Isotropic Ornstein-Uhlenbeck Flows

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IRTG Summer School
Disentis
Outline

1. Stochastic Flows And Stochastic Differential Equations
   - Stochastic Flows: A First Example
   - SDEs And Spatial Semimartingales
   - Conclusion

2. IBFs and IOUFs
   - Isotropic Brownian Flows
   - Isotropic Ornstein-Uhlenbeck Flows

3. Spatial Regularity
   - Statement Of The Result
   - Sketch Of Proof
Motivating Example

Consider the following stochastic differential equation

$$d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = A \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} 17 & 0 \\ 0 & 42 \end{pmatrix} d \begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix}$$

$X_s = x$, $Y_s = y$, $A$ is a real matrix.

The Solution to this equation is of course:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = e^{(t-s)A} \begin{pmatrix} x \\ y \end{pmatrix} + e^{(t-s)A} \int_0^t e^{-(u-s)A} \begin{pmatrix} 17 \\ 42 \end{pmatrix} dW_u$$
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The flow property

- Consider the solution as a function of the initial value.

\[ \Phi_{s,t} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto e^{(t-s)A} \begin{pmatrix} x \\ y \end{pmatrix} + e^{(t-s)A} \int_0^t e^{-uA} \begin{pmatrix} 17 \\ 42 \end{pmatrix} dW_u \]

- The function \( \Phi = \Phi_{s,t}(\cdot, \omega) \) satisfies:
  - it is a diffeomorphism for any \( \omega, s, t \)
  - \( \Phi_{t,t}(\cdot, \omega) \) is the identity for all \( \omega \) and \( t \)
  - \( \Phi_{s,t}(\cdot, \omega) = \Phi_{u,t}(\cdot, \omega) \circ \Phi_{s,u}(\cdot, \omega) \)
  - These properties state that \( \Phi \) is a stochastic flow of diffeomorphisms.
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Let us write

\[ M(t, \begin{pmatrix} x \\ y \end{pmatrix}) = A \begin{pmatrix} x \\ y \end{pmatrix} t + \begin{pmatrix} 17 & 0 \\ 0 & 42 \end{pmatrix} W_t \]

Then the SDE becomes

\[ d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = M(dt, \begin{pmatrix} X_t \\ Y_t \end{pmatrix}), \quad X_s = x, \quad Y_s = y \]
Kunita-Type SDEs

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Semimartingale Fields

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- This states that \( M \) is a semimartingale field
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**Definition**

A function $b : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is an isotropic covariance tensor if:

1. $x \mapsto b(x)$ is smooth enough and the derivatives are bounded.
2. $b(0) = E_d$ (the $d$-dimensional identity)
3. $x \mapsto b(x)$ is not constant.
4. $b(x) = O^* b(Ox) O$ for any $x \in \mathbb{R}^d$ and $O \in O(d)$

**Definition**

Let $b$ be as above. \( \{ M(t, x) : t \geq 0, x \in \mathbb{R}^d \} \) is an isotropic Brownian field if:

1. $(t, x) \mapsto M(t, x)$ is a centered Gaussian process.
2. $\text{cov}(M(s, x), M(t, y)) = (s \wedge t) b(x - y)$
3. $(t, x) \mapsto M(t, x)$ is continuous for almost all $\omega$. 
Isotropic Covariance Tensors, Isotropic Brownian Fields

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Isotropic Brownian Flows (IBFs)

Properties

- translation invariance
- rotation invariance
- one-point motion is a $d$-dimensional standard Brownian Motion
- SDEs for two-point-distance, ...
- Lyapunov-Exponents are known, deterministic and constant
- Lebesgue measure is invariant
- No straightforward entropy definition
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Isotropc Ornstein-Uhlenbeck Flows (IOUFs)

Introduce a drift into the SDE

$$\phi_{s,t}(x) = x + \int_s^t M(du, \phi_{s,u}(x)) - c \int_s^t \phi_{s,u}(x)du$$  \hspace{1cm} (1)

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Spatial Regularity Lemma

Let $\phi = \phi_{0,1} : \mathbb{R}^d \to \mathbb{R}^d$ be as in (1) (with $s = 0$ and $t = 1$). Then we have a.s.

1. $\lim_{R \to \infty} \sup_{||x|| \geq R} \frac{||\phi(x) - e^{-c}x||}{||x||} = 0$ (2)

2. $\lim_{R \to \infty} \sup_{||x|| \geq R} \frac{||\phi(x)||}{||x||} = e^{-c}$ (3)
It is sufficient to show

$$\lim_{R \to \infty} \sup_{R \leq \|x\| \leq R+1} \frac{\|\phi(x) - e^{-c}x\|}{\|x\|} = 0$$

- $x \mapsto \frac{\|\phi(x) - e^{-c}x\|}{\|x\|}$ is continuous
- $X := \{x \in \mathbb{R}^d : R \leq \|x\| \leq R + 1\}$ is compact
Reformulation To Compact State Space

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The Chaining Lemma

\((X, d)\) compact metric space, \(\phi: X \to \mathbb{R}_+\) be a.s. cont. func.,

\((\delta_i)_{i \geq 0}\) positive real numbers with \(\sum_{i=0}^{\infty} \delta_i < \infty\),

\((\chi_i)_{i=0}^{\infty}\) \(\delta_i\)-dense in \(X\) with \(\chi_0 = \{x_0\}\), with \(d(x, x_0) \leq \delta_0 \forall x \in X\).

**Lemma (Cranston, Scheutzow and Steinsaltz ’00)**

For arbitrary positive \(\epsilon, z \geq 0\) and an arbitrary sequence of positive reals \((\epsilon_i)_{i \geq 0}\) such that \(\epsilon + \sum_{i=0}^{\infty} \epsilon_i = 1\) we have

\[
\mathbb{P}\left( \sup_{x \in X} \phi(x) > z \right) \\
\leq \mathbb{P}\left( \phi(x_0) > \epsilon z \right) + \sum_{i=0}^{\infty} |\chi_{i+1}| \sup_{d(x,y) \leq \delta_i} \mathbb{P}\left( |\phi(x) - \phi(y)| > \epsilon_i z \right).
\]
Estimates

\[
\mathbb{P}\left[ \|\phi(x_0) - e^{-c}x_0\| > \frac{\tilde{\epsilon}R}{2} \right] \leq c_4 e^{-\frac{\tilde{\epsilon}^2}{8d^2}R^2}
\]

\[
\mathbb{P}\left[ \|\|\phi(x) - e^{-c}x\| - \|\phi(y) - e^{-c}y\|\| > 2^{-j-2}\tilde{\epsilon}R \right]
\leq \mathbb{P}\left[ \|\|\phi(x) - e^{-c}x\| - \|\phi(y) - e^{-c}y\|\| > 2^{-j-2}\tilde{\epsilon}R3^j|x - y| \right]
\leq \mathbb{P}\left[ \|\phi(x) - \phi(y)\| > 2^{-j-3}\tilde{\epsilon}R3^j|x - y| \right]
\leq \mathbb{P}\left[ B_1^* \geq \frac{\log(2^{-3-j}\tilde{\epsilon}R3^j)}{\sigma} - \lambda \right]
\leq c_5(2^{-j-3}\tilde{\epsilon}R3^j)^{-\frac{\log(2^{-3-j}\tilde{\epsilon}R3^j)-2\lambda}{2\sigma^2}}
Estimates

\[
P \left[ \| \phi(x_0) - e^{-c} x_0 \| > \tilde{\epsilon} R \right] \leq c_4 e^{-\frac{\tilde{\epsilon}^2}{8d^2}} R^2
\]

\[
P \left[ \| \phi(x) - e^{-c} x \| - \| \phi(y) - e^{-c} y \| > 2^{-j} \tilde{\epsilon} R \right]
\leq P \left[ \| \phi(x) - e^{-c} x \| - \| \phi(y) - e^{-c} y \| > 2^{-j} \tilde{\epsilon} R 3^j \| x - y \| \right]
\leq P \left[ \| \phi(x) - \phi(y) \| > 2^{-j} \tilde{\epsilon} R 3^j \| x - y \| \right]
\leq P \left[ B_1^* \geq \frac{\log(2^{-3-j} \tilde{\epsilon} R 3^j) - \lambda}{\sigma} \right]
\leq c_5 (2^{-j} \tilde{\epsilon} R 3^j) \frac{\log(2^{-3-j} \tilde{\epsilon} R 3^j) - 2\lambda}{2\sigma^2}
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Open Problems - Directions Of Research

- Pesin Formula for IOUFs
- Meaningful definition of the entropy for IBFs
- Pesin Theory for IBFs
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References