Optimal execution strategies in limit order books

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Outline

- Problem
- Limit order book model
- Optimal execution strategy
- Examples
- Sketch of the proof
- Model ramifications

Problem

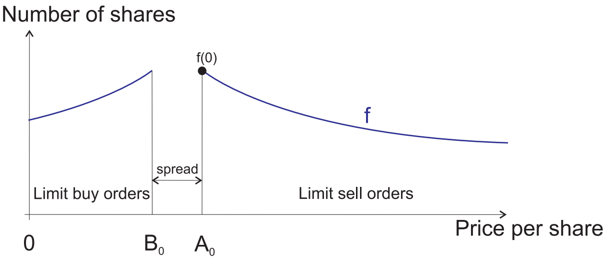
- Trade a big position of a **single asset** in fixed time $\rightsquigarrow$ **Price impact!**
- More precisely: Buy $X \in \mathbb{N}$ shares over $[0, T]$ at equidistant trading times $(t_n)_{n=0,\ldots,N}$
- Find **optimal strategy** $\xi_0, \ldots, \xi_N$ with $\sum_{n=0}^{N} \xi_n = X$ such that expected costs are minimized $\rightsquigarrow$ risk neutral investor

$$\min_{\xi} \mathbb{E}\Big[\sum_{n=0}^{N} \pi_{t_n}(\xi_n) \Big]$$

- We need a market model for the **transaction cost** π !

Market: Limit order book (LOB)

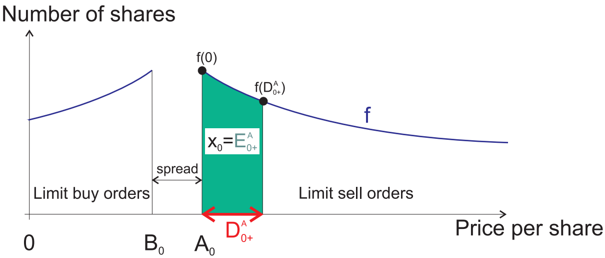
- Snapshot of a LOB in $t = 0$:



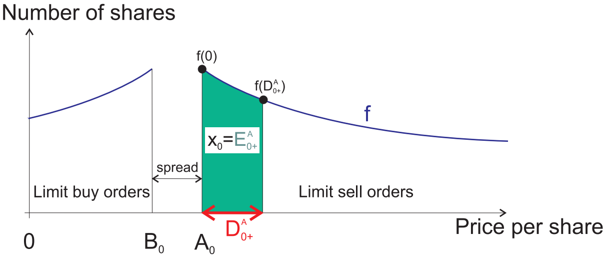
- LOB form: $f : \mathbb{R} \to]0, \infty[$ continuous
- Unaffected best ask A_t is a martingale and the best bid satisfies $B_t \leq A_t$

Market: Limit order book (LOB)

- Price impact of a market buy order x_0



- **Resilience** of the LOB



- Exponential resilience with resilience speed ρ

Model E	Model D
$E_{t_1} = e^{-\rho\tau} E_{t_0+}$	$D_{t_1} = e^{-\rho\tau} D_{t_0+}$

- Our model is a generalization of **Obizhaeva, Wang** (2005)

Model

Cost of transaction of size x_t at time t

$$\pi_t(x_t) := \left\{ \begin{array}{ll} A_t x_t + \int_{D_t^A}^{D_{t+}^A} x f(x) dx & \text{buy order} \\ B_t x_t + \int_{D_t^B}^{D_{t+}^B} x f(x) dx & \text{sell order} \end{array} \right\}$$

Stochastic optimization problem (risk neutral investor)

$$\min_{\xi} \mathbb{E}\Big[\sum_{n=0}^{N} \pi_{t_n}(\xi_n) \Big]$$

for all adapted strategies $\xi = (\xi_0, ..., \xi_N)$ such that ξ_n is bounded from below and $\sum_{n=0}^{N} \xi_n = X$

Optimal strategy

Theorem

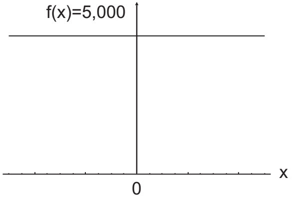
Under some technical assumptions, there exists a **unique optimal strategy** ξ *in both models. It is* **deterministic**, *consists only of buy orders and is determined by:*

	Model E	**Model D**
ξ_0	$\tilde{h}_E(\xi_0) = 0$	$\tilde{h}_D(\xi_0) = 0$
$\xi_1 = \ldots = \xi_{N-1}$	$\xi_0(1 - e^{-\rho\tau})$	$\xi_0 - F(e^{-\rho\tau} F^{-1}(\xi_0))$
ξ_N	$X - \xi_0 - (N-1)\xi_1$	

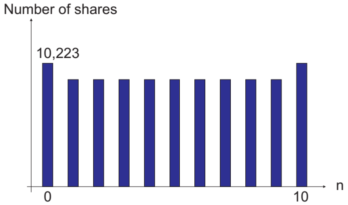
- **Interpretation:** $E_{t_1} = \ldots = E_{t_N} \rightsquigarrow$ "Optimal level of E", trade-off between price impact and attracting new limit sell orders

Example 1

- Constant limit order book form:

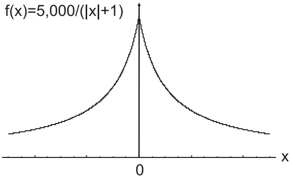


- Same optimal strategy for Model E and D: $\xi_0 = \xi_N = \frac{X}{(N-1)(1-e^{-\rho\tau})+2}$

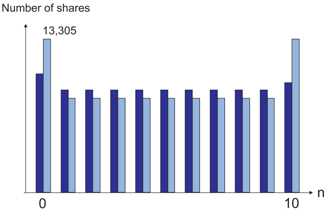


Example 2

- Limit order book form:



- Optimal strategy for Model E and D:



Proof for Model E !

$$\pi_t(x_t) := \left\{ \begin{array}{ll} A_t x_t + \int_{D_t^A}^{D_{t+}^A} x f(x) dx & \text{buy order} \\ B_t x_t + \int_{D_t^B}^{D_{t+}^B} x f(x) dx & \text{sell order} \end{array} \right\}$$

$$\min_{\xi} \mathbb{E}\Big[\sum_{n=0}^{N} \pi_{t_n}(\xi_n) \Big]$$

1. Reduction to deterministic strategies
2. Lagrange method to determine optimal strategy
3. Uniqueness and positivity of the strategy

Proof: 1. Reduction to deterministic strategies

- W.l.o.g consider only buy orders
- Martingale property of A and integrating by parts yields:

$$\mathbb{E}\Big[\sum_{n=0}^{N} \pi_{t_n}(\xi_n)\Big] = XA_0 + \mathbb{E}\Big[\underbrace{\sum_{n=0}^{N} \int_{D^A_{t_n}}^{D^A_{t_{n+}}} xf(x)dx}_{=:C(\xi_0,\dots,\xi_N)}\Big]$$

- Show C has unique minimum in $\{(x_0,\dots,x_N) \in \mathbb{R}^{N+1}_{>0} \mid \sum_{n=0}^{N} x_n = X\}$

Proof: 2. Lagrange method

- Show $C(x) \overset{|x|\to\infty}{\longrightarrow} \infty$ to guarantee the existence of a Lagrange multiplier $\nu \in \mathbb{R}$ with

$$\nu = \frac{\partial}{\partial x_n} C(x_0^*, ..., x_N^*)$$
$$= a \left[\frac{\partial}{\partial x_{n+1}} C - F^{-1}\left(a(a^n x_0^* + ... + x_n^*)\right) \right] + F^{-1}(a^n x_0^* + ... + x_n^*)$$

  with **resilience coefficient** $a := e^{-\rho\tau}$

- This leads to the system $h_E(a^n x_0^* + ... + x_n^*) = \nu(1-a)$ for $n = 0, ..., N-1$ which is explicitly solved by

$$\begin{aligned} x_0^* &= h_E^{-1}(\nu(1-a)) \\ x_n^* &= x_0^*(1-a) \text{ for } n = 1, ..., N-1 \\ x_N^* &= X - x_0^* - (N-1)x_n^* \end{aligned}$$

- Find $x_0^* : C(x_0^*, ..., x_N^*) = \overline{C}(x_0^*)$ with $\frac{\partial}{\partial x}\overline{C}(x) = \widetilde{h}_E(x)$

Ramifications

- Inhomogeneous trading times $(t_n)_{n=0,...,N}$ and time varying resilience $(\rho_t)_{t\in[0,T]}$

$$a_n := e^{-\int_{t_{n-1}}^{t_n} \rho_t dt}$$

- If $f(x) \equiv$ const., then the optimization can be reduced to a quadratic form $\min_x \frac{1}{2}\langle x, Mx\rangle$ with

$$M := \begin{bmatrix} 1 & a_1 & a_1 a_2 & \cdots & a_1 \dots a_N \\ a_1 & 1 & a_2 & & \vdots \\ a_1 a_2 & a_2 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_N \\ a_1 \dots a_N & \cdots & \cdots & a_N & 1 \end{bmatrix} \in]0,1]^{N+1,N+1}$$

Ramifications

Optimal strategy without constraints

There is a unique, deterministic, positive optimal strategy:

$$\xi_0 = \frac{c}{1+a_1},\ \xi_n = c\left(\frac{1}{1+a_n} - \frac{a_{n+1}}{1+a_{n+1}}\right) \text{ for } n = 1, \dots, N-1,\ \xi_N = \frac{c}{1+a_N}$$

Optimal strategy with constraints

Linear constraints $\left\{x \in \mathbb{R}^{N+1} \middle| \sum_{n=0}^{N} x_n = X, \langle v^j, x \rangle \geq 0 \right\}$
Then the optimal strategy is

$$x = cM^{-1}\mathbf{1} + \sum_j c_j M^{-1} v^j$$

for constants c, c_j uniquely determined by a system of linear equations.

Conclusion

- Market microstructure model for LOB
- Improvements compared to Obizhaeva, Wang:
  - LOB form not necessarily constant $\rightsquigarrow$ nonlinear price impact
  - Explicit optimal strategies with similar qualities ("Optimal level of E")
  - More general unaffected best ask, bid

Thank you for your attention!

[1] Alfonsi, A., Fruth, A., Schied, A. *Optimal execution strategies in limit order books with general shape functions.* Preprint, TU Berlin (2007)

[2] Alfonsi, A., Fruth, A., Schied, A. *Constrained portfolio liquidation in a limit order book model.* Preprint, forthcoming in Banach Center Publications, TU Berlin (2007)

[3] Obizhaeva, A., Wang, J. *Optimal trading strategy and supply/demand dynamics.* Preprint, forthcoming in Journal of Financial Markets