Differentiability of reflected BSDEs with quadratic growth

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IRTG Stochastic Models of Complex Processes

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Outline

What is a BSDE?

Parameters:

- ξ r.v. $\mathcal{F}_T$-measurable
- $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ predictable mapping

A BSDE with *terminal condition* ξ and *generator/driver* f is an equation of the type

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds. \tag{1}$$

A solution is a *pair* of adapted processes (Y, Z) such that (1) makes sense.

Utility maximization

- incomplete financial market, i.e. $d < m$ stocks

$$dS_t^i = S_t^i(b_t^i dt + \sigma_t^i dW_t), \; i = 1, \dots d,$$

where $b \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d,m}$.

- small investor: wealth process ($p_s := \pi_s \sigma_s, \; \theta_s := \sigma_s^{-1} b_s$)

$$V_t^p = v + \int_0^t \pi_s \frac{dS_s}{S_s} = v + \int_0^t p_s(dW_s + \theta_s ds)$$

- utility function

$$U(x) = -\exp^{-\alpha x} \; (\alpha > 0 \text{ risk aversion})$$

- Optimization problem under constraint C

$$Val(v) = \sup_{p \in C} E\left[U(V_T^p)\right]$$

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- utility function

$$U(x) = -\exp^{-\alpha x} \; (\alpha > 0 \text{ risk aversion})$$

- ξ European Option
- Optimization problem under constraint C

$$Val(v) = \sup_{p \in C} E\left[U(V_T^p + \xi)\right]$$

Utility maximization

Optimization problem: $Val(v) = \sup_{p \in C} E\left[U(V_T^p + \xi)\right]$

Idea: Find a process Y with terminal condition $Y_T = \xi$ such that

- $U(V_t^p + Y_t)$ is a supermartingale for all p
- $U(V_t^{p^{opt}} + Y_t)$ is a martingale for one p^{opt}

$\rightarrow$ BSDE with terminal condition ξ

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s) ds.$$

Utility maximization

Optimization problem: $Val(v) = \sup_{p \in C} E\left[U(V_T^p + \xi)\right]$

Theorem (Hu, Imkeller, Müller 2005)

$$Val(v) = U(v + Y_0)$$

where (Y, Z) is the unique solution of

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s) ds$$
$$\text{and } f(\cdot, z) = -\frac{\alpha}{2} dist^2(\frac{1}{\alpha}\theta - z, C) - z\theta + \frac{1}{2\alpha}|\theta|^2.$$

!f grows quadratically in z!

What is a RBSDE?

Parameters:

- $(\xi_t)_{t\in[0,T]}$ continuous on $[0,T[$ and $\lim_{t\to T}\xi_t \leq \xi_T$
- $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ predictable mapping

A RBSDE with *barrier* ξ and *generator/driver* f is an equation of the type

$$Y_t = \xi_T - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t, \qquad (2)$$

$$Y_t \geq \xi_t, \quad \int_0^T (Y_t - \xi_t)dK_t = 0,$$

where K is a continuous nondecreasing process.
A solution is a *triple* of adapted processes (Y, Z, K) such that (2) makes sense.

Utility maximization

Same setting as before:

- wealth process $V_t^p = v + \int_0^t p_s(dW_s + \theta_s ds)$
- utility function $U(x) = -e^{-\alpha x}$ ($\alpha > 0$ risk aversion)

Question: What happens if the investor holds an American option with payoff function $(\xi_t)_{t\in[0,T]}$?

Optimization problem:

$$Val(v) = \sup_{\nu,p} E\left[U(V_T^p + \xi_\nu)\right]$$

Utility maximization

Optimization problem: $Val(v) = \sup_{\nu,p} E\left[U(V_T^p + \xi_\nu)\right]$

Theorem (A.R.)

$$Val(v) = U(v + Y_0)$$

where (Y, Z, K) *is the unique solution of*

$$Y_t = \xi_T - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s)ds + K_T - K_t,$$

$Y_t \geq \xi_t$, $\int_0^T (Y_t - \xi_t)dK_t$, *with* K *continuous, nondecreasing and*

$$f(\cdot, z) = -\frac{\alpha}{2} dist^2(\frac{1}{\alpha}\theta - z, C) - z\theta + \frac{1}{2\alpha}|\theta|^2.$$

!f grows quadratically in z!

Parameterized RBSDE

Parameter dependence on $x \in \mathbb{R}$

$$Y_t^x = \xi_T(x) - \int_t^T Z_s^x dW_s + \int_t^T f(s, Z_s^x) ds + K_T^x - K_t^x.$$

$$Y_t^x \geq \xi_t(x), \quad \int_0^T (Y_t^x - \xi_t(x)) dK_t^x = 0,$$

Question: Are the solution processes Y^x, Z^x and K^x continuous or even differentiable with respect to x?

Our setting: Quadratic RBSDEs

- Consider RBSDE

$$Y_t = \xi_T - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s) ds + K_T - K_t,$$
$$Y_t \geq \xi_t, \quad \int_0^T (Y_t - \xi_t) dK_t = 0,$$

  with
  - ξ bounded adapted process, continuous on $[0, T[$ and $\lim_{t \to T} \xi_t \leq \xi_T$
  - f s.t. $\forall (t, z)$: $|f(t, z)| \leq M(1 + |z|^2)$, and continuous in z
- Kobylanski (02) proved solution processes are $\sup_t |Y_t| < \infty$ and $E[\int Z_s^2 ds] < \infty$

BMO Martingales

Definition (BMO)

Uniformly integrable martingales M with $M_0 = 0$ and

$$\| M \|_{BMO} = \sup_{\tau} \| E[\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau]^{\frac{1}{2}} \|_\infty < \infty$$

$\mathcal{E}(M) := exp\{M - \frac{1}{2}\langle M \rangle\}$

Theorem (Kazamaki 1994)

- $M\ BMO \Longrightarrow dQ = \mathcal{E}(M)_T dP$ *is a probability measure*
- $M\ BMO \Longrightarrow \exists p > 1$ *such that* $\mathcal{E}(M) \in L^p$

Theorem (A.R.)

(Y, Z, K) *solution of the above RBSDE* $\Longrightarrow \int ZdW$ *is BMO*

Moment estimates

Using Itô formula, the BMO property of $\int ZdW$ and inequalities of Hölder, BDG, Doob,Young, for $p > 1$:

Theorem (A.R.)

$$E^P\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right] + E^P\left[\Big(\int_0^T |Z_s|^2 ds\Big)^p\right] + E^P\left[K_T^{2p}\right]$$
$$\leq CE^P\left[\xi_T^{2pq^2} + \sup_{t\in[0,T]} |\xi_t|^{2pq^2} + \Big(\int_0^T f(s,0)ds\Big)^{2pq^2}\right]^{\frac{1}{q^2}}.$$

With similar methods we can estimate the variation in the solution induced by a variation in the data!

Results

Theorem (A.R.)

Let ξ be differentiable in x, lipschitz in norm, f be differentiable in z, $\nabla_z f$ of linear growth in z,
Then for $p > 1$ and $|x - x'| < 1$

$$E\left[\sup_{t\in[0,T]} |Y_t^x - Y_t^{x'}|^{2p}\right] \leq C|x - x'|^p$$

$$E\left[\left(\int_0^T |Z_t^x - Z_t^{x'}|^2 ds\right)^p\right] \leq C|x - x'|^p$$

$$E\left[\sup_{t\in[0,T]} |K_t^x - K_t^{x'}|^{2p}\right] \leq C|x - x'|^p.$$

Spaces:

- $\mathcal{S}^p$ space of predictable processes X such that

$$\| X \|_{\mathcal{S}^p} = E\left[\sup_t |X_t|^p\right]^{\frac{1}{p}} < \infty$$

- $\mathcal{H}^p$ space of predictable processes X such that

$$\| X \|_{\mathcal{H}^p} = E\left[\left(\int_0^T |X_t|^2 dt\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty$$

Corollary (A.R.)

- *(Y_t^x) and (K_t^x) are continuous in t and x.*
- *$\mathbb{R} \to \mathcal{H}^{2p} : x \mapsto Z^x$ is Hölder continuous with $\alpha = \frac{1}{2}$.*
- *$\mathbb{R} \to \mathcal{S}^{2p} : x \mapsto Y^x$ is Hölder continuous with $\alpha = \frac{1}{2}$.*

Differentiability

BUT:
We can't prove Differentiability of Y^x in x in the classical sense
Reason:

$$E\left[\sup_{t\in[0,T]}|Y_t^x - Y_t^{x'}|^{2p}\right] \leq C|x-x'|^p$$

We would like to prove:

Theorem

There exists a version of (Y_t^x, Z_t^x, K_t^x) *such that a.s.*

- Y^x *continuously differentiable in a weak sense*
- Z^x *is differentiable in a weak sense*

Thank you!