbb{Z})^d$.
The law $\mathbb{Q}_u$ on $\{0,1\}^{\mathbb{Z}^d}$ of the vacant set of the random interlacement at level $u > 0$ is characterized by

$$\mathbb{Q}_u[\omega(x) = 1, \text{ for all } x \in K] = \exp\{-u \operatorname{cap}(K)\},$$

for all finite subsets $K \subset \mathbb{Z}^d$.

Theorem ($u > 0$, $d \geq 3$)

Consider $M \geq 1$ and for each $N \geq 1$, sites $x_1, \ldots, x_M$ in $(\mathbb{Z}/N\mathbb{Z})^d$ such that $\lim_N |x_i - x_j|_\infty = \infty$ for $i \neq j$.

Then $(\omega_{x_1,uN^d}, \ldots, \omega_{x_M,uN^d})$ converges in distribution to $\mathbb{Q}_u^{\otimes M}$ under P, as $N \to \infty$.

For finite subsets $K_1, \dots, K_M$ of $\mathbb{Z}^d$, set $B = \bigcup_{1 \le i \le M} (x_i + K_i)$.

$$P[H_B > uN^d] \sim \exp\left\{-uN^d / E[H_B]\right\}$$
$$\to \exp\{-u \sum_{1 \le i \le M} \mathrm{cap}(K_i)\} = \prod_{1 \le i \le M} \mathbb{Q}_u[\omega(x) = 1, \text{ for all } x \in K_i].$$

$$N^d / E[H_B] = \inf\{\mathcal{E}(f,f), f : (\mathbb{Z}/N\mathbb{Z})^d \to \mathbb{R}, f \equiv 1 \text{ on } B, \sum_x f(x) = 0\}$$
$$= \sup\{1/(I,I), I \text{ unit flow on } (\mathbb{Z}/N\mathbb{Z})^d \text{ from } B \text{ to the uniform distribution}\}.$$

For a finite subset A of $\mathbb{Z}^d$:

$$\mathrm{cap}(A) = \inf\{\mathcal{E}(f,f), f \in C_c, f = 1 \text{ on } A\}$$
$$= \sup\{1/(I,I), I \text{ a unit flow from } A \text{ to infinity}\}.$$

- I. Benjamini, A.S. Sznitman.
  Giant component and vacant set for random walk on a discrete torus.
  J. Eur. Math. Soc. (JEMS), 10(1):133-172, 2008.
- D. Windisch.
  Logarithmic components of the vacant set for random walk on a discrete torus.
  Electronic Journal of Probability, 13, p. 880-897, 2008.
- D. Windisch.
  Random walk on a discrete torus and random interlacements.
  Electronic Communications in Probability, 13, p. 140-150, 2008.