Dynamic Risk Measures and Conditional Robust Utility Representation
How can we Understand Risk in a Dynamic Setting?

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IRTG — Disentis Summer School 2008

Juli 22th 2007
Outline

1. Dynamic Risk Measures: Disappointment
2. Preference Orders
3. Conditional Preference Orders
4. Dynamic of Preferences
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1. Dynamic Risk Measures: Disappointment
2. Preference Orders
3. Conditional Preference Orders
4. Dynamic of Preferences
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

**Definition (Convex Risk Measure — Artzner & al, Föllmer & Schied)**

A functional $\rho : \mathbb{L}^\infty \to \mathbb{R}$ is a convex risk measure if it is:

- **Monotone**: For $X, Y \in \mathbb{L}^\infty$, $X \geq Y$ then $\rho(X) \leq \rho(Y)$
- **Translation invariant**: For $X \in \mathbb{L}^\infty$ and $m \in \mathbb{R}$, $\rho(X + m) = \rho(X) - m$
- **Convex**: For $X, Y \in \mathbb{L}^\infty$ and $\lambda \in [0, 1]$:
  $$\rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$$
- **Normalized**: $\rho(0) = 0$
Dynamic Risk Measures: Disappointment
Definition - Conditional case

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\mathcal{F}_t\) a sub-\(\sigma\)-algebra of \(\mathcal{F}\).

**Definition (Conditional Convex Risk Measure)**

A functional \(\rho_t : L^\infty \to L^\infty_t\) is a conditional convex risk measure if it is:

- **Monotone:** For \(X, Y \in L^\infty\), \(X \geq Y\) then \(\rho_t (X) \leq \rho_t (Y)\) \(P\)-a.s.

- **Conditionally translation invariant:** For \(X \in L^\infty\) and \(m_t \in L^\infty_t\),
  \(\rho_t (X + m_t) = \rho_t (X) - m_t\) \(P\)-a.s.

- **Conditionally convex:** For \(X, Y \in L^\infty\) and \(0 \leq \lambda_t \leq 1\) \(\mathcal{F}_t\)-measurable:
  \[\rho_t (\lambda_t X + (1 - \lambda_t) Y) \leq \lambda_t \rho_t (X) + (1 - \lambda_t) \rho_t (Y)\] \(P\)-a.s.

- **Normalized:** \(\rho_t (0) = 0\) \(P\)-a.s.
An important result concerning convex risk measures is the dual representation (Static case: Föllmer and Schied. Conditional case: Detlefsen and Scandolo).

**Theorem**

If a conditional convex risk measure is continuous from below (i.e. \( X_n \downarrow X \) implies \( \rho_t(X_n) \nearrow \rho_t(X) \)) the following representation holds:

\[
\rho_t(X) = \operatorname{ess sup} \left\{ E_Q \left[ -X \mid \mathcal{F}_t \right] - \alpha_t(Q) \right\}
\]

where \( \alpha_t : \mathcal{M}_1(\Omega, \mathcal{F}, P) \rightarrow L_\infty(\Omega, \mathcal{F}_t, P) \cup \infty \) is a penalty function.
Considering a family of conditional risk measures \((\rho_t)_{t\in[0,T]}\) on a filtrated probability space, the property of time consistency is understood as follow:

**Definition**

The family of conditional convex risk measures, is said to be time consistent if for all \(X, Y \in L^\infty\) and times \(0 \leq t \leq s \leq T\), holds:

\[
\rho_s (X) \geq \rho_s (Y) \quad P\text{-a.s.} \quad \Rightarrow \quad \rho_t (X) \geq \rho_t (Y) \quad P\text{-a.s.}
\]

This definition is equivalent to the following dynamic programing principle:

\[
\rho_t (X) = \rho_t (-\rho_s (X))
\]
Dynamic Risk Measures: Disappointment

Why are we so disappointed?
The time consistency together with cash invariance impose some very strong conditions in the continuous case such that infinitely many of them lead to some entropic-"like" risk measures, i.e. $\rho_t(X) = 1/\gamma \ln \left( E \left[ e^{-\gamma X} \big| \mathcal{F}_t \right] \right)$. 
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For a subdivision $\sigma_n$ of the interval $[0, T]$, take as penalty function $\alpha_t(Q) = E \left[ \varphi \left( \frac{Z}{Z_t} \right) \mid \mathcal{F}_t \right]$ for a positive convex function $\varphi$ twice differentiable in a neighborhood of 1 and with $\inf \varphi(x) = \varphi(1) = 0$. The filtration is generated by a Brownian motion.

If we imposed for the corresponding discrete family of risk measures $\rho_{\sigma_n}^{\sigma_n}$ to be time consistent we have:

\begin{equation}
\rho_{\sigma_n}^{\sigma_n}(X) \frac{dP \otimes dt}{|\sigma_n| \to 0} \to \frac{1}{\gamma} \ln \left( E \left[ e^{-\gamma X} \mid \mathcal{F}_t \right] \right) \tag{2.1}
\end{equation}

where $\gamma = 2/\varphi''(1)$.
Moreover, KUPPER and SCHACHERMAYER proved in the restrictive framework of law invariance a general result:

**Theorem**

For an infinite family $\rho_n$ of law invariant risk measures on an atom free filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$. If the family is time consistent, there exists then $\gamma \in \mathbb{R}^+ \cup \infty$ such that:

$$\rho_n(X) = \frac{1}{\gamma} \ln \left( E \left[ e^{-\gamma X} \mid \mathcal{F}_n \right] \right)$$
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4 Dynamic of Preferences
Preference Orders

von Neumann J. & Morgenstern O. (1944)[7]

The preference order is defined by a binary relation $\succeq$ on the set of measures with bounded support $\mathcal{M}_b(S, \mathcal{F}) \equiv \mathcal{M}$.
Preference Orders

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Preference Axioms

- **Weak Preference Order**: $\succeq$ is reflexive, transitive and complete.
- **Independance**: For any $\mu \succ \nu$ holds:
  \[ \alpha \mu + (1 - \alpha) \lambda \succ \alpha \nu + (1 - \alpha) \lambda \]
  for any $\lambda \in \mathcal{M}$ and $\alpha \in ]0, 1]$.

- **Continuity**: The restriction of $\succeq$ to $\mathcal{M}(B(0, r))$ is continuous w.r.t. the weak topology for any $r > 0$.

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ such that:

\[ \mu \succeq \nu \iff U(\mu) \geq U(\nu) \]

where:

\[ U(\mu) = \int u(x) \mu(dx) \]
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  for any \( Y \in \mathcal{X} \) and \( \alpha \in ]0, 1] \).
- Several other technical axioms (archimedian, monotonicity, . . .)

**Numerical Representation**

There exist a continuous function \( u : \mathbb{R} \mapsto \mathbb{R} \) and a probability measure \( Q \in M_1(\Omega, \mathcal{F}) \) such that:

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X \succeq Y \iff U(X) \geq U(Y)
\]

where:

\[
U(X) = E_Q[u(X)]
\]
Preference Orders

Savage L. (1954)

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Preference Orders

Robust version: Gilboa & Schmeidler (89)[3], Maccheroni... (04)[5], Föllmer... (07)[1][2]...

To overcome Elsberg’s paradox, the independence axiom will be weakened.
To overcome Elsberg’s paradox, the independence axiom will be weakened. The preference order are now defined on the space $\tilde{X}$ of uniformly bounded stochastic kernels on the real line $\tilde{X} (\omega, dx)$ in which $\mathcal{X}$ and $\mathcal{M}_b (\mathbb{R})$ are embedded.
Preference Axioms

- **Weak Preference Order:** $\succeq$ is reflexive, transitive and complete.
- **Weak Certainty Independence:**
  \[ \alpha \tilde{X} + (1 - \alpha) \mu \succ \alpha \tilde{Y} + (1 - \alpha) \mu \]
  \[ \Downarrow \]
  \[ \alpha \tilde{X} + (1 - \alpha) \nu \succ \alpha \tilde{Y} + (1 - \alpha) \nu \]
  for any $\nu \in \mathcal{M}_b (\mathbb{R})$.
- **Uncertainty Aversion:** For $\tilde{X} \sim \tilde{Y}$ and $\alpha \in [0, 1]$ holds:
  \[ \alpha \tilde{X} + (1 - \alpha) \tilde{Y} \succeq \tilde{X} \]
- + technical axioms (archimedean, monotonicity, continuity from above)

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ and a penalty function $\alpha : \mathcal{M}_1 (\Omega, \mathcal{F}) \mapsto \mathbb{R} \cup \infty$ such that:

\[ X \succeq Y \iff U(X) \geq U(Y) \]

where:

\[ U(X) = \inf_{Q \in \mathcal{M}_1 (\Omega, \mathcal{F})} \{ E_Q [u(X)] + \alpha(Q) \} \]

In particular:

\[ U(X) = -\rho^{\text{conv}} (u(X)) \]
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Conditional Preference Orders

What is a Conditional Preference Order.

The question of a conditional preference order has already emerged in the literature (Kreps & Porteus [4], Skiadas [6], Macheroni & al.) but their axiomatic is highly disputable, and is strongly related to their basic setting (Trees).
Conditional Preference Orders

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The key question to address is the completeness, and they are beyond the conditional concept in stochastic many reasons for doubting of this assumption:

Indeed, Incompleteness does not reflects an unexceptional trait as pointed out by Aumann R.J.:

*Of all the axiom of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life, but unlike them we find it hard to accept even from a normative viewpoint. [...] For example, certain decisions that an individual is asked to make might involve highly hypothetical situations, which he will never face in real life. He might feel that he cannot reach an “honest” decision in such cases. Other decision problems might be extremely complex, too complex for intuitive “insight”, and our individual might prefer to make no decision at all in these problems. Is it “rational” to force decision in such cases?*
Conditional Preference Orders

Axiomatic

- **Partial Weak Order**: $\succeq^\mathcal{G}$ is $P$-a.s. reflexive and transitive.
- **$\mathcal{G}$-consistency**: For all $\tilde{X}, \tilde{Y}$ and family $(A_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{G}$ holds:
  - Intersection consistency: $\exists n \in \mathbb{N}, \tilde{X} \succeq^\mathcal{G} A_n \tilde{Y} \implies \tilde{X} \succeq^\mathcal{G} \left\{ \bigcap_{n \in \mathbb{N}} A_n \right\} \tilde{Y}$
  - Union consistency: $\forall n \in \mathbb{N}, \tilde{X} \succeq^\mathcal{G} A_n \tilde{Y} \implies \tilde{X} \succeq^\mathcal{G} \left\{ \bigcup_{n \in \mathbb{N}} A_n \right\} \tilde{Y}$
  - Least comparison: There exists $A \in \mathcal{G}$ with $P[A] > 0$ such that: $\tilde{X} \succeq^A \tilde{Y}$ or $\tilde{X} \preceq^A \tilde{Y}$

- **$\mathcal{G}$-Uncertainty Aversion**: For $\tilde{X} \sim^\mathcal{G} \tilde{Y}$ holds $\alpha \tilde{X} + (1 - \alpha) \tilde{Y} \succeq^\mathcal{G} \tilde{X}$ for all $\mathcal{G}$-measurable function $\alpha$ with $0 \leq \alpha \leq 1$

- **Monotonicity**: If $\tilde{Y}(\omega) \succeq^\mathcal{G} \tilde{X}(\omega)$ $P$-a.s., then $\tilde{Y} \succeq^\mathcal{G} \tilde{X}$. Moreover, for reals $x, y, x < y$ iff $\delta_x \prec^\mathcal{G} \delta_y$

- **Weak Certainty Independence**: For $\tilde{X}, \tilde{Y} \in \mathcal{X}$, $\tilde{Z}_i \equiv \mu_i \in \mathcal{M}_b(\mathbb{R}, \mathcal{G})$ for $i = 1, 2$ and a $\mathcal{G}$-measurable function $\alpha$ such that $0 < \alpha \leq 1$ we have:
  $$\alpha \tilde{X} + (1 - \alpha) \tilde{Z}_1 \succeq^\mathcal{G} \alpha \tilde{Y} + (1 - \alpha) \tilde{Z}_1 \implies \alpha \tilde{X} + (1 - \alpha) \tilde{Z}_2 \succeq^\mathcal{G} \alpha \tilde{Y} + (1 - \alpha) \tilde{Z}_2$$

- **Continuity**: If $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{X}$ are such that $\tilde{Z} \succeq^\mathcal{G} \tilde{Y} \succeq^\mathcal{G} \tilde{X}$, there exists then $\mathcal{G}$-measurable functions $\alpha, \beta$ with $0 < \alpha, \beta < 1$ such that:
  $$\alpha \tilde{Z} + (1 - \alpha) \tilde{X} \succeq^\mathcal{G} \tilde{Y} \succeq^\mathcal{G} \beta \tilde{Z} + (1 - \beta) \tilde{X}$$

Moreover for all $c > 0$, the restriction of $\succeq^\mathcal{G}$ to $\mathcal{M}_1([-c, c], \mathcal{G})$ is continuous with respect to the $P$-a.s. weak topology.
Even if we loose completeness, we can manage to deal with in a good way:

**Lemma**

*Suppose given a weak partial preference order satisfying the first and second axiom aforementioned, then for each $\tilde{X}, \tilde{Y} \in \tilde{X}$ there exists a partition $A, B, C \in \mathcal{G}$ of $\Omega$ such that:*

$$
\begin{align*}
\tilde{X} &\succ^G_A \tilde{Y} \\
\tilde{X} &\prec^G_B \tilde{Y} \\
\tilde{X} &\sim^G_C \tilde{Y}
\end{align*}
$$
Considering the restriction of $\succeq^G$ on $\mathcal{M}_b(\mathbb{R}, \mathcal{G})$ we get a conditional version of the theorem of von Neumann J. & Morgenstern O.:

**Theorem**

If $\succeq^G$ verify the first, second, fifth and sixth axiom aforementioned, there exists then a conditional von Neumann and Morgenstern representation of $\succeq^G$:

\[
\forall \mu \in \mathcal{M}_b(\mathbb{R}, \mathcal{G}) \ , \text{ for } P\text{-almost all } \omega \in \Omega \ , \ U(\mu, \omega) = \int u(x, \omega) \mu(dx, \omega)
\]

(4.1)

where $U(\mu, \cdot)$ is a $\mathcal{G}$-measurable random variable, for all $\omega \in \Omega$, $u(\cdot, \omega)$ is continuous and for all $x \in \mathbb{R}$, $u(x, \cdot)$ is $\mathcal{G}$-measurable.
Theorem

If the preference order \( \succeq^G \) fulfills all the axioms aforementioned, there exists then a conditional numerical representation \( \tilde{U} \) which restriction on \( \mathcal{M}_b \left( \mathcal{R}, G \right) \) is a conditional von Morgenstern and Neumann representation. If moreover the range of \( u \) is \( P \)-a.s. equal to \( \mathbb{R} \) and the induced preference order \( \succeq^G \) on \( X \), viewed as a subset of \( \tilde{X} \) satisfies the following additional continuity property:

\[
X \succeq^G Y \text{ and } X_n \nearrow X \text{ } P\text{-a.s.} \quad \implies \quad X_n \succeq^G Y \text{ for all large } n \quad (4.2)
\]

There exists then a penalty function

\[
\alpha_{\text{min}}^G : \mathcal{M}_1 \left( \Omega, \mathcal{F} \right) \rightarrow [L^\infty \left( \Omega, G, P \right) \cup \{+\infty\}] \text{ such that we get for the induced preference relation a generalised robust Savage representation on } X:
\]

\[
U(X) = \text{ess inf}_{Q \in \mathcal{M}_1(\Omega, \mathcal{F}, \equiv P \text{ on } \mathcal{F}_t)} \left\{ E_Q \left[ u(X) \mid G \right] + \alpha_{\text{min}}^G(Q) \right\} \quad (4.3)
\]
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We consider here some processes \((X_t)_{t=0,1,...,T}\).

- **Temporal Consistency:** If \(X \succeq^{t+1} Y\) and \(X = Y\) up to time \(t\), then \(X \succeq^t Y\).
  This should deliver the time consistency of the risk measure \(\rho_t\) and a recursive definition of the utility function.

- **Information Preference:** For an increasing function \(f : \mathbb{N} \mapsto \mathbb{N}\) with \(f(s) = s\) for \(s \leq t\) and \(f(s) \geq s\) for \(s > t\), then for any adapted process \(Y\) equal to \(X\) up to time \(t\) and with \(\mathcal{L}aw\left( Y \mid \mathcal{F}_t \right) \sim \mathcal{L}aw\left( X_{f(\cdot)} \mid \mathcal{F}_t \right)\) we should have \(X \succeq^t Y\).
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