RAMSEYAN ULTRAFILTERS

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Abstract

We investigate families of partitions of ω which are related to special coideals, so-called happy families, and give a dual form of Ramsey ultrafilters in terms of partitions. The combinatorial properties of these partition-ultrafilters, which we call Ramseyan ultrafilters, are similar to those of Ramsey ultrafilters. For example it will be shown that dual Mathias forcing restricted to a Ramseyan ultrafilter has the same features as Mathias forcing restricted to a Ramsey ultrafilter. Further we introduce an ordering on the set of partition-filters and consider the dual form of some cardinal characteristics of the continuum.

0 Introduction

The Stone-Čech compactification $\beta\mathbb{N}$ of the natural numbers, or equivalently, the ultrafilters over ω, is a well-studied space (*cf. e.g.* [vM90] and [CN74]) which has a lot of interesting topological and combinatorial features (*cf.* [HS98] and [To97]). In the late 1960's, a partial ordering on the non-principal ultrafilters $\beta\mathbb{N}\setminus\mathbb{N}$, the so-called Rudin-Keisler ordering, was established and "small" points with respect to this ordering were investigated rigorously (*cf.* [Bo70], [Bl73], [Bl81$_1$] and [La89]). The minimal points have a nice combinatorial characterization which is related to Ramsey's Theorem (*cf.* [Ra29, Theorem A]) and so, the ultrafilters which are minimal with respect to the Rudin-Keisler ordering are also called Ramsey ultrafilters (for further characterizations of Ramsey ultrafilters see [BJ95, Chapter 4.5]). Families,

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not necessarily filters, having similar combinatorial properties as Ramsey ultrafilters, are the so-called happy families (*cf.* [Ma77]), which are very important in the investigation of Mathias forcing (*cf.* [Ma77]).

From the category theoretical point of view, subsets of ω and partitions of ω are dual to each other (see *e.g.* [HLö∞_1, Introduction]), and therefore, it is natural to look for the dualization of statements about subsets of ω in terms of partitions of ω. In this dualization process, a lot of work is already done. Confer: [HLö∞_1] for a dualization of $\beta\mathbb{N}$; [CaSi84], [Ha98$_1$] and [HLö∞_2] for the dualization of the Ramsey property and of Mathias forcing; [CaSi84] for a dualization of Ramsey's Theorem; [CKMW00] and [Ha98$_2$] for the dualization of some cardinal characteristics of the continuum.

To investigate partition-filters, a useful tool is missing: the dualization of Ramsey ultrafilters. The aim of this paper is to fill this gap.

1 Partition-filters

1.1 Notations and definitions

Most of our set-theoretic notation is standard and can be found in textbooks like [Je78], [Ku83] or [BJ95]. So, we consider a natural number n as an ordinal, in particular $n = \{k : k < n\}$ and $0 = \emptyset$, and consequently, the set of natural numbers is denoted by ω. For a set S, $\mathcal{P}(S)$ denotes the power-set of S. The notation concerning partitions is not yet standardized. However, we will use the notation introduced in [Ha98$_1$].

A **partition** X of a set S consisting of pairwise disjoint, non-empty sets, such that $\bigcup X = S$. The elements of a partition are called **blocks**. Mostly, we will consider partitions of ω, so, if not specified otherwise, the word "partition" refers to a *partition of* ω.

Most of the partitions in consideration are infinite, or in other words, contain infinitely many blocks. However, at some places we also have to consider finite partitions, this means, partitions containing only finitely many blocks. The unique partition containing just one block is denoted by $\{\omega\}$. The set of all partitions is denoted by $(\omega)^{\leq\omega}$ and the set of all partitions containing infinitely many blocks is denoted by $(\omega)^{\omega}$.

Let X and Y be two partitions of a set S. We say X is **coarser** than Y, or that Y is **finer** than X (and write $X \sqsubseteq Y$), if each block of X is the union of blocks of Y. Let $X \sqcap Y$ denote the finest partition of S which is coarser than X and Y.

Further, for $n \in \omega$ and a partition $X \in (\omega)^{\leq\omega}$, let $X \sqcap \{n\}$ be the partition we get, if we glue all blocks of X together which contain a member of n. If X and Y are two partitions, then we write $X \sqsubseteq^* Y$ if there is an $n \in \omega$ such that $(X \sqcap \{n\}) \sqsubseteq Y$.

A set $\mathscr{F} \subseteq (\omega)^{\leq\omega}$ is a **partition-filter**, if the following holds:

(a) $\{\omega\} \notin \mathscr{F}$.

(b) For any $X, Y \in \mathscr{F}$ we have $X \sqcap Y \in \mathscr{F}$.

(c) If $X \in \mathscr{F}$ and $X \sqsubseteq Y \in (\omega)^{\leq\omega}$, then $Y \in \mathscr{F}$.

A partition-filter $\mathscr{F} \subseteq (\omega)^{\leq\omega}$ is called **principal**, if there is a partition $X \in (\omega)^{\leq\omega}$ such that $\mathscr{F} = \{Y : X \sqsubseteq Y\}$.

A set $\mathscr{U} \subseteq (\omega)^{\leq\omega}$ is a **partition-ultrafilter**, if $\mathscr{U}$ is a partition-filter which is not properly contained in any partition-filter.

Notice that a partition-ultrafilter $\mathscr{U}$ which does not contain a finite partition is always non-principal, and vice versa, a principal partition-ultrafilter always contains a finite partition, in fact it contains a 2-block partition (see [HLö∞_1, Fact 3.1]). Thus, if $\mathscr{U}$ is a non-principal partition-ultrafilter, $X \in \mathscr{U}$ and $X \sqsubseteq^* Y$, then $Y \in \mathscr{U}$.

In the sequel we are mostly interested in partition-filters which don't contain a finite partition, or in other words, in partition-filters $\mathscr{F} \subseteq (\omega)^{\omega}$.

For the sake of convenience, we defined the notion of partition-filter only for partition-filters over ω, but it is obvious how to generalize this notion for partition-filters over arbitrary sets S (see also [HLö∞_1]).

1.2 An ordering on the set of partition-filters

Let $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$ denote the set of all partition-filters. We define a partial ordering on $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$ which has some similarities with the Rudin-Keisler ordering on $\beta\mathbb{N} \setminus \mathbb{N}$.

To keep the notation short, for $\mathscr{H} \subseteq \mathcal{P}\big(\mathcal{P}(\omega)\big)$ and a function $f : \omega \to \omega$ we define

$$f^{-1}(\mathscr{H}) := \{f^{-1}(X) : X \in \mathscr{H}\},$$

where for $X \in \mathscr{H}$ we define

$$f^{-1}(X) := \{f^{-1}(b) : b \in X\},$$

where for $b \subseteq \omega$, $f^{-1}(b) := \{n : f(n) \in b\}$.

Let $f:\omega \twoheadrightarrow \omega$ be any surjection from ω onto ω and let $X \in (\omega)^{\leq\omega}$ be any partition. Then $f(X)$ denotes the finest partition such that whenever n and m lie in the same block of X, then $f(n)$ and $f(m)$ lie in the same block of $f(X)$.

For any partition-filter $\mathscr{F} \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$ define

$$f(\mathscr{F}) := \big\{Y \in (\omega)^{\leq\omega} : \exists X \in \mathscr{F}\big(f(X) \sqsubseteq Y\big)\big\}.$$

We define the ordering "$\lesssim$" on $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$ as follows:

$$\mathscr{F} \lesssim \mathscr{G} \quad \text{if and only if} \quad \mathscr{F} = f(\mathscr{G}) \text{ for some surjection } f:\omega \twoheadrightarrow \omega\,.$$

Since the identity map is a surjection and the composition of two surjections is again a surjection, the partial ordering "$\lesssim$" is reflexive and transitive.

FACT 1.2.1 Let $\mathscr{F}, \mathscr{G} \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$ and assume $f(\mathscr{G}) = \mathscr{F}$ for some surjection $f:\omega \twoheadrightarrow \omega$. Then $\mathscr{G} \subseteq f^{-1}(\mathscr{F})$ and $f^{-1}(\mathscr{F}) \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$.

PROOF: Let $\mathscr{H} = f^{-1}(\mathscr{F})$, where $f:\omega \twoheadrightarrow \omega$ is such that $f(\mathscr{G}) = \mathscr{F}$. Since $\mathscr{F}$ is a partition-filter and f is a function, for any $X_1, X_2 \in \mathscr{F}$ we have $X_1 \sqcap X_2 \in \mathscr{F}$ and $f^{-1}(X_1 \sqcap X_2) = f^{-1}(X_1) \sqcap f^{-1}(X_2)$, and therefore, $\mathscr{H}$ is a partition-filter. Further, for any $Y \in \mathscr{G}$ we get $f(Y) \in \mathscr{F}$ and $f^{-1}(f(Y)) \sqsubseteq Y$, which implies $\mathscr{G} \subseteq \mathscr{H}$. $\dashv$

The ordering "$\lesssim$" induces in a natural way an equivalence relation "$\simeq$" on the set of partition-filters $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$:

$$\mathscr{F} \simeq \mathscr{G} \quad \text{if and only if} \quad \mathscr{F} \lesssim \mathscr{G} \ \text{ and } \ \mathscr{G} \lesssim \mathscr{F}\,.$$

So, the ordering "$\lesssim$" induces a partial ordering of the set of equivalence classes of partition-filters. Concerning partition-ultrafilters, we get the following.

FACT 1.2.2 Let $\mathscr{U}, \mathscr{V} \in \mathtt{PUF}\big((\omega)^{\leq\omega}\big)$ and assume that $\mathscr{U}$ is principal or contains a partition, all of whose blocks are infinite. If $\mathscr{U} \simeq \mathscr{V}$, then there is a permutation h of ω such that $h(\mathscr{U}) = \mathscr{V}$.

PROOF: Because $\mathscr{U} \lesssim \mathscr{V}$ and $\mathscr{V} \lesssim \mathscr{U}$, there are surjections f and g from ω onto ω such that $\mathscr{V} = f(\mathscr{U})$ and $\mathscr{U} = g(\mathscr{V})$, and because $\mathscr{U}$ and $\mathscr{V}$ are both partition-ultrafilters, by Fact 1.2.1 we get $\mathscr{U} = f^{-1}(\mathscr{V})$ and $\mathscr{V} = g^{-1}(\mathscr{U})$.
First assume that $\mathscr{U}$ is principal and therefore contains a 2-block partition $X = \{b_0, b_1\}$. Because $g^{-1}(X) \in \mathscr{V}$, the partition-ultrafilter $\mathscr{V}$ is also principal and we get $\mathscr{V} = \{Y \in (\omega)^{\leq\omega} : g^{-1}(X) \sqsubseteq Y\}$, where $g^{-1}(X) = \{g^{-1}(b_0), g^{-1}(b_1)\} =:$

$\{c_0, c_1\}$. Now, because $\mathscr{U} = f^{-1}(\mathscr{V})$, we must have $f^{-1}\big(g^{-1}(X)\big) = X$, which implies $f^{-1}\big(g^{-1}(b_i)\big) \in \{b_0, b_1\}$ (for $i \in \{0,1\}$). If one of the blocks of X is finite, say b_0, then $f|_{b_0}$ as well as $g|_{f(b_0)}$ must be one-to-one, and therefore, b_0 has the same cardinality as c_0. Hence, no matter if one of the blocks of X is finite or not, we can define a permutation h of ω such that $h(b_0) = c_0$ and $h(b_1) = c_1$, which implies $h(\mathscr{U}) = \mathscr{V}$.
Now assume that $\mathscr{U}$ contains a partition $X = \{b_i : i \in \omega\}$, all of whose blocks b_i are infinite. Because g is a surjection, $g^{-1}(X)$, which is a member of $\mathscr{V}$, is a partition, all of whose blocks are infinite. Let h be a permutation of ω such that $h(b_i) = g^{-1}(b_i)$. Take any $Y \in \mathscr{V}$ with $Y \sqsubseteq g^{-1}(X)$. By the definition of h we have $h^{-1}(Y) = g(Y)$ and since $\mathscr{U} = g(\mathscr{V})$ there is a $Z \in \mathscr{U}$ such that $g(Y) = Z$, which implies $h(Z) = Y$, hence, $h(\mathscr{U}) = \mathscr{V}$. ⊣

The following proposition shows that "$\lesssim$" is directed upward (for a similar result concerning the Rudin-Keisler ordering see [Bl73, p. 147]).

FACT 1.2.3 For any partition-filters $\mathscr{D}$, $\mathscr{E} \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$, there is a partition-filter $\mathscr{F} \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$, such that $\mathscr{D} \lesssim \mathscr{F}$ and $\mathscr{E} \lesssim \mathscr{F}$.

PROOF: Let ρ_1 and ρ_2 be two functions from ω into ω defined by $\rho_1(n) := 2n$ and $\rho_2(n) := 2n+1$. For a partition X and $i \in \{0,1\}$, let $\rho_i(X) := \{\rho_i(b) : b \in X\}$, where $\rho_i(b) := \{\rho_i(n) : n \in b\}$. Now, take any two partition-filters $\mathscr{D}$, $\mathscr{E} \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$ and define $\mathscr{F}$ by

$$\mathscr{F} := \big\{\rho_1(X) \cup \rho_2(Y) : X \in \mathscr{D} \wedge Y \in \mathscr{E}\big\}.$$

Clearly, this defines a partition-filter. Define two surjections f and g from ω onto ω as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

$$g(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $f(\mathscr{F}) = \mathscr{D}$ and $g(\mathscr{F}) = \mathscr{E}$, which implies $\mathscr{D} \lesssim \mathscr{F}$ and $\mathscr{E} \lesssim \mathscr{F}$. ⊣

2 Ramseyan ultrafilters

2.1 Coloring segments

If X is a partition of a set S, then we say that S is the **domain** of X, written $\operatorname{dom}(X) = S$. The set of all partitions of natural numbers $n \in \omega$, called **segments**, is denoted by $(\mathbb{N})$. Thus, $s \in (\mathbb{N})$ implies $\operatorname{dom}(s) \in \omega$. In particular, $\emptyset$ is the unique partition of 0 and $\{\{\emptyset\}\} = \{1\}$ is the unique partition of 1. For $s \in (\mathbb{N})$, $|s|$ denotes the cardinality of s, which simply means the number of blocks of s, and $\bigcup s := \{\operatorname{dom}(s)\}$.

For a set $b \subseteq \omega$, let $\min(b)$ be the least element of b and for a set $P \subseteq \mathcal{P}(\omega)$, let $\operatorname{Min}(P) := \{\min(b) : b \in P\}$. Further, for a finite set $b \subseteq \omega$, let $\max(b)$ be the greatest element of b. For $X \in (\omega)^{\leq\omega}$, $s \in (\mathbb{N})$ and $n \in \omega$, let $X(n)$ and $s(n)$ be the nth block of X and s, respectively, where we start counting with 0 and assume that the blocks are ordered by their least element.

Let $s, t \in (\mathbb{N})$ and $X \in (\omega)^{\leq\omega}$: We write $s \sqsubseteq X$, if each block $b \in s$ is the union of some sets $b_i \cap \operatorname{dom}(s)$, where each b_i is a block of X; we write $s \preccurlyeq t$ and $s \preccurlyeq X$, respectively, if for each $b \in s$ there is a $c_b \in t$ and a $d_b \in X$, respectively, such that $b = c_b \cap \operatorname{dom}(s) = d_b \cap \operatorname{dom}(s)$ (notice that $s \preccurlyeq t$ implies $\operatorname{dom}(s) \subseteq \operatorname{dom}(t)$); and for $s \sqsubseteq X$, $s \sqcap X$ denotes the finest partition $Y \in (\omega)^{\leq\omega}$, such that $s \preccurlyeq Y \sqsubseteq X$.

For $s \in (\mathbb{N})$, let s^* denote the partition $s \cup \big\{\{\operatorname{dom}(s)\}\big\}$. In particular, $\emptyset^* = \{1\}$. Notice that $|s^*| = |s| + 1$.

For $s \in (\mathbb{N})$ and $X \in (\omega)^\omega$ with $s \sqsubseteq X$, let

$$(s, X)^\omega := \{Y \in (\omega)^\omega : s \preccurlyeq Y \sqsubseteq X\}.$$

A set $(s, X)^\omega$, where s and X are as above, is called a **dual Ellentuck neighborhood** (*cf.* [CaSi84, p. 275]). In particular, $(\emptyset, X)^\omega = (\{1\}, X)^\omega =: (X)^\omega$.

For $n \in \omega$, $(\omega)^{n*}$ denotes the set of all $u \in (\mathbb{N})$ such that $|u| = n$. Further, for $n \in \omega$ and $X \in (\omega)^\omega$ let

$$(X)^{n*} := \big\{u \in (\mathbb{N}) : |u| = n \wedge u^* \sqsubseteq X\big\};$$

and if $s \in (\mathbb{N})$ is such that $|s| \leq n$ and $s \sqsubseteq X$, let

$$(s, X)^{n*} := \big\{u \in (\mathbb{N}) : |u| = n \wedge s \preccurlyeq u \wedge u^* \sqsubseteq X\big\}.$$

From the so-called Dual Ramsey Theorem of Carlson and Simpson, which is Theorem 1.2 of [CaSi84], we get the following.

Proposition 2.1.1 For any coloring of $(\omega)^{(n+1)*}$ with $r+1$ colors, where $r, n \in \omega$, and for any $Z \in (\omega)^{\omega}$, there is an infinite partition $X \in (Z)^{\omega}$ such that $(X)^{(n+1)*}$ is monochromatic.

This combinatorial result is the dualization of Ramsey's Theorem, [Ra29, Theorem A], in terms of partitions.

We say that a surjection $f : \omega \twoheadrightarrow \omega$ **respects** the partition $X \in (\omega)^{\omega}$, if we have $f^{-1}(f(X)) = X$, otherwise, we say that it **disregards** the partition X. If $f^{-1}(f(X)) = \{\omega\}$, then we say that f **completely disregards** the partition X.

Lemma 2.1.2 For any surjection $f : \omega \twoheadrightarrow \omega$ and for any $Z \in (\omega)^{\omega}$, there is an $X \in (Z)^{\omega}$ such that f either respects or completely disregards the partition X.

Proof: For a surjection $f : \omega \twoheadrightarrow \omega$, define the coloring $\pi : (\omega)^{2*} \to \{0, 1\}$ as follows. $\pi(s) := 0$ if and only if $f(s(0)) \cap f(s(1)) = \emptyset$. By Proposition 2.1.1, there is a partition $X \in (Z)^{\omega}$ such that $(X)^{2*}$ is monochromatic with respect to π, which implies that f respects X in case of $\pi|_{(X)^{2*}} = \{0\}$, and f completely disregards X is case of $\pi|_{(X)^{2*}} = \{1\}$. $\dashv$

In the sequel we will use a slightly stronger version of Proposition 2.1.1, which is given in the following two corollaries.

Corollary 2.1.3 For any coloring of $(\omega)^{(n+k+1)*}$ with $r+1$ colors, where $r, n, k \in \omega$, and for any dual-Ellentuck neighborhood $(s, Y)^{\omega}$, where $|s| = n+1$, there is an infinite partition $X \in (s, Y)^{\omega}$ such that $(s, X)^{(n+k+1)*}$ is monochromatic.

Proof: Let $(s, Y)^{\omega}$ be any dual-Ellentuck neighborhood, with $|s| = n+1 \geq 1$. Set $Y' := s \sqcap Y$, $R := \bigcup_{i<n+1} Y'(i)$ and $Y_R := Y' \setminus \{Y'(i) : i < n+1\}$, and take any order-preserving bijection $f : \omega \setminus R \to \omega$. Then $Z := f(Y_R)$ is an infinite partition of ω. For $u \in (Z)^{n+k+1*}$ we define $\xi(u) \in (s, Y)^{n+k+1*}$ as follows. $\mathrm{dom}(\xi(u)) := f^{-1}(\mathrm{dom}(u))$ and for $i < n+k+1$,

$$\xi(u)(i) := \begin{cases} \big(Y'(i) \cap \mathrm{dom}(u)\big) \cup f^{-1}(u(i)) & \text{for } i < n+1, \\ f^{-1}(u(i)) & \text{otherwise.} \end{cases}$$

Let $\pi : (\omega)^{(n+k+1)*} \to r+1$ be any coloring. Define $\tau : (\omega)^{(n+k+1)*} \to r+1$ by stipulating $\tau(u) := \pi(\xi(u))$. By Proposition 2.1.1 there is an infinite partition $X' \in (Z)^{\omega}$ such that $(X')^{n+k+1*}$ is monochromatic with respect to the coloring τ. Now let $X \in (\omega)^{\omega}$ be such that

$$X(i) := \begin{cases} Y'(i) \cup f^{-1}(X'(i)) & \text{for } i < n+1 \\ f^{-1}(X'(i)) & \text{otherwise.} \end{cases}$$

Then, by definition of τ and X', $X \in (s,Y)^\omega$ and $(s,X)^{(n+k+1)*}$ is monochromatic with respect to π. $\dashv$

Corollary 2.1.4 For any coloring of $\bigcup_{n\in\omega}(\omega)^{(n+k+1)*}$ with $r+1$ colors, where $r,k\in\omega$, and for any $Z\in(\omega)^\omega$, there is an infinite partition $X\in(Z)^\omega$ such that for any $n\in\omega$ and for any $s\preccurlyeq X$ with $|s|=n+1$, $(s,X)^{(n+k+1)*}$ is monochromatic.

Proof: Using Corollary 2.1.3 repeatedly, we can construct the partition $X\in(\omega)^\omega$ straight forward by induction on n. $\dashv$

We say that a family $\mathscr{C}\subseteq(\omega)^\omega$ has the **segment-coloring-property**, if for every coloring of $\bigcup_{n\in\omega}(\omega)^{(n+k+1)*}$ with $r+1$ colors, where $r,k\in\omega$, and for any $Z\in\mathscr{C}$, there is an infinite partition $X\in(Z)^\omega\cap\mathscr{C}$, such that for any $n\in\omega$ and for any $s\preccurlyeq X$ with $|s|=n+1$, $(s,X)^{(n+k+1)*}$ is monochromatic.

If a partition-ultrafilter $\mathscr{U}\in\mathtt{PUF}\big((\omega)^\omega\big)$ has the segment-coloring-property, then it is called a **Ramseyan ultrafilter**.

The next lemma shows that every partition-filter $\mathscr{F}\in\mathtt{PF}\big((\omega)^\omega\big)$ which has the segment-coloring-property is a partition-ultrafilter. A similar result we have for Ramsey filters over ω, since every Ramsey filter is an ultrafilter.

Lemma 2.1.5 If $\mathscr{F}\subseteq(\omega)^\omega$ is a partition-filter which has the segment-coloring-property, then $\mathscr{F}\subseteq(\omega)^\omega$ is a partition-ultrafilter.

Proof: Take any $Z\in(\omega)^\omega$ such that for any $X\in\mathscr{F}$, $Z\sqcap X\in(\omega)^\omega$. Define the coloring $\pi:(\omega)^{2*}\to\{0,1\}$ by stipulating $\pi(u)=0$ if and only if $u\in(Z)^{2*}$. Because $\mathscr{F}$ has the segment-coloring-property, there is a partition $X\in\mathscr{F}$ such that $(X)^{2*}$ is monochromatic with respect to π, which implies that $X\sqsubseteq Z$ in case of $\pi|_{(X)^{2*}}=\{0\}$, and $X\sqcap Z=\{\omega\}$ in case of $\pi|_{(X)^{2*}}=\{1\}$. By the choice of Z we must have $X\sqsubseteq Z$, thus, since $\mathscr{F}$ is a partition-filter, $Z\in\mathscr{F}$. $\dashv$

The following lemma gives a relation between Ramseyan and Ramsey ultrafilters.

Lemma 2.1.6 If $\mathscr{U}$ is a Ramseyan ultrafilter, then $\{\mathrm{Min}(X)\setminus\{0\}:X\in\mathscr{U}\}$ is a Ramsey ultrafilter over ω (to be pedantic, one should say "over $\omega\setminus\{0\}$").

Proof: Let $\tau:[\omega]^n\to r$ be any coloring of the n-element subsets of ω with r colors, where n and r are positive natural numbers. Define $\pi:(\omega)^{n*}\to r$ by stipulating $\pi(s):=\tau(\mathrm{Min}(s^*)\setminus\{0\})$. Take $X\in\mathscr{U}$ such that $(X)^{n*}$ is monochromatic with respect to π, then, by the definition of π, the set $[\mathrm{Min}(X)\setminus\{0\}]^n$ is monochromatic with respect to τ. $\dashv$

Ramsey ultrafilters over ω build the minimal points of the Rudin-Keisler ordering on $\beta\mathbb{N}\setminus\mathbb{N}$. This fact can also be expressed by saying that a non-principal ultrafilter $\mathcal{U}$ is a Ramsey ultrafilter if and only if any function $g:\omega\to\omega$ is either constant or one-to-one on some set of $\mathcal{U}$. By Lemma 2.1.2, we get a similar result for Ramseyan ultrafilters with respect to the ordering "$\lesssim$".

THEOREM 2.1.7 If $\mathscr{U}$ is a Ramseyan ultrafilter, then for any surjection $f:\omega\twoheadrightarrow\omega$ there is an $X\in\mathscr{U}$ such that f either respects or completely disregards X.

PROOF: The proof is the same as the proof of Lemma 2.1.2, but restricted to the partition-ultrafilter $\mathscr{U}$. $\dashv$

2.2 On the existence of Ramseyan ultrafilters

As we have seen in Lemma 2.1.6, every Ramseyan ultrafilter induces a Ramsey ultrafilter over ω. It is not clear if the converse holds as well. However, Ramseyan ultrafilters are always forceable: Let $\mathbb{U}^\flat$ be the forcing notion consisting of infinite partitions, stipulating $X\leq Y\Leftrightarrow X\sqsubseteq^* Y$. $\mathbb{U}^\flat$ is the natural dualization of the forcing notion $\langle\mathcal{P}(\omega)/\mathrm{fin},\subseteq^*\rangle$, in the sequel denoted by $\mathbb{U}$, and it is not hard to see that if $\mathscr{G}$ is $\mathbb{U}^\flat$-generic over $\mathbf{V}$, then $\mathscr{G}$ is a Ramseyan ultrafilter in $\mathbf{V}[\mathscr{G}]$. Since $\mathbb{U}^\flat$ is σ-closed, as a consequence we get that Ramseyan ultrafilters exist if we assume the continuum hypothesis (denoted by CH). On the other hand we know by Lemma 2.1.6 that Ramseyan ultrafilters cannot exist if there are no Ramsey ultrafilters. Kenneth Kunen proved (*cf.* [Je78, Theorem 91]) that it is consistent with ZFC that Ramsey ultrafilters don't exist. We like to mention that Saharon Shelah showed that even p-points, which are weaker ultrafilters than Ramsey ultrafilters, may not exist (see [Sh98, VI §4]). He also proved that it is possible that—up to isomorphisms—there exists a unique Ramsey ultrafilter (see [Sh98, VI §5]).

In the following, $\mathfrak{c}$ denotes the cardinality of the continuum and $2^{\mathfrak{c}}$ denotes the cardinality of its power-set.

Andreas Blass proved that Martin's Axiom, denoted by MA, implies the existence of $2^{\mathfrak{c}}$ Ramsey ultrafilters (see [Bl73, Theorem 2]). He mentions in this paper that with CH in place of MA, this result is due to Keisler and with 1 in place of $2^{\mathfrak{c}}$, it is due to Booth (*cf.* [Bo70, Theorem 4.14]). Further he mentions that his proof is essentially the union of Keisler's and Booth's proof. However, Blass' proof uses at a crucial point that MA implies that the tower number is equal to $\mathfrak{c}$. Such a result we don't have for partitions, because Timothy Carlson proved that the dual-tower number is equal to $\aleph_1$ (see [Mt86, Proposition 4.3]). So, concerning the existence of Ramseyan ultrafilters under MA, we cannot simply translate the proof of Blass, and it seems that MA and sets of partitions are quite unrelated. But as mentioned above, if one assumes CH, then Ramseyan ultrafilters exist. Moreover, with respect to the equivalence relation "$\simeq$" (defined in section 1.2) we get the following (for a similar result w.r.t. the Rudin-Keisler ordering see [Bl73, p. 149]).

THEOREM 2.2.1 CH implies the existence of $2^{\mathfrak{c}}$ pairwise non-equivalent Ramseyan ultrafilters.

PROOF: Assume $\mathbf{V} \models \mathsf{CH}$. Let χ be large enough such that $\mathcal{P}((\omega)^\omega) \in H(\chi)$, *i.e.*, the power set of $(\omega)^\omega$ (in $\mathbf{V}$) is hereditarily of size $< \chi$. Let $\mathbf{N}$ be an elementary submodel of $\langle H(\chi), \in \rangle$ with $|\mathbf{N}| = \aleph_1$, containing all reals (or equivalently, all partitions) of $\mathbf{V}$. We consider the forcing notion $\mathbb{U}^\flat$ in the model $\mathbf{N}$. Since $|\mathbf{N}| = \aleph_1$, in $\mathbf{V}$ there is an enumeration $\{D_\alpha \subseteq (\omega)^\omega : \alpha < \omega_1\}$ of all dense sets of $\mathbb{U}^\flat$ which lie in $\mathbf{N}$. For any $Z \in (\omega)^\omega \cap \mathbf{V}$, let $Y_Z^{\alpha,0}, Y_Z^{\alpha,1} \in D_\alpha$ be such that $Y_Z^{\alpha,0} \sqsubseteq^* Z$, $Y_Z^{\alpha,1} \sqsubseteq^* Z$ and $Y_Z^{\alpha,0} \sqcap Y_Z^{\alpha,1} \notin (\omega)^\omega$ (since D_α is dense, such partitions exist). For any function $\zeta : \mathfrak{c} \to \{0,1\}$ we can construct a set $H_\zeta = \{X_\alpha : \alpha < \omega_1\}$ in $\mathbf{V}$ such that for all $\beta < \alpha < \omega_1$ we have $X_\alpha \sqsubseteq^* Y_{X_\beta}^{\beta,\zeta(\beta)}$. By construction, for any function ζ, the set $G_\zeta := \{X \in (\omega)^\omega : X_\alpha \sqsubseteq^* X \text{ for some } X_\alpha \in H_\zeta\}$ is $\mathbb{U}^\flat$-generic over $\mathbf{N}$, thus, a Ramseyan ultrafilter in $\mathbf{N}[G_\zeta]$, and since $\mathbb{U}^\flat$ is σ-closed and therefore adds no new reals, G_ζ is also a Ramseyan ultrafilter in $\mathbf{V}$. Furthermore, if $\zeta \neq \zeta'$, then the two Ramseyan ultrafilters G_ζ and $G_{\zeta'}$ are different (consider the two partitions $X_{\beta+1} \in H_\zeta$ and $X'_{\beta+1} \in H_{\zeta'}$, where $\zeta(\beta) \neq \zeta'(\beta)$). Hence, in $\mathbf{V}$, there are $2^{\mathfrak{c}}$ Ramseyan ultrafilters. Because there are only $\mathfrak{c}$ surjections from ω onto ω, no equivalence class (w.r.t. "$\simeq$") can contain more than $\mathfrak{c}$ Ramseyan ultrafilters, so, in $\mathbf{V}$, there must be $2^{\mathfrak{c}}$ pairwise non-equivalent Ramseyan ultrafilters. $\dashv$

3 The happy families' relatives

3.1 Relatively happy families

As we will see below, the partition-families which have the segment-coloring-property are related to special coideals, so-called happy families, which are introduced and rigorously investigated by Adrian Mathias in [Ma77]. So, partition-families with the segment-coloring-property can be considered as "relatives of happy families".

Let us first consider the definition of Mathias' happy families.

Let $[\omega]^\omega$ be the set of all infinite subsets of ω, and let $[\omega]^{<\omega}$ be the set of all finite subsets of ω. A set $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a **free ideal**, if $\mathcal{I}$ is an ideal which contains the **Fréchet ideal** $[\omega]^{<\omega}$. A set $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a **free filter**, if $\{y : \omega \setminus y \in \mathcal{F}\}$ is an ideal containing the Fréchet ideal. For $a \in [\omega]^{<\omega}$, let $a^* := \max\{n+1 : n \in a\}$, in particular, $0^* = 0$. For $x, y \in \mathcal{P}(\omega)$ we write $y \subseteq^* x$ if $(y \setminus x) \in [\omega]^{<\omega}$. For a set

$\mathcal{B}\subseteq\mathcal{P}(\omega)$, let $\mathrm{fil}(\mathcal{B})$ be the free filter generated by $\mathcal{B}$, so, $x \in \mathrm{fil}(\mathcal{B})$ if and only if there is a finite set $y_0, \dots, y_n \in \mathcal{B}$ such that $(y_0 \cap \dots \cap y_n)\subseteq^* x$.

A set $x\subseteq\omega$ is said to **diagonalize** the family $\{x_a : a \in [\omega]^{<\omega}\}$, if $x\subseteq x_0$ and for all $a \in [\omega]^{<\omega}$, if $\max(a) \in x$, then $(x \setminus a^*)\subseteq x_a$.

The family $\mathcal{A}\subseteq\mathcal{P}(\omega)$ is **happy**, if $\mathcal{P}(\omega) \setminus \mathcal{A}$ is a free ideal and whenever $\mathrm{fil}\{x_a : a \in [\omega]^{<\omega}\}\subseteq\mathcal{A}$, there is an $x \in \mathcal{A}$ which diagonalizes $\{x_a : a \in [\omega]^{<\omega}\}$.

In terms of happy families one can define Ramsey ultrafilters as follows: A Ramsey ultrafilter is an ultrafilter that is also a happy family.

Now we turn back to partitions. The Fréchet ideal corresponds to the set of finite partitions, and therefore, the notion of a free filter corresponds to partition-filters containing only infinite partitions, hence, to partition-filters $\mathscr{F}\subseteq(\omega)^\omega$. For a set $\mathscr{B}\subseteq(\omega)^\omega$, let $\mathrm{fil}(\mathscr{B})$ be the partition-filter generated by $\mathscr{B}$, so, $X \in \mathrm{fil}(\mathscr{B})$ if and only if there is a finite set of partitions $Y_0, \dots, Y_n \in \mathscr{B}$ such that $(Y_0\sqcap\dots\sqcap Y_n) \sqsubseteq^* X$.

A partition X is said to diagonalize the family $\{X_s : s \in (\mathbb{N})\}$, if $X \sqsubseteq X_\emptyset$ and for all $s \in (\mathbb{N})$, if $s^* \preccurlyeq X$, then $\left(\bigcup s^* \sqcap X\right) \sqsubseteq X_s$.

The family $\mathscr{A}\subseteq(\omega)^\omega$ is **relatively happy**, if whenever $\mathrm{fil}\{X_s : s \in (\mathbb{N})\}\subseteq\mathscr{A}$, there is an $X \in \mathscr{A}$ which diagonalizes $\{X_s : s \in (\mathbb{N})\}$.

An example of a relatively happy family is $(\omega)^\omega$, the set of all infinite partitions (compare with [Ma77, Example 0.2]). Another example of a much smaller relatively happy family is given in the following theorem (compare with [Ma77, p. 63]).

Theorem 3.1.1 Every Ramseyan ultrafilter is relatively happy.

Proof: Let $\mathscr{U}\subseteq(\omega)^\omega$ be a partition-ultrafilter which has the segment-coloring-property and let $\{X_s : s \in (\mathbb{N})\}\subseteq\mathscr{U}$ be any family. Since $\mathscr{U}$ is a partition-filter, we obviously have $\mathrm{fil}\{X_s : s \in (\mathbb{N})\}\subseteq\mathscr{U}$. For $t \in (\mathbb{N})$ with $|t| \geq 2$, let s_t be such that $s_t^* \preccurlyeq t$ and $|s_t| = |t| - 2$. Define the coloring $\pi : \bigcup_{n\in\omega}(\omega)^{(n+2)*} \to \{0,1\}$ by stipulating

$$\pi(t) := \begin{cases} 0 & \text{if } \left(\bigcup s_t^* \sqcap t^*\right) \sqsubseteq X_{s_t}, \\ 1 & \text{otherwise.} \end{cases}$$

Let $X \in (X_\emptyset)^\omega \cap \mathscr{U}$ be such that for any $n \in \omega$ and for any $s^* \preccurlyeq X$ with $|s| = n$, $(s^*, X)^{(n+2)*}$ is monochromatic with respect to π. Take any $s^* \preccurlyeq X$. Since $(s^*, X)^{(|s|+2)*}$ is monochromatic with respect to π, each $t^* \sqsubseteq X$ with $s^* \preccurlyeq t$ and $|t| = |s| + 2$ gets the same color. Hence, for all such t's we have either $\left(\bigcup s^* \sqcap t^*\right) \sqsubseteq X_s$, which implies $X \sqsubseteq^* X_s$, or $\left(\bigcup s^* \sqcap t^*\right) \not\sqsubseteq X_s$, which implies $X \sqcap X_s \notin (\omega)^\omega$. The latter is impossible, since it contradicts the assumption that $\mathscr{U}$ is a partition-filter. So, we are always in the former case, which completes the proof. $\dashv$

3.2 A game characterization

There is a characterization of happy ultrafilters over ω, *i.e.*, of Ramsey ultrafilters, in terms of games (*cf.* [BJ95, Theorem 4.5.3]). A similar characterization we get for relatively happy partition-ultrafilter.

Let $\mathscr{U}$ be a partition-ultrafilter. Define a game $G(\mathscr{U})$ played by players I and II as follows:

$$\begin{array}{lccccccc} \mathrm{I} & X_1 & & X_2 & & X_3 & & \\ & & & & & & & \dots \\ \mathrm{II} & & s_1 & & s_2 & & s_3 & \end{array}$$

Player I on the n-th move plays a partition $X_n \in \mathscr{U}$. Player II responds with a segment $s_n \in (\mathbb{N})$ such that $|s_n| = n$, $s^*_{n-1} \preccurlyeq s_n$ and for all $m < n$, $\left(\bigcup s^*_m \sqcap s^*_n\right) \sqsubseteq X_{m+1}$, where $s_0 := \emptyset$. Player I wins if and only if the unique partition X with $s_n \preccurlyeq X$ (for all n) is not in $\mathscr{U}$.

THEOREM 3.2.1 Let $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$, then player I has a winning strategy in $G(\mathscr{U})$ if and only if $\mathscr{U}$ is not relatively happy.

PROOF: Assume first that the partition-ultrafilter $\mathscr{U}$ is relatively happy and that $\{X_s : s \in (\mathbb{N})\}$ is a strategy for player I. This means, player I begins with $X_\emptyset$ and then, if s_n is the n-th move of player II, player I plays X_{s_n}. Because $\mathscr{U}$ is relatively happy, there is a partition $X \in \mathscr{U}$ which diagonalizes the family $\{X_s : s \in (\mathbb{N})\}$, in particular, $X \sqsubseteq X_\emptyset$. Now, by the definition of X and by the rules of the game $G(\mathscr{U})$, player II can play the segments of X. More precisely, player II plays on the n-th move the segment s_n, so that $|s_n| = n$ and $s^*_n \preccurlyeq X$. Since $X \in \mathscr{U}$, the strategy $\{X_s : s \in (\mathbb{N})\}$ was not a winning strategy for player I.

Now assume that the strategy $\sigma = \{X_s : s \in (\mathbb{N})\}$ is not a winning strategy for player I. Consider the game where player I is playing according to the strategy σ. In this game, player II can play segments s_n such that the unique partition X with $s_n \preccurlyeq X$ (for all n) is in $\mathscr{U}$. We have to show that X diagonalizes the family $\{X_s : s \in (\mathbb{N})\}$. For $n \in \omega$, let $s_n \in (\mathbb{N})$ be such that $s^*_n \preccurlyeq X$ and $|s_n| = n$. Fix $m \in \omega$, then, by the rules of the game, for any $n > m$ we have $\left(\bigcup s^*_m \sqcap s^*_n\right) \sqsubseteq X_{m+1}$, which implies $\left(\bigcup s^*_m \sqcap X\right) \sqsubseteq X_{m+1}$. Since player I follows the strategy σ, $X_{m+1} = X_{s_m}$, and because m was arbitrary, for all $m \in \omega$ we get $\left(\bigcup s^*_m \sqcap X\right) \sqsubseteq X_{s_m}$. Hence, X diagonalizes the family $\{X_s : s \in (\mathbb{N})\}$. $\dashv$

4 The combinatorics of dual Mathias forcing

First we recall the Ellentuck topology on $[\omega]^\omega$. For $x \in [\omega]^\omega$ and $a \in [\omega]^{<\omega}$ with $x \cap (\max(a)+1) = a$, let $[a,x]^\omega := \{y \in [\omega]^\omega : a \subseteq y \subseteq x\}$, and let the basic open sets on $[\omega]^\omega$ be the sets $[a,x]^\omega$. These sets are called **Ellentuck neighborhoods**. The topology induced by the Ellentuck neighborhoods is called **Ellentuck topology** (*cf.* [El74]).

The **Mathias forcing** $\mathbb{M}$, introduced in [Ma77], consists of ordered pairs $\langle a,x \rangle$ such that $[a,x]^\omega$ is an Ellentuck neighborhood and the ordering on $\mathbb{M}$ is defined by stipulating $\langle a,x \rangle \leq \langle b,y \rangle \Leftrightarrow [a,x]^\omega \subseteq [b,y]^\omega$.

Mathias forcing restricted to a non-principal ultrafilter $\mathcal{U}$, denoted by $\mathbb{M}_\mathcal{U}$, consists of the ordered pairs $\langle a,x \rangle \in \mathbb{M}$, where in addition we require that $x \in \mathcal{U}$.

Mathias forcing has a lot of nice combinatorial properties (some of them are mentioned below) which also hold for Mathias forcing restricted to a Ramsey ultrafilter (see [Ma77]).

The **dual Ellentuck topology** on $(\omega)^\omega$ is the topology induced by the dual Ellentuck neighborhoods (defined in section 2.1). Now, the **dual Mathias forcing** $\mathbb{M}^\flat$, introduced in [CaSi84], is defined similarly to Mathias forcing $\mathbb{M}$, using the dual Ellentuck topology instead of the Ellentuck topology. So, $\mathbb{M}^\flat$ consists of ordered pairs $\langle s,X \rangle$ such that $(s,X)^\omega$ is a dual Ellentuck neighborhood and the ordering on $\mathbb{M}^\flat$ is defined by stipulating $\langle s,X \rangle \leq \langle t,Y \rangle \Leftrightarrow (s,X)^\omega \subseteq (t,Y)^\omega$.

Dual Mathias forcing restricted to a partition-ultrafilter $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$, denoted by $\mathbb{M}^\flat_\mathscr{U}$, consists of the ordered pairs $\langle s,X \rangle \in \mathbb{M}^\flat$, where in addition we require that $X \in \mathscr{U}$ (see *e.g.* [Ha98$_1$] and [HLö∞_2]).

Both, Mathias forcing as well as dual Mathias forcing, are proper forcings. Moreover, both have (i) a decomposition, (ii) pure decision and (iii) the homogeneity property (see *e.g.* [Ma77], [CaSi84] and [Ha98$_1$]):

(i) **Decomposition**: $\mathbb{M} \approx \mathbb{U} * \mathbb{M}_{\check{\mathbf{U}}}$, where $\check{\mathbf{U}}$ is the canonical $\mathbb{U}$-name for the $\mathbb{U}$-generic object ($\mathbb{U}$ as in section 2.2).

$\mathbb{M}^\flat \approx \mathbb{U}^\flat * \mathbb{M}^\flat_{\check{\mathbf{U}}}$, where $\check{\mathbf{U}}$ is the canonical $\mathbb{U}^\flat$-name for the $\mathbb{U}^\flat$-generic object ($\mathbb{U}^\flat$ as in section 2.2).

(ii) **Pure decision**: For any $\mathbb{M}$-condition $\langle a,x \rangle$ and any sentence Φ of the forcing language $\mathbb{M}$, there is an $\mathbb{M}$-condition $\langle a,y \rangle \leq \langle a,x \rangle$ such that either $\langle a,y \rangle \Vdash_\mathbb{M} \Phi$ or $\langle a,y \rangle \Vdash_\mathbb{M} \neg\Phi$.

For any $\mathbb{M}^\flat$-condition $\langle s,X \rangle$ and any sentence Φ of the forcing language $\mathbb{M}^\flat$, there is an $\mathbb{M}^\flat$-condition $\langle s, Y\rangle \le \langle s, X\rangle$ such that either $\langle s, Y\rangle \Vdash_{\mathbb{M}^\flat} \Phi$ or $\langle s, Y\rangle \Vdash_{\mathbb{M}^\flat} \neg\Phi$.

(iii) **Homogeneity property**: If x_G is $\mathbb{M}$-generic over $\mathbf{V}$ and $y \in [x_G]^\omega$, then y is also $\mathbb{M}$-generic over $\mathbf{V}$.

If X_G is $\mathbb{M}^\flat$-generic over $\mathbf{V}$ and $Y \in (X_G)^\omega$, then Y is also $\mathbb{M}^\flat$-generic over $\mathbf{V}$.

In [Ha98$_1$] it is shown that if $\mathscr{F} \subseteq (\omega)^\omega$ is a so-called *game-family*, then $\mathbb{M}^\flat_{\mathscr{F}}$ has pure decision and the homogeneity property ([Ha98$_1$, Thm. 4.3 & 4.4]). Game-families have the segment-coloring-property and therefore, the so-called *game-filters*, *i.e.*, game-families which are partition-filters, are Ramseyan ultrafilters. Unlike for Ramseyan ultrafilters, it is not clear if CH implies the existence of game-filters, so, it seems that game-filters are stronger than Ramseyan ultrafilters. However, in the sequel we show that if $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$ is a Ramseyan ultrafilter, then $\mathbb{M}^\flat_{\mathscr{U}}$ has pure decision and the homogeneity property.

Recently, Stevo Todorčević gave an abstract presentation of Ellentuck's theorem by introducing the notion of a *quasi ordering with approximations* which *admits a finitization* and the notion of a *Ramsey space*. The ABSTRACT ELLENTUCK THEOREM says that a quasi ordering with approximations which admits a finitization and satisfies certain axioms is a Ramsey space.

Let $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$ be a partition-ultrafilter and let "$\sqsubseteq$" be the quasi ordering on $\mathscr{U}$. For each $n \in \omega$, let the function $p_n : \mathscr{U} \to (\mathbb{N})$ be such that $p_n(X)$ is the unique s with $s^* \preccurlyeq X$ and $|s| = n$. Let p be the sequence $(p_n)_{n\in\omega}$. It is easy to verify that the triple $(\mathscr{U}, \sqsubseteq, p)$ is a **quasi ordering with approximations**. For $n, m \in \omega$ and $X, Y \in \mathscr{U}$ define: $p_n(X) \sqsubseteq_{\text{fin}} p_m(Y)$ if and only if $\operatorname{dom}\big(p_n(X)\big) = \operatorname{dom}\big(p_m(Y)\big)$ and $p_n(X) \sqsubseteq p_m(Y)$. This definition verifies that $(\mathscr{U}, \sqsubseteq, p)$ **admits a finitization**. If $(s, X)^\omega$ is a dual Ellentuck neighborhood and $X \in \mathscr{U}$, then $(s, X)^\omega \cap \mathscr{U}$ is called a **$\mathscr{U}$-dual Ellentuck neighborhood**. The topology on $\mathscr{U}$, induced by the $\mathscr{U}$-dual Ellentuck neighborhoods, is called the **$\mathscr{U}$-dual Ellentuck topology**. With respect to the $\mathscr{U}$-dual Ellentuck topology, the topological space $\mathscr{U}$ is a **Ramsey space**, if for any subset $S \subseteq \mathscr{U}$ which has the Baire property with respect to the $\mathscr{U}$-dual Ellentuck topology, and for any $\mathscr{U}$-dual Ellentuck neighborhood $(s, Y)^\omega \cap \mathscr{U}$, there is a partition $X \in (s, Y)^\omega \cap \mathscr{U}$ such that either $(s, X)^\omega \cap \mathscr{U} \subseteq S$ or $(s, X)^\omega \cap \mathscr{U} \subseteq \mathscr{U} \setminus S$.

Let $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$ be a Ramseyan ultrafilter. Since the triple $(\mathscr{U}, \sqsubseteq, p)$ satisfies certain axioms, by Todorčević's ABSTRACT ELLENTUCK THEOREM, the Ramseyan ultrafilter $\mathscr{U}$ with respect to the $\mathscr{U}$-dual Ellentuck topology is a Ramsey space. Moreover, we get the following two results.

THEOREM 4.1 If $\mathscr{U}$ is a Ramseyan ultrafilter, then $\mathbb{M}^\flat_{\mathscr{U}}$ has pure decision.

PROOF: Let Φ be any sentence of the forcing language $\mathbb{M}^\flat_{\mathscr{U}}$. With respect to Φ we define

$$D_0 := \{Y \in \mathscr{U} : \text{ for some } t \preccurlyeq Y,\, \langle t, Y\rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \neg\Phi\}\,,$$

and

$$D_1 := \{Y \in \mathscr{U} : \text{ for some } t \preccurlyeq Y,\, \langle t, Y\rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \Phi\}\,.$$

Clearly D_0 and D_1 are both open (w.r.t. the $\mathscr{U}$-dual Ellentuck topology) and $D_0 \cup D_1$ is dense (w.r.t. the partial order in $\mathbb{M}^\flat_{\mathscr{U}}$). Because $\mathscr{U}$ is a Ramsey space, for any $\mathscr{U}$-dual Ellentuck neighborhood $(s, Y)^\omega \cap \mathscr{U}$ there is an $X \in (s, Y)^\omega \cap \mathscr{U}$ such that $(s, X)^\omega \cap \mathscr{U} \subseteq D_0$ or $(s, X)^\omega \cap \mathscr{U} \cap D_0 = \emptyset$. In the former case we have $\langle s, X\rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \neg\Phi$ and we are done. In the latter case we find $X' \in (s, X)^\omega \cap \mathscr{U}$ such that $(s, X')^\omega \cap \mathscr{U} \subseteq D_1$. (Otherwise we would have $(s, X')^\omega \cap \mathscr{U} \cap (D_0 \cup D_1) = \emptyset$, which is impossible by the density of $D_0 \cup D_1$.) Hence, $\langle s, X'\rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \Phi$. $\dashv$

THEOREM 4.2 If $\mathscr{U}$ is a Ramseyan ultrafilter, then $\mathbb{M}^\flat_{\mathscr{U}}$ has the homogeneity property.

PROOF: For a dense set $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$, let

$$\bigcup D := \{X \in (\omega)^\omega : X \in (s, Y)^\omega \text{ for some } \langle s, Y\rangle \in D\}\,.$$

It is clear that a partition X_G is $\mathbb{M}^\flat_{\mathscr{U}}$-generic if and only if $X_G \in \bigcup D$ for each dense set $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$. Let $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$ be an arbitrary dense set and let D' be the set of all $\langle s, Z\rangle \in \mathbb{M}^\flat_{\mathscr{U}}$ such that $(t, Z)^\omega \subseteq \bigcup D$ for all $t \sqsubseteq s$ with $\mathrm{dom}(t) = \mathrm{dom}(s)$.

First we show that D' is dense in $\mathbb{M}^\flat_{\mathscr{U}}$. For this, take an arbitrary $\langle s, W\rangle \in \mathbb{M}^\flat_{\mathscr{U}}$ and let $\{t_i : 0 \le i \le m\}$ be an enumeration of all $t \in (\mathbb{N})$ such that $t \sqsubseteq s$ and $\mathrm{dom}(t) = \mathrm{dom}(s)$. Because D is dense in $\mathbb{M}^\flat_{\mathscr{U}}$, $\bigcup D$ is open (w.r.t. the $\mathscr{U}$-dual Ellentuck topology), and since $\mathscr{U}$ is a Ramsey space, for every t_i we find a $W' \in \mathscr{U}$ such that $t_i \sqsubseteq W'$ and $(t_i, W')^\omega \subseteq \bigcup D$. Moreover, if we define $W_{-1} := W$, for every $i \le m$ we can choose a partition $W_i \in \mathscr{U}$ such that $W_i \sqsubseteq W_{i-1}$, $s \preccurlyeq W_i$ and $(t_i, W_i)^\omega \subseteq \bigcup D$. Thus, $\langle s, W_m\rangle \in D'$, and because $\langle s, W_m\rangle \le \langle s, W\rangle$, D' is dense in $\mathbb{M}^\flat_{\mathscr{U}}$.

Let X_G be $\mathbb{M}^\flat_{\mathscr{U}}$-generic and let $Y \in (X_G)^\omega$ be arbitrary. Since D' is dense, there is a condition $\langle s, Z\rangle \in D'$ such that $s \preccurlyeq X_G \sqsubseteq Z$. Since $Y \in (X_G)^\omega$, we have $t \preccurlyeq Y \sqsubseteq Z$ for some $t \sqsubseteq s$ with $\mathrm{dom}(t) = \mathrm{dom}(s)$, and because $(t, Z)^\omega \subseteq \bigcup D$, we get $Y \in \bigcup D$. Hence, $Y \in \bigcup D$ for each dense set $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$, which completes the proof. $\dashv$

Appendix

In this section we are gathering some results concerning the dual form of some cardinal characteristics of the continuum. For the definition of the classical cardinal characteristics, as well as for the relation between them, we refer the reader to [Va90].

First we consider the shattering cardinal $\mathfrak{h}$. This cardinal was introduced in [BPS80] as the minimal height of a tree π-base of $\beta\mathbb{N}\setminus\mathbb{N}$. Later it was shown by Szymon Plewik in ([Pl86]) that $\mathfrak{h}=\mathrm{add}(r^0)=\mathrm{cov}(r^0)$, where r^0 denotes the ideal of Ramsey-null sets. It is easy to see that $\mathfrak{p}\leq\mathfrak{h}$, and therefore, $\mathsf{MA}(\sigma\text{-centered})$ implies $\mathfrak{h}=\mathfrak{c}$.

The dual form of the classical cardinal characteristics were introduced and investigated in [CKMW00] and further investigated in [Ha98$_2$]. Concerning the dual-shattering cardinal $\mathfrak{H}$, one easily gets $\aleph_1\leq\mathfrak{H}\leq\mathfrak{h}$, and in [Ha98$_2$] it is shown that $\mathfrak{H}>\aleph_1$ is consistent relative to ZFC and that $\mathfrak{H}=\mathrm{add}(R^0)=\mathrm{cov}(R^0)$, where R^0 denotes the ideal of dual Ramsey-null sets. After all these symmetries, one would not expect the following: $\mathsf{MA}+(\mathfrak{c}>\mathfrak{H})$ is consistent relative to ZFC. This was proved by Jörg Brendle in [Br00] and implies that $\mathfrak{H}<\mathfrak{p}$ is consistent relative to ZFC.

Concerning the reaping and the dual-reaping number $\mathfrak{r}$ and $\mathfrak{R}$, respectively, the situation looks different. It is shown in [Ha98$_1$] that $\mathfrak{p}\leq\mathfrak{R}\leq\min\{\mathfrak{r},\mathfrak{i}\}$, and thus we get $\mathsf{MA}(\sigma\text{-centered})$ implies $\mathfrak{R}=\mathfrak{c}$. Further, it is easy to show that $\mathfrak{R}\leq\mathfrak{U}$, where $\mathfrak{U}$ denotes the partition-ultrafilter base number, *i.e.*, the dual form of $\mathfrak{u}$, and consequently, $\mathsf{MA}(\sigma\text{-centered})$ implies $\mathfrak{U}=\mathfrak{c}$.

For a Ramsey ultrafilter $\mathcal{U}$, Brendle introduced in [Br95] the ideal $r^0_{\mathcal{U}}$, which is the ideal of Ramsey-null sets with respect to the ultrafilter $\mathcal{U}$. Concerning this ideal $r^0_{\mathcal{U}}$, he showed for example that $\mathfrak{hom}\leq\mathrm{non}(r^0_{\mathcal{U}})$, where $\mathfrak{hom}$ is the homogeneity number investigated by Blass in [Bl93, Section 6]. There, Blass also investigated the so-called partition number $\mathfrak{par}$ and showed that $\mathfrak{par}=\min\{\mathfrak{b},\mathfrak{s}\}$. Now, replacing the Ramsey ultrafilter $\mathcal{U}$ by a Ramseyan ultrafilter $\mathscr{U}$, one obtains the ideal $R^0_{\mathscr{U}}$ of dual Ramsey-null sets with respect to $\mathscr{U}$ as the dualization of the ideal $r^0_{\mathcal{U}}$, and replacing the colorings of $[\omega]^2$—involved in the definition of $\mathfrak{hom}$ and $\mathfrak{par}$—by colorings of $(\omega)^{2*}$, one obtains the cardinal characteristics $\mathfrak{Hom}$ and $\mathfrak{Par}$ and could begin to investigate them. But this is left to the reader. ☺

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