# A SET-THEORETIC APPROACH TO COMPLETE MINIMAL SYSTEMS IN BANACH SPACES OF BOUNDED FUNCTIONS*

Lorenz Halbeisen

*Department of Pure Mathematics*
*Queen's University Belfast*
*Belfast BT7 1NN*
*Northern Ireland*

*email*: halbeis@qub.ac.uk

**2000 Mathematics Subject Classification: 46B15** 03E05 46B26
**Keywords:** independent families, complete minimal systems

Abstract

Using independent families from combinatorial set theory, it is shown that for every infinite cardinal $\kappa$, $\ell_\infty(\kappa)^*$ contains a subspace which is isomorphic to a Hilbert space of dimension $2^\kappa$. This provides a new proof for the first step in the construction of complete minimal systems in Banach spaces of bounded functions.

## 1. Introduction

Let $X$ be a Banach space and let $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ be an arbitrary set of vectors of $X$. Let $[x_\lambda : \lambda \in \Lambda]$ denote the **closure of the linear span** of $\{x_\lambda : \lambda \in \Lambda\}$. A set $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ is called a **complete system** if $[x_\lambda : \lambda \in \Lambda] = X$, and it is called a **minimal system** if for every $\lambda' \in \Lambda$, $x_{\lambda'} \notin \big[x_\lambda : \lambda \in \Lambda \setminus \{\lambda'\}\big]$. A **complete minimal system**, abbreviated *c.m.s.*, is a complete system which is also minimal.

Using functionals, we can characterize minimal systems (and consequently *c.m.s.*) also in the following way: Let $X$ be a Banach space. A pair of sequences $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ and $\{f_\lambda : \lambda \in \Lambda\} \subseteq X^*$ is called a **biorthogonal system** if $f_{\lambda'}(x_\lambda) = \delta^\lambda_{\lambda'}$. Now, a sequence $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ is minimal if and only if there is a sequence $\{f_\lambda : \lambda \in \Lambda\} \subseteq X^*$, such that the pair $\big(\{x_\lambda : \lambda \in \Lambda\}, \{f_\lambda : \lambda \in \Lambda\}\big)$ is a biorthogonal

*The research for this paper was resumed during the *Workshop on Set Theory, Topology, and Banach Space Theory*. The workshop took place in June 2003 at *Queen's University Belfast* and was supported by the author's *Nuffield Foundation Grant* NAL/00513/G.

system. A biorthogonal system which corresponds to a complete minimal system is called **complete biorthogonal system**.

Even though not every Banach space has a *c.m.s.* (see *e.g.*, [Pl80] or [GK80]), it is known that $\ell_\infty$ has a *c.m.s.*. The first (not completely correct) proof for the existence of a *c.m.s.* in $\ell_\infty$ was given by William Davis and William Johnson in [DJ73]. Later, Borys Godun gave a correct (and slightly easier) proof in [Go83]. However, the crucial point in both proofs is the following result due to Haskel Rosenthal (*cf.* [Ro69, Proposition 3.4]):

PROPOSITION 1. The space $\ell_\infty^*$ contains a subspace isomorphic to a Hilbert space of dimension the continuum.

Let us briefly sketch why Proposition 1 implies the existence of a *c.m.s.* in $\ell_\infty$: Let $Y \subseteq \ell_\infty$ be isomorphic to a Hilbert space of dimension the continuum. Since $Y$ is reflexive, $Y$ is weakly$^*$ closed (*cf. e.g.*, [Ro69, Proposition 1.2]), and therefore, $({}_\perp Y)^\perp = Y$, where ${}_\perp Y = \big\{x \in \ell_\infty : \forall y \in Y(y(x) = 0)\big\}$ and $({}_\perp Y)^\perp := \big\{x^* \in \ell_\infty^* : \forall x \in {}_\perp Y\big(x^*(x) = 0\big)\big\}$. Thus, $\big(\ell_\infty / {}_\perp Y\big)^*$ is isomorphic to the Hilbert space $Y$, which implies that also $\ell_\infty / {}_\perp Y$ is isomorphic to $Y$. Now, following [Go83], with the orthonormal basis in $Y$ we can easily construct a *c.m.s.* in $\ell_\infty$. At this point we like to mention that starting with generalized version of Proposition 1 (*cf.* [Ro69, p. 203, Remark 2]), a similar construction yields a *c.m.s.* in $\ell_\infty(\kappa)$ for any infinite cardinal $\kappa$.

Rosenthal's proof of Proposition 1 involves some deep results from functional analysis. On the other hand, from a set-theoretical point of view a *c.m.s.* in $\ell_\infty$ is just a set of bounded real-valued sequences, and therefore, it was natural to seek a more combinatorial or set-theoretical proof of Proposition 1 and the aim of this paper is to provide such a proof.

ACKNOWLEDGEMENT: I like to thank Anatolij Plichko and Stephanie Halbeisen for pointing me out and explaining me the various steps in the construction of a *c.m.s.* in $\ell_\infty$.

## 2. Some set theory

2.1. **Set-theoretic terminology.** Our set-theoretical axioms are the axioms of Zermelo and Fraenkel including the Axiom of Choice. All our set-theoretical notations and definitions are standard and can be found in textbooks like [Ku83].

For a set $x$, the **cardinality** of $x$, denoted by $|x|$, is the least ordinal number $\alpha$ for which there exists a bijection $f : \alpha \to x$; such an ordinal number $\alpha$ is called a **cardinal number** (or just a **cardinal**). The least infinite ordinal number, which is also a cardinal, is denoted by $\omega$, thus, $|\omega| = \omega$. In particular, $\omega = \{0, 1, 2, \dots\}$ is the set of natural numbers. A set $x$ is called finite, if $|x| \in \omega$, otherwise it is called infinite. Further it is called countable, if $|x| \leq \omega$. For a set $x$, $\mathcal{P}(x)$ denotes the power set of $x$ and $[x]^{<\omega}$ denotes the set of all finite subsets of $x$. For a cardinal $\kappa$, $|\mathcal{P}(\kappa)|$ is denoted by $2^\kappa$. For example there exists a bijection between the reals $\mathbb{R}$ and $\mathcal{P}(\omega)$, hence $|\mathbb{R}| = |\mathcal{P}(\omega)| = 2^\omega$. For every infinite cardinal we have $2^\kappa > \kappa$ and $\left|[\kappa]^{<\omega}\right| = \kappa$.

2.2. **Independent families.** Let $\kappa$ be an infinite cardinal and let $\mathcal{I} \subseteq \mathcal{P}(\kappa)$, then $\mathcal{I}$ is called an **independent family** (on $\kappa$), if whenever $m$ and $n-1$ belong to $\omega$, and $x_0, \ldots, x_m, \ldots, x_{m+n}$ are distinct members of $\mathcal{I}$, then

$$\Big| \bigcap_{0 \le i \le m} x_i \setminus \bigcup_{1 \le j \le n} x_{m+j} \Big| = \kappa\,.$$

To make this paper self-contained, let us prove the following result due to Felix Hausdorff (*cf.* [Ha36]):

PROPOSITION 2. For any infinite cardinal $\kappa$, there is an independent family on $\kappa$ of cardinality $2^\kappa$.

*Proof.* We just follow Exercise (A6) on p. 288 of [Ku83]. Let

$$J = \big\{ \langle s, A\rangle : s \subseteq \kappa \textit{ and } |s| < \omega \textit{ and } A \subseteq \mathcal{P}(s) \big\}\,.$$

Notice that $|J| = \kappa$, so, it is enough to construct an independent family of cardinality $2^\kappa$ on $J$. For $x' \subseteq \kappa$, let $x := \big\{\langle s, A\rangle \in J : x' \cap s \in A\big\}$. Then $\mathcal{I} = \big\{x : x' \in \mathcal{P}(\kappa)\big\}$ is an independent family on $J$ of cardinality $2^\kappa$. Indeed, let $x'_0, \ldots, x'_m, \ldots, x'_{m+n}$ be distinct members of $\mathcal{P}(\kappa)$ (for some $m$ and $n-1$ in $\omega$). Then there is a finite set $s \subseteq \kappa$ such that for all $i, j$ with $0 \le i < j \le m+n$ we have $x'_i \cap s \ne x'_j \cap s$. Let $A = \{s \cap x'_i : 0 \le i \le m\} \subseteq \mathcal{P}(s)$, and for every $\alpha \in \kappa \setminus s$, let $s_\alpha = s \cup \{\alpha\}$ and $A_\alpha = A \cup \big\{t \cup \{\alpha\} : t \in A\big\}$. Then

$$\big\{\langle s_\alpha, A_\alpha\rangle : \alpha \in \kappa \setminus s\big\} \subseteq \bigcap_{0 \le i \le m} x_i \setminus \bigcup_{1 \le j \le n} x_{m+j}\,,$$

which implies that $\Big|\bigcap_{i \le m} x_i \setminus \bigcup_{m < j \le n} x_j\Big| = \kappa$, and therefore, $\mathcal{I}$ is an independent family on $J$ of cardinality $2^\kappa$. $\dashv$

As an easy consequence we get the following

FACT. If $\mathcal{I} = \{x_\alpha : \alpha \in 2^\kappa\}$ is an independent family on $\kappa$ and $\alpha_1, \ldots, \alpha_n$ are finitely many distinct elements of $2^\kappa$, then $\big|\bigcap_{1 \le i \le n} y_{\alpha_i}\big| = \kappa$, where for every $1 \le i \le n$, the set $y_{\alpha_i}$ is either equal to the set $x_{\alpha_i}$, or to its complement $\kappa \setminus x_{\alpha_i}$.

2.3. **The Banach spaces $\ell_2(\kappa)$ and $\ell_\infty(\kappa)$.** Let $\kappa$ be an infinite cardinal. The Banach space $\ell_\infty(\kappa)$ is the set of all bounded functions from $\kappa$ to $\mathbb{R}$, where for $x \in \ell_\infty(\kappa)$, $\|x\| = \sup\{x(\alpha) : \alpha \in \kappa\}$. The Banach space $\ell_2(\kappa)$ is the set of all functions $x$ from $\kappa$ to $\mathbb{R}$ such that $\sum_{\alpha \in \kappa} x(\alpha)^2 =: \|x\|^2 < \infty$. It is common to write $\ell_2$ and $\ell_\infty$ instead of $\ell_2(\omega)$ and $\ell_\infty(\omega)$ respectively. Like for $\ell_2$ and $\ell_\infty$, one can show that $\ell_2(\kappa)^* = \ell_2(\kappa)$ and that $\ell_\infty(\kappa)^*$ is isometric to the space of all finitely additive signed measures $\mu$ of bounded variation on $\mathcal{P}(\kappa)$, supplied with the norm $\|\mu\| = |\mu|(\kappa)$, where $|\mu|$ is the total variation of $\mu$.

For $\alpha, \beta \in \kappa$, let

$$\delta_\alpha^\beta = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and let $e_\alpha : \kappa \to \{0, 1\}$ be such that $e_\alpha(\beta) = \delta_\alpha^\beta$. It is easy to see that the set of vectors $\{e_\alpha : \alpha \in \kappa\}$ is a *c.m.s.* of $\ell_2(\kappa)$. On the other hand, the set $\{e_\alpha : \alpha \in \kappa\}$ is much too small to be a *c.m.s.* of $\ell_\infty(\kappa)$. In general, the cardinality of a complete minimal system $S$ of an infinite dimensional real Banach space $X$ is always equal to the density character of $X$. Indeed, on the one hand, the set of all finite linear combinations of $S$ with rational coefficients is dense in $X$, and on the other hand, $S$ is discrete in $X$. In particular, the density character of $\ell_\infty(\kappa)$ is $2^\kappa$, so, any *c.m.s.* of $\ell_\infty(\kappa)$ must have cardinality $2^\kappa$.

## 3. $\ell_\infty(\kappa)^*$ CONTAINS AN ISOMORPHIC COPY OF $\ell_2(2^\kappa)$

Now we are ready to prove the main result.

THEOREM. Let $\kappa$ be an infinite cardinal. Then any independent family on $\kappa$ of cardinality $2^\kappa$ induces a subspace of $\ell_\infty(\kappa)^*$ which is isomorphic to the Hilbert space $\ell_2(2^\kappa)$.

*Proof.* Let $\mathcal{I} = \{x_\alpha : \alpha \in 2^\kappa\}$ be an independent family on $\kappa$ of cardinality $2^\kappa$ (which exists by Proposition 2). Define a measure $\hat{\mu}$ on the set $B$ of all Boolean combinations of elements of $\mathcal{I}$ by stipulating

- $\hat{\mu}(x_\alpha) = \hat{\mu}(\kappa \setminus x_\alpha) = 1/2$ (for all $x_\alpha \in \mathcal{I}$),
- $\hat{\mu}(x_\alpha \cap x_\beta) = \hat{\mu}\big(x_\alpha \cap (\kappa \setminus x_\beta)\big) = 1/4$ (for all distinct $x_\alpha, x_\beta \in \mathcal{I}$),

and in general, if $\alpha_1, \dots, \alpha_n$ are finitely many distinct elements of $2^\kappa$ and $0 \le j \le n$, then

$$\hat{\mu}\Big(\bigcap\nolimits_{1 \le i \le j} x_{\alpha_i} \cap \bigcap\nolimits_{j < i \le n} (\kappa \setminus x_{\alpha_i})\Big) = 2^{-n}\,.$$

The measure $\hat{\mu}$ induces a normalized linear functional $\varphi_{\hat{\mu}}$ on a subspace of $\ell_\infty(\kappa)$. Thus, by the normed space version of the Hahn-Banach Extension Theorem, there is a normalized functional on all of $\ell_\infty(\kappa)$ which extends the functional $\varphi_{\hat{\mu}}$. In particular, there is a measure $\mu$ on $\mathcal{P}(\kappa)$ with $\|\mu\| = 1$, such that $\mu|_B \equiv \hat{\mu}$. For every $\alpha \in 2^\kappa$ let $f_\alpha : \kappa \to \{1, -1\}$ such that

$$f_\alpha(\lambda) = \begin{cases} 1 & \text{if } \lambda \in x_\alpha, \\ -1 & \text{otherwise.} \end{cases}$$

Now, for every $\alpha \in 2^\kappa$, let the measure $\mu_\alpha$ on $\mathcal{P}(\kappa)$ be defined by

$$\mu_\alpha(E) = \mu(E \cap x_\alpha) - \mu\big(E \cap (\kappa \setminus x_\alpha)\big)\,,$$

and let $\varphi_\alpha$ be the linear functional on $\ell_\infty(\kappa)$ induced by the measure $\mu_\alpha$. It is not hard to see that for all $\alpha, \beta \in 2^\kappa$, $\varphi_\alpha(f_\beta) = \delta_\alpha^\beta$ and that $\|\varphi_\alpha\|_{\ell_\infty(\kappa)^*} = 1$. Let $Y = [\varphi_\alpha : \alpha \in 2^\kappa] \subseteq \ell_\infty(\kappa)^*$, and let $\sum_{i=1}^n a_i \varphi_{\alpha_i} \in Y$.

Claim. For each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$ we have

$$\Big\| \sum_{i=1}^n a_i \varphi_{\alpha_i} \Big\|_{\ell_\infty(\kappa)^*} = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \Big| \sum_{i=1}^n \varepsilon_i a_i \Big|\,.$$

*Proof.* For each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$ let $E_\varepsilon = \bigcap_{1 \le i \le n} y_{\alpha_i}$, where

$$y_{\alpha_i} = \begin{cases} x_{\alpha_i} & \text{if } \varepsilon_i a_i \ge 0, \\ \kappa \setminus x_{\alpha_i} & \text{otherwise.} \end{cases}$$

By the fact mentioned above, $|E_\varepsilon| = \kappa$, and by the properties of the measure $\mu$ we get $\mu(E) = 2^{-n}$. Notice that for any distinct $\varepsilon$ and $\varepsilon'$ in $\{-1,1\}^n$ we have $E_\varepsilon \cap E_{\varepsilon'} = \emptyset$ and that $\kappa = \bigcup_{\varepsilon \in \{-1,1\}^n} E_\varepsilon$. Further, for every $\varepsilon \in \{-1,1\}^n$ let $f_\varepsilon : \kappa \to \{\pm 1, 0\}$ be such that

$$f_\varepsilon(\lambda) = \begin{cases} 1 & \text{if } \lambda \in E_\varepsilon \text{ and } \sum_{i=1}^n \varepsilon_i a_i \ge 0, \\ -1 & \text{if } \lambda \in E_\varepsilon \text{ and } \sum_{i=1}^n \varepsilon_i a_i < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and let $f = \sum_{\varepsilon \in \{-1,1\}^n} f_\varepsilon$. It is not hard to verify that for each $\varepsilon \in \{-1,1\}^n$ we have

$$\big(a_1 \varphi_{\alpha_1} + \ldots + a_n \varphi_{\alpha_n}\big)(f_\varepsilon) = 2^{-n} \left| \varepsilon_1 a_1 + \ldots + \varepsilon_n a_n \right|,$$

and therefore,

$$\Big( \sum_{i=1}^n a_i \varphi_{\alpha_i} \Big)(f) = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \Big| \sum_{i=1}^n \varepsilon_i a_i \Big|\,.$$

Now, since the $E_\varepsilon$'s are pairwise disjoint, $\|f\|_{\ell_\infty(\kappa)} = 1$, and by the construction of $f$ we finally get

$$\Big\| \sum_{i=1}^n a_i \varphi_{\alpha_i} \Big\|_{\ell_\infty(\kappa)^*} = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \Big| \sum_{i=1}^n \varepsilon_i a_i \Big|\,.$$

$\dashv_{\text{Claim}}$

Hence, by Khintchine's inequality, there is a constant $c = 1/\sqrt{2}$ such that

$$c \cdot \sqrt{\sum_{i=1}^n a_i^2} \le \Big\| \sum_{i=1}^n a_i \varphi_{\alpha_i} \Big\|_{\ell_\infty(\kappa)^*} \le \sqrt{\sum_{i=1}^n a_i^2}\,,$$

which implies that the space $Y \subseteq \ell_\infty(\kappa)^*$ is isomorphic to the Hilbert space $\ell_2(2^\kappa)$ and completes the proof. $\dashv$

REMARK. For an infinite cardinal $\kappa$, the Banach space $c_0(\kappa)$ is the set of all functions $x$ from $\kappa$ to $\mathbb{R}$ such that for every $\varepsilon > 0$, the set $\{\alpha < \kappa : |x(\alpha)| > \varepsilon\}$ is finite. Now, the Theorem admits the following generalization: *Let $\kappa$ be an infinite cardinal. Then the space $\big(\ell_\infty(\kappa)/c_0(\kappa)\big)^*$ contains a subspace which is isomorphic to $\ell_2(2^\kappa)$.* Consequently we get: *For every infinite cardinal $\kappa$, the space $\ell_\infty(\kappa)/c_0(\kappa)$ has a complete minimal system.*

## References


[DJ73] William J. Davis and William B. Johnson: *On the existence of fundamental and total bounded biorthogonal systems in Banach spaces*, ***Studia Mathematica***, vol. 45 (1973), 173–179.

[Go83] Borys V. Godun: *Complete biorthogonal systems in Banach spaces,* (Russian) ***Funktsional'nyi Analiz i Ego Prilozheniya*** vol. 17 (1) (1983), 1–7; translation in ***Functional Analysis and its Applications*** vol. 17 (1) (1983), 1–5.

[GK80] Borys V. Godun and Mikhail I. Kadets: *Banach spaces without complete minimal systems,* (Russian) ***Funktsional'nyi Analiz i Ego Prilozheniya*** vol. 14 (4) (1980), 67–68; translation in ***Functional Analysis and its Applications*** vol. 14 (4) (1980), 301–302.

[Ha36] Felix Hausdorff: *Über zwei Sätze von G. Fichtenholz und L. Kantorovitch*, ***Studia Mathematica***, vol. 6 (1936), 18–19.

[Ku83] Kenneth Kunen: ***Set Theory, an Introduction to Independence Proofs***, North Holland, Amsterdam (1983).

[Pl80] Anatolij Plichko: *A Banach space without a fundamental biorthogonal system,* (Russian) ***Doklady Akademii Nauk SSSR*** vol. 254 (4) (1980), 798–801; translation in ***Soviet Mathematical Doklady*** vol. 22 (2) (1980), 450–453.

[Ro69] Haskel P. Rosenthal: *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\mu)$*, ***Journal of Functional Analysis***, vol. 4 (1969), 176–214.