On continuously Urysohn and strongly separating spaces

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Abstract

A topological space X is *continuously Urysohn* if for each pair of distinct points $x, y \in X$ there is a continuous real-valued function $f_{x,y} \in C(X)$ such that $f_{x,y}(x) \neq f_{x,y}(y)$ and the correspondence $(x,y) \to f_{x,y}$ is a continuous function from $X^2 \setminus \Delta$ to $C(X)$, where $C(X)$ carries the topology of uniform convergence and $\Delta = \{(x,x) : x \in X\}$. Metric spaces are examples of continuously Urysohn spaces with the additional property that the functions $f_{x,y}$ depend on just one parameter. We show that spaces with this property are precisely the spaces that have a weaker metric topology. However, to find an example of a continuously Urysohn space where the functions $f_{x,y}$ cannot be chosen independently of one of their parameters, it is easier to consider a much simpler property than "continuously Urysohn", given by the following definition: A topological space X is *strongly separating* if for each point $x \in X$ there is a continuous, real-valued function g_x such that for any $z \in X$, $g_x(x) = g_x(z)$ implies $x = z$. We show that a continuously Urysohn space may fail to be strongly separating. In particular, the example that we present is a continuously Urysohn space, where the Urysohn functions $f_{x,y}$ cannot be chosen independently of y. This answers a question raised by David Lutzer.

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1 Basics and Background

Most of our notations and notions are standard and can be found in books like [3] and [4]. A notion which has a different meaning depending on the context is the notion of a P-space. According to [4] we define a P-space (with capital P) as follows:

A ***P*-space** X is a space in which every point is a P-point, where a point $x \in X$ is a ***P*-point** if the intersection of any family of countably many neighborhoods of x is still a neighborhood of x. Note that every isolated point x, *i.e.*, every point x with the property that $\{x\}$ is open, is a P-point.

Searching for generalized metric spaces, Alexander V. Arkhangel′skiĭ introduced in [1] a certain type of spaces, which he could characterize as the class of preimages of metric spaces under perfect surjections and which, in his terminology, are paracompact p-spaces. Following [4], we will refer to these spaces as **paracompact *p*-spaces** (with a small p).

About 30 years after Arkhangel′skiĭ's work, in the early 1990s, E. N. Stepanova introduced in [6] and [7] a property which is necessary and sufficient for a paracompact p-space to be metrizable, namely:

Definition: A topological space X is called **continuously Urysohn** if:

(i) For each pair of distinct points $x, y \in X$ there is a function $f_{x,y} \in C(X)$, where $C(X)$ is the set of all continuous real-valued functions on X, such that $f_{x,y}(x) \neq f_{x,y}(y)$.

(ii) The correspondence $(x, y) \to f_{x,y}$ is a continuous function from $X^2 \setminus \Delta$ to $C(X)$, where $C(X)$ carries the topology of uniform convergence and $\Delta = \{(x, x) : x \in X\}$.

For paracompact p-spaces, Stepanova found the following characterization (see [6, p. 314]): *A paracompact p-space is continuously Urysohn if and only if it is metrizable.*

The following definition will be useful in the investigation of the notion of continuously Urysohn:

Definition: If X is a continuously Urysohn space, then we call the corresponding family $\{f_{x,y} : (x, y) \in X^2 \setminus \Delta\}$ a **continuous separating family** for X.

Remark: In any continuous separating family we may replace the functions $f_{x,y}$ by $h_{x,y}(z) = f_{x,y}(z) - f_{x,y}(x)$ so that we may always assume $f_{x,y}(x) = 0$.

Lemma 1.1 If X is metrizable, then X admits a continuous separating family.

PROOF. Let $d(x, y)$ be the metric on X. For $x \neq y$ define $f_{x,y}(z) := d(x, z)$. Then $f_{x,y} \in C(X)$ and since $|f_{x,y}(z) - f_{x',y}(z)| = |d(x, z) - d(x', z)| \leq d(x, x')$, we get that the correspondence $(x,y) \to f_{x,y}$ is a continuous function from $X^2 \setminus \Delta$ to $C(X)$. Hence, $\{f_{x,y} : (x,y) \in X^2 \setminus \Delta\}$ is a continuous separating family. $\dashv$

Notice that the continuous separating family in Lemma 1.1 really depends on only one of its parameters, namely on x.

As mentioned above, Stepanova showed in [6] that a paracompact p-space is metrizable if and only if it is continuously Urysohn. Since then, the concept of continuously Urysohn spaces has been intensively studied. During his investigations of the matter, David Lutzer observed that he did not know of any continuously Urysohn space X in which one could prove that both parameters are *required* in describing a continuous separating family for X.

This leads to the following definition:

Definition: If X is a continuously Urysohn space, where the corresponding continuous separating family depends on only one of its parameters, say x, then this family is called a **one-parameter continuous separating family** for X.

We can characterize topological spaces that admit one-parameter continuous separating families as follows:

Proposition 1.2 A space X admits a continuous separating family $\{f_{x,y} : (x,y) \in X^2 \setminus \Delta\}$ that depends on just one parameter if and only if X has a weaker metric topology.

PROOF. If X has a weaker topology induced by a metric on X, then the one-parameter continuous separating family that works for the metric topology, also works for the given space X. Conversely, suppose X has a continuous separating family $\{f_{x,y} : (x,y) \in X^2 \setminus \Delta\}$ that does not depend on the second parameter. Then, for any $y, z \in X \setminus \{x\}$, $f_{x,y} \equiv f_{x,z}$. Define $h_x := f_{x,y}$ (for any y). Since the correspondence $(x,y) \to f_{x,y}$ is a continuous function from $X^2 \setminus \Delta$ to $C(X)$ (where $C(X)$ carries the topology of uniform convergence), the function $x \to h_x$ from X to $C(X)$ is continuous as well. As mentioned in the remark above, we may assume without loss of generality that $f_{x,y}(x) = 0$, which implies $h_x(y) = 0$ if and only if $y = x$. Thus, for any two distinct points $x, y \in X$ we have $0 = h_x(x) \neq h_y(x)$. Hence, the correspondence $x \to h_x$ is continuous, one-to-one, and since the topology on $C(X)$ is metrizable, the topology on X induced by the metric $d_X(x,y) := d_{C(X)}(h_x, h_y)$ is a weaker metric topology on X. $\dashv$

In finding examples of continuously Urysohn spaces that cannot admit one-parameter continuous separating families, it is often easier to use a much more simple topological property.

Definition: A topological space X is called **strongly separating** if for each point $x \in X$ there is a $g_x \in C(X)$, such that $g_x^{-1}[0] = \{x\}$, where $g_x^{-1}[r] := \{z \in X : g_x(z) = r\}$. In other words, a space is strongly separating if each point is a zero-set of some continuous real-valued function.

The next result will be the key to showing that the space constructed in Section 2 is not strongly separating.

Proposition 1.3 If X contains a non-isolated P-point, then X is not strongly separating.

PROOF. Let x be a P-point of X which is not an isolated point and take any continuous real-valued function g_x with $g_x(x) = 0$. For each $n \geq 1$, the set $O_n = g_x^{-1}[(\frac{-1}{n}, \frac{1}{n})]$ is an open neighborhood of x. Hence, since x is a P-point, $g_x^{-1}[0] = \bigcap\{O_n : n \geq 1\}$ is a neighborhood of x, and thus, because x is not an isolated point, $g_x^{-1}[0] \neq \{x\}$. $\dashv$

As an easy consequence we get the fact that a P-space, which does not contain isolated points, cannot be strongly separating.

Remark: In Section 3 we will see that the converse of Proposition 1.3 does not hold, even if we assume that the space is compact Hausdorff, *i.e.*, we give two examples of topological spaces which are compact Hausdorff, not strongly separating and do not contain a P-point.

In the next section we construct an example of a continuously Urysohn space which is not strongly separating. Moreover, the space is a paracompact P-space which is not metrizable. Therefore, this is an example of a continuously Urysohn space which does not allow to choose the Urysohn functions $f_{x,y}$ independently from y. Hence, this answers Lutzer's question mentioned above.

2 The space $\boldsymbol{S}$ and its properties

For an ordinal number α, let ${}^{\alpha}2$ be the set of all functions $\mu : \alpha \to \{0, 1\}$. If $\mu \in {}^{\alpha}2$, then $\mathrm{dom}(\mu) := \alpha$. Further, ${}^{\alpha}2$ denotes the set of all 0-1 sequences of length α.

Let $S := \{\mu : \mu \in {}^{\alpha}2 \text{ for some } \alpha < \omega_1\}$ and let $\bar{S} := \{\bar{\mu} : \bar{\mu} \in {}^{\omega_1}2\}$. For $\mu \in S$ let $O_\mu := \{\bar{\mu} \in \bar{S} : \mu = \bar{\mu} \restriction \mathrm{dom}(\mu)\}$, where $\bar{\mu} \restriction \alpha$ is the restriction of the function $\bar{\mu}$ to the set α. On the set S we define a partial order as follows: $\nu \preccurlyeq \mu$ if and only if $\mathrm{dom}(\nu) \leq \mathrm{dom}(\mu)$ and $\mu \restriction \mathrm{dom}(\nu) = \nu$. We write $\nu \prec \mu$ for $\nu \preccurlyeq \mu$ and $\nu \neq \mu$. Further, use $\{O_\mu : \mu \in S\}$ as the base for a topology τ on the set $\bar{S}$ and define $\boldsymbol{S} := (\bar{S}, \tau)$. It is easy to see that $\boldsymbol{S}$ is a topological space which does not contain isolated points.

The next few lemmata give some more properties of the space $\boldsymbol{S}$.

Lemma 2.1 $\boldsymbol{S}$ is paracompact.

PROOF. Clearly, $\boldsymbol{S}$ is Hausdorff. To see that $\boldsymbol{S}$ is paracompact we will show that every open cover of $\bar{S}$ has an open locally finite refinement. In fact we show that for each open cover $\mathcal{C}$ of $\bar{S}$ we find an open refinement $\mathcal{F}$ of $\mathcal{C}$ such that each $\bar{\mu} \in \bar{S}$ is in exactly one member of $\mathcal{F}$.
Let $\mathcal{C}$ be an arbitrary open cover of $\bar{S}$. Let $T := \{\mu \in S : \exists O \in \mathcal{C}\,(O_\mu \subseteq O)\}$. Further, let $\min T := \{\mu \in S : \mu \in T \wedge \forall \nu \in S(\nu \prec \mu \to \nu \notin T)\}$. Now, $\mathcal{F} := \{O_\mu : \mu \in \min T\}$ is obviously an open refinement of $\mathcal{C}$ and it is still an open cover. To see this, take an arbitrary $\bar{\nu} \in \bar{S}$. Because $\mathcal{C}$ is an open cover, there is an $O \in \mathcal{C}$ such that $\bar{\nu} \in O$. Thus we find a $\mu' \in T$ such that $\bar{\nu} \in O_{\mu'}$ and therefore also a $\mu \preccurlyeq \mu'$ such that $\mu \in \min T$ and $\bar{\nu} \in O_\mu$. Finally, each $\bar{\mu} \in \bar{S}$ is in exactly one member of $\mathcal{F}$: If $O_\mu \cap O_\nu \neq \emptyset$ for two distinct $\mu, \nu \in \min T$, then we have either $\mu \prec \nu$ or $\nu \prec \mu$, but in both cases, either μ or ν does not belong to $\min T$. Thus, the open sets of $\mathcal{F}$ are pairwise disjoint, which implies that each $\bar{\mu} \in \bar{S}$ is in exactly one member of $\mathcal{F}$. $\dashv$

Lemma 2.2 $\boldsymbol{S}$ is continuously Urysohn.

PROOF. First we define for each ordered pair of distinct points $\bar{\mu}, \bar{\nu} \in \bar{S}$ an element $\varphi(\bar{\mu}, \bar{\nu}) \in S$ as follows: $\varphi(\bar{\mu}, \bar{\nu}) = \eta$ if and only if $\bar{\mu} \in O_\eta$, $\bar{\nu} \notin O_\eta$ and for all $\eta' \in S$ with this property we have $\eta \preccurlyeq \eta'$. In other words, let α be the first ordinal such that $\bar{\mu}(\alpha) \neq \bar{\nu}(\alpha)$ and let $\eta := \bar{\mu} \restriction \alpha + 1$. Now, for two distinct points $\bar{\mu}, \bar{\nu} \in \bar{S}$ we define $f_{\bar{\mu},\bar{\nu}} : \bar{S} \to \{0, 1\}$ as follows:

$$f_{\bar{\mu},\bar{\nu}}(\bar{\eta}) = \begin{cases} 1 & \text{if } \bar{\eta} \in O_{\varphi(\bar{\mu},\bar{\nu})}, \\ 0 & \text{otherwise.} \end{cases}$$

It remains to show that this function has the desired properties.

For fixed $\bar{\mu}, \bar{\nu} \in \bar{S}$, the function $f_{\bar{\mu},\bar{\nu}}$ is a real-valued continuous function: Obviously, $f_{\bar{\mu},\bar{\nu}}$ is real-valued, and because each set O_η (for η in S) is both open and closed, it is also continuous.

For each $\bar{\mu}' \in O_{\varphi(\bar{\mu},\bar{\nu})}$ and $\bar{\nu}' \in O_{\varphi(\bar{\nu},\bar{\mu})}$ we have $\varphi(\bar{\mu}', \bar{\nu}') = \varphi(\bar{\mu}, \bar{\nu})$, and hence we get $f_{\bar{\mu}',\bar{\nu}'} \equiv f_{\bar{\mu},\bar{\nu}}$. Thus, the correspondence $(\bar{\mu}, \bar{\nu}) \to f_{\bar{\mu},\bar{\nu}}$ is a continuous function from $\bar{S}^2 \setminus \Delta$ to the space $C(\bar{S})$ of all continuous real-valued functions on $\bar{S}$, where $C(\bar{S})$ carries the topology of uniform convergence. $\dashv$

Combining the lemmata above we get the following:

Theorem 2.3 The space $\boldsymbol{S}$ is a continuously Urysohn space that is paracompact, is a P-space, has no isolated points, and is not strongly separating. In particular, $\boldsymbol{S}$ does not admit a one-parameter continuous separating family.

PROOF. It remains to show that $\boldsymbol{S}$ is a P-space, has no isolated points, and is not strongly separating.
Clearly, $\boldsymbol{S}$ has no isolated points. To see that each point $\bar{\mu} \in \bar{S}$ is a P-point, let $\{U_n : n \in \omega\}$ be an arbitrary set of neighborhoods of some $\bar{\mu} \in \bar{S}$. For $n \in \omega$ let $\alpha_n := \min\{\beta : \bar{\mu} \in O_\nu \subseteq U_n \wedge \operatorname{dom}(\nu) = \beta\}$. Because ω_1 is regular and uncountable, there is an $\alpha < \omega_1$ such that $\alpha > \alpha_n$ for all $n \in \omega$. By construction, $O_{\bar{\mu}\restriction\alpha} \subseteq U_n$, for all $n \in \omega$, and we obviously have $\bar{\mu} \in O_{\bar{\mu}\restriction\alpha}$, hence, $\boldsymbol{S}$ is a P-space.
It now follows from Proposition 1.3 that $\boldsymbol{S}$ is not strongly separating, which completes the proof of the theorem. $\dashv$

3 Notes on strongly separating spaces

By definition we get that any completely regular space in which points are G_δ-sets, must be strongly separating. Further, we get that continuously Urysohn spaces, as well as strongly separating spaces, are always Hausdorff. But a strongly separating space is not necessarily metrizable. To see this, take any metric space which has a finer, non-metrizable topology. Then the space with respect to the finer topology is still strongly separating, but by construction not metrizable. As an example we like to mention the Baire space of all functions from ω to ω: With the usual topology, this space is a complete separable metric space, and the finer Ellentuck topology (introduced by Erik Ellentuck in [2]) is not metrizable.

As noted in Proposition 1.3, the existence of non-isolated P-points is enough to prevent a space from being strongly separating. But there are also spaces without P-points that fail to be strongly separating (see examples below).

For an infinite discrete set S, let the topological space βS be Stone-Čech compactification of S, or in other words, the space of all ultrafilters over S. The topology on βS is induced by the basic open sets A^*, where $A \subseteq S$ and A^* is the set of all ultrafilters containing A. Further, let $\beta S \setminus S$ be the remainder of the Stone-Čech compactification of S, or in other words, the space of all non-principal ultrafilters over S.

In the following two examples of compact Hausdorff spaces without P-points that fail to be strongly separating, the spaces $\beta\omega_1$ and $\beta\omega \setminus \omega$ are involved. These spaces are compact Hausdorff spaces which are not strongly separating. To see the latter property, remember that if a space X is strongly separating, then each point of X would be a zero-set of some continuous real-valued function and hence, a G_δ-set, which would mean that the space X is first-countable. But neither $\beta\omega_1$ nor $\beta\omega \setminus \omega$ is first-countable.

Let I denote the closed unit interval. Obviously, I has no isolated points and does not contain P-points. So, with the facts concerning $\beta\omega_1$ mentioned above, we get:

Example 3.1 The space $\beta\omega_1 \times I$ is a compact Hausdorff space having no P-points and no isolated points, and is not strongly separating.

Saharon Shelah has shown in [5, Chapter VI, §4] that it is consistent with the usual axioms of set theory, denoted by ZFC, that $\beta\omega \setminus \omega$ contains no P-point. Since $\beta\omega \setminus \omega$ has no isolated points, this leads to the following:

Example 3.2 It is consistent with ZFC that the compact Hausdorff space $\beta\omega \setminus \omega$ has no P-points and no isolated points, and is not strongly separating.

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