On shattering, splitting and reaping partitions

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Abstract

In this article we investigate the dual-shattering cardinal $\mathfrak{H}$, the dual-splitting cardinal $\mathfrak{S}$ and the dual-reaping cardinal $\mathfrak{R}$, which are dualizations of the well-known cardinals $\mathfrak{h}$ (the shattering cardinal, also known as the distributivity number of $\mathcal{P}(\omega)/\mathit{fin}$), $\mathfrak{s}$ (the splitting number) and $\mathfrak{r}$ (the reaping number). Using some properties of the ideal $\mathfrak{J}$ of nowhere dual-Ramsey sets, which is an ideal over the set of partitions of ω, we show that $\mathbf{add}(\mathfrak{J}) = \mathbf{cov}(\mathfrak{J}) = \mathfrak{H}$. With this result we can show that $\mathfrak{H} > \omega_1$ is consistent with ZFC and as a corollary we get the relative consistency of $\mathfrak{H} > \mathfrak{t}$, where $\mathfrak{t}$ is the tower number. Concerning $\mathfrak{S}$ we show that $\mathbf{cov}(\mathcal{M}) \leq \mathfrak{S}$ (where $\mathcal{M}$ is the ideal of the meager sets). For the dual-reaping cardinal $\mathfrak{R}$ we get $\mathfrak{p} \leq \mathfrak{R} \leq \mathfrak{r}$ (where $\mathfrak{p}$ is the pseudo-intersection number) and for a modified dual-reaping number $\mathfrak{R}'$ we get $\mathfrak{R}' \leq \mathfrak{d}$ (where $\mathfrak{d}$ is the dominating number). As a consistency result we get $\mathfrak{R} < \mathbf{cov}(\mathcal{M})$.

1 The set of partitions

A *partial partition* X (of ω) consisting of pairwise disjoint, nonempty sets, such that $\mathrm{dom}(X) := \bigcup X \subseteq \omega$. The elements of a partial partition X are called the blocks of X and $\mathrm{Min}(X)$ denotes the set of the least elements of the blocks of X. If $\mathrm{dom}(X) = \omega$, then X is called a *partition*. $\{\omega\}$ is the partition such that each block is a singleton and $\{\{\omega\}\}$ is the partition containing only one block. The set of all partitions containing infinitely (resp. finitely) many blocks is denoted by $(\omega)^\omega$ (resp. $(\omega)^{<\omega}$). By $(\omega)^{\underline{\omega}}$ we denote the set of all infinite partitions such that at least one block is infinite. The set of all partial partitions with $\mathrm{dom}(X) \in \omega$ is denoted by $(\mathbb{N})$.

Let X_1, X_2 be two partial partitions. We say that X_1 is *coarser* than X_2, or that X_2 is *finer* than X_1, and write $X_1 \sqsubseteq X_2$ if for all blocks $b \in X_1$ the set $b \cap \mathrm{dom}(X_2)$ is the union of some sets $b_i \cap \mathrm{dom}(X_1)$, where each b_i is a block of X_2. (Note that if X_1 is coarser than X_2, then X_1 is in a natural way also contained in X_2.) Let $X_1 \sqcap X_2$ denotes the finest partial partition which is coarser than X_1 and X_2 such that $\mathrm{dom}(X_1 \sqcap X_2) = \mathrm{dom}(X_1) \cup \mathrm{dom}(X_2)$. Similarly $X_1 \sqcup X_2$ denotes the coarsest partial partition which is finer than X_1 and X_2 such that $\mathrm{dom}(X_1 \sqcup X_2) = \mathrm{dom}(X_1) \cup \mathrm{dom}(X_2)$.

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If f is a finite subset of ω, then $\{f\}$ is a partial partition with $\mathrm{dom}(\{f\}) = f$. For two partial partitions X_1 and X_2 we write $X_1 \sqsubseteq^* X_2$ if there is a finite set $f \subseteq \mathrm{dom}(X_1)$ such that $X_1 \sqcap \{f\} \sqsubseteq X_2$ and say that X_1 is coarser* than X_2. If $X_1 \sqsubseteq^* X_2$ and $X_2 \sqsubseteq^* X_1$ then we write $X_1 \stackrel{*}{=} X_2$. If $X \stackrel{*}{=} \{\omega\}$, then X is called *trivial.*

Let X_1, X_2 be two partial partitions. If each block of X_1 can be written as the intersection of a block of X_2 with $\mathrm{dom}(X_1)$, then we write $X_1 \preceq X_2$. Note that $X_1 \preceq X_2$ implies $\mathrm{dom}(X_1) \subseteq \mathrm{dom}(X_2)$.

We define a topology on the set of partitions as follows. Let $X \in (\omega)^\omega$ and $s \in (\mathbb{N})$ such that $s \sqsubseteq X$, then $(s,X)^\omega := \{Y \in (\omega)^\omega : s \preceq Y \wedge Y \sqsubseteq X\}$ and $(X)^\omega := (\emptyset,X)^\omega$. Now let the basic open sets on $(\omega)^\omega$ be the sets $(s,X)^\omega$ (where X and s as above). These sets are called the *dual Ellentuck neighborhoods.* The topology induced by the dual Ellentuck neighborhoods is called the *dual Ellentuck topology* (cf. [CS]).

2 On the dual-shattering cardinal $\mathfrak{H}$

Four cardinals

We first give the definition of the dual-shattering cardinal $\mathfrak{H}$.

Two partitions $X_1, X_2 \in (\omega)^\omega$ are called *almost orthogonal* ($X_1 \perp_* X_2$) if $X_1 \sqcap X_2 \notin (\omega)^\omega$, otherwise they are *compatible* ($X_1 \parallel X_2$). If $X_1 \sqcap X_2 = \{\{\omega\}\}$, then they are called *orthogonal* ($X_1 \perp X_2$). We say that a family $\mathcal{A} \subseteq (\omega)^\omega$ is *maximal almost orthogonal* (*mao*) if $\mathcal{A}$ is a maximal family of pairwise almost orthogonal partitions. A family $\mathcal{H}$ of *mao* families of partitions *shatters* a partition $X \in (\omega)^\omega$, if there are $H \in \mathcal{H}$ and two distinct partitions in H which are both compatible with X. A family of *mao* families of partitions is *shattering* if it shatters each member of $(\omega)^\omega$. The dual-shattering cardinal $\mathfrak{H}$ is the least cardinal number κ, for which there exists a shattering family of cardinality κ.

One can show that $\mathfrak{H} \leq \mathfrak{h}$ and $\mathfrak{H} \leq \mathfrak{S}$ (cf. [CMW]), (where $\mathfrak{S}$ is the dual-splitting cardinal).

Two cardinals related to the ideal of nowhere dual-Ramsey sets

Let $C \subseteq (\omega)^\omega$ be a set of partitions, then we say that C has the *dual-Ramsey property* or that C is *dual-Ramsey*, if there is a partition $X \in (\omega)^\omega$ such that $(X)^\omega \subseteq C$ or $(X)^\omega \cap C = \emptyset$. If the latter case holds, we also say that C is *dual-Ramsey*$_\circ$. If for each dual Ellentuck neighborhood $(s,Y)^\omega$ there is an $X \in (s,Y)^\omega$ such that $(s,X)^\omega \subseteq C$ or $(s,X)^\omega \cap C = \emptyset$, we call C *completely dual-Ramsey.* If for each dual Ellentuck neighborhood the latter case holds, we say that C is *nowhere dual-Ramsey.*

Remark 1: In [CS] it is proved, that a set is completely dual-Ramsey if and only if it has the Baire property and it is nowhere dual-Ramsey if and only if it is meager with respect to the dual Ellentuck topology. From this it follows, that a set is nowhere dual-Ramsey if and only if the complement contains a dense and open subset (with respect to the dual Ellentuck topology).

Let $\mathfrak{J}$ be set of partitions which are completely dual-Ramsey$_0$. The set $\mathfrak{J} \subseteq \mathcal{P}((\omega)^\omega)$ is an ideal which is not prime. The cardinals $\mathbf{add}(\mathfrak{J})$ and $\mathbf{cov}(\mathfrak{J})$ are two cardinals related to this ideal.
$\mathbf{add}(\mathfrak{J})$ is the smallest cardinal κ such that there exists a family $\mathcal{F} = \{J_\alpha \in \mathfrak{J} : \alpha < \kappa\}$ with $\bigcup \mathcal{F} \notin \mathfrak{J}$.
$\mathbf{cov}(\mathfrak{J})$ is the smallest cardinal κ such that there exists a family $\mathcal{F} = \{J_\alpha \in \mathfrak{J} : \alpha < \kappa\}$ with $\bigcup \mathcal{F} = (\omega)^\omega$.
Because $(\omega)^\omega \notin \mathfrak{J}$, it is clear that $\mathbf{add}(\mathfrak{J}) \leq \mathbf{cov}(\mathfrak{J})$. Further it is easy to see that $\omega_1 \leq \mathbf{add}(\mathfrak{J})$. In the next section we will show that $\mathbf{add}(\mathfrak{J}) = \mathbf{cov}(\mathfrak{J})$.

The distributivity number $\mathbf{d}(\mathfrak{W})$

A complete Boolean algebra $\langle B, \leq \rangle$ is called κ-distributive, where κ is a cardinal, if and only if for every family $\langle u_{\alpha i} : i \in I_\alpha, \alpha < \kappa \rangle$ of members of B the following holds:

$$\prod_{\alpha<\kappa} \sum_{i \in I_\alpha} u_{\alpha i} = \sum_{f \in \prod\limits_{\alpha<\kappa} I_\alpha} \prod_{\alpha<\kappa} u_{\alpha f(\alpha)}.$$

It is well known (cf. [Je2]) that for a forcing notion $\langle P, \leq \rangle$ the following statements are equivalent:

- r.o.(P) is κ-distributive.
- The intersection of κ open dense sets in P is dense.
- Every family of κ maximal anti-chains of P has a common refinement.
- Forcing with P does not add a new subset of κ.

Let $\mathcal{J}$ be the ideal of all finite sets of ω and let $\langle (\omega)^\omega / \mathcal{J}, \leq \rangle =: \mathfrak{W}$ be the partial order defined as follows:

$$p \in \mathfrak{W} \ \Leftrightarrow \ p \in (\omega)^\omega,$$

$$p \leq q \ \Leftrightarrow \ p \sqsubseteq^* q.$$

The distributivity number $\mathbf{d}(\mathfrak{W})$ is defined as the least cardinal κ for which the Boolean algebra r.o.$(\mathfrak{W})$ is not κ-distributive.

The four cardinals are equal

Now we will show, that the four cardinals defined above are all equal. This is a similar result as in the case when we consider infinite subsets of ω instead of infinite partitions (cf. [Pl] and [BPS]).

FACT 2.1 *If $T \subseteq (\omega)^\omega$ is an open and dense set with respect to the dual Ellentuck topology, then it contains a* mao *family.*

PROOF: First choose an almost orthogonal family $\mathcal{A} \subseteq T$ which is maximal in T. Now for an arbitrary $X \in (\omega)^\omega$, $T \cap (X)^\omega \neq \emptyset$. So, X must be compatible with some $A \in \mathcal{A}$ and therefore $\mathcal{A}$ is *mao*. $\dashv$

LEMMA 2.2 $\mathfrak{H} \leq \mathbf{add}(\mathfrak{J})$.

PROOF: Let $\langle S_\alpha : \alpha < \lambda < \mathfrak{H}\rangle$ be a sequence of nowhere dual-Ramsey sets and let $T_\alpha \subseteq (\omega)^\omega \setminus S_\alpha$ $(\alpha < \lambda)$ be such that T_α is open and dense with respect to the dual Ellentuck topology (which is always possible by the Remark 1). For each $\alpha < \lambda$ let

$$T_\alpha^* := \{X \in (\omega)^\omega : \exists Y \in T_\alpha(X \sqsubseteq^* Y \wedge \neg(X \stackrel{*}{=} Y))\}.$$

It is easy to see, that for each $\alpha < \lambda$ the set T_α^* is open and dense with respect to the dual Ellentuck topology.
Let $U_\alpha \subseteq T_\alpha^*$ $(\alpha < \lambda)$ be *mao*. Because $\lambda < \mathfrak{H}$, the set $\langle U_\alpha : \alpha < \lambda\rangle$ can not be shattering. Let for $\alpha < \lambda$ $U_\alpha^* := \{X \in (\omega)^\omega : \exists Z_\alpha \in U_\alpha(X \sqsubseteq^* Z_\alpha)\}$, then $U_\alpha^* \subseteq T_\alpha$ and $\bigcap_{\alpha<\lambda} U_\alpha^*$ is open and dense with respect to the dual Ellentuck topology:
$\bigcap\limits_{\alpha<\lambda} U_\alpha^*$ is open: clear.
$\bigcap\limits_{\alpha<\lambda} U_\alpha^*$ is dense: Let $(s,Z)^\omega$ be arbitrary. Because $\langle U_\alpha : \alpha < \lambda\rangle$ is not shattering, there is a $Y \in (s,Z)^\omega$ such that $\forall \alpha < \lambda \exists X_\alpha \in U_\alpha(Y \sqsubseteq^* X_\alpha)$. Hence, $Y \in \bigcap_{\alpha<\lambda} U_\alpha^*$.
Further we have by construction

$$\bigcap_{\alpha<\lambda} U_\alpha^* \cap \bigcup_{\alpha<\lambda} S_\alpha = \emptyset,$$

which completes the proof. ⊣

LEMMA 2.3 $\mathfrak{H} \leq \mathbf{d}(\mathfrak{W})$.

PROOF: Let $\langle T_\alpha : \alpha < \lambda < \mathfrak{H}\rangle$ be a sequence of open and dense sets with respect to the dual Ellentuck topology. Now the set $\bigcap_{\alpha<\lambda} U_\alpha^*$, constructed as in Lemma 2.2, is dense (and even open) and a subset of $\bigcap_{\alpha<\lambda} T_\alpha$. Therefore $\mathfrak{H} \leq \mathbf{d}(\mathfrak{W})$. ⊣

LEMMA 2.4 $\mathbf{add}(\mathfrak{J}) \leq \mathfrak{H}$.

PROOF: Let $\langle R_\alpha : \alpha < \mathfrak{H}\rangle$ be a shattering family and $P_\alpha := \{X : \exists Y \in R_\alpha(X \sqsubseteq^* Y)\}$.
For each $\alpha < \mathfrak{H}$, P_α is dense and open with respect to the dual Ellentuck topology:
P_α is open: clear.
P_α is dense: Let $(s,Z)^\omega$ be arbitrary and $X \in (s,Z)^\omega$. Because R_α is *mao*, there is a $Y \in R_\alpha$ such that $X' := X \sqcup Y \in (\omega)^\omega$. Now let $X'' \stackrel{*}{=} X'$ such that $X'' \in (s,Z)^\omega$, then $X'' \sqsubseteq^* Y$.
Now we show that $\bigcap_{\alpha<\mathfrak{H}} P_\alpha = \emptyset$ and therefore $\bigcup_{\alpha<\mathfrak{H}}((\omega)^\omega \setminus P_\alpha) = (\omega)^\omega$. Assume there is an $X \in \bigcap_{\alpha<\mathfrak{H}} P_\alpha$, then $\forall \alpha < \mathfrak{H} \exists \mathfrak{Y}_\alpha \in \mathfrak{R}_\alpha(\mathfrak{X} \sqsubseteq^* \mathfrak{Y}_\alpha)$. But this contradicts that $\langle R_\alpha : \alpha < \mathfrak{H}\rangle$ is shattering. ⊣

LEMMA 2.5 $\mathbf{d}(\mathfrak{W}) \leq \mathfrak{H}$.

PROOF: In the proof of Lemma 2.4 we constructed a sequence $\langle P_\alpha : \alpha < \mathfrak{H}\rangle$ of open and dense sets with an empty intersection. Therefore $\bigcap_{\alpha<\mathfrak{H}} P_\alpha$ is not dense. ⊣

COROLLARY 2.6 $\mathbf{cov}(\mathfrak{J}) \leq \mathfrak{H}$.

PROOF: In the proof of Lemma 2.4, in fact we proved that $\mathbf{cov}(\mathfrak{J}) \leq \mathfrak{H}$. ⊣

COROLLARY 2.7 $\mathbf{add}(\mathfrak{J}) = \mathbf{cov}(\mathfrak{J}) = \mathbf{d}(\mathfrak{W}) = \mathfrak{H}$.

PROOF: It is clear that $\mathbf{add}(\mathfrak{J}) \leq \mathbf{cov}(\mathfrak{J})$. By the Lemmas 2.3 and 2.5 we know that $\mathfrak{H} = \mathbf{d}(\mathfrak{W})$. Further by the Lemma 2.2 and the Corollary 2.6 it follows that $\mathfrak{H} \leq \mathbf{add}(\mathfrak{J}) \leq \mathbf{cov}(\mathfrak{J}) \leq \mathfrak{H}$. Hence we have $\mathbf{add}(\mathfrak{J}) = \mathbf{cov}(\mathfrak{J}) = \mathbf{d}(\mathfrak{W}) = \mathfrak{H}$. ⊣

COROLLARY 2.8 *The union of less than $\mathfrak{H}$ completely dual-Ramsey sets is dual-Ramsey, but the union of $\mathfrak{H}$ completely dual-Ramsey sets can be a set, which does not have the dual-Ramsey property.*

PROOF: Follows from Remark 1 and Corollary 2.7. ⊣

On the consistency of $\mathfrak{H} > \omega_1$

First we give some facts concerning the dual-Mathias forcing.

The conditions of dual-Mathias forcing are pairs $\langle s, X\rangle$ such that $s \in (\mathbb{N})$, $X \in (\omega)^\omega$ and $s \sqsubseteq X$, stipulating $\langle s, X\rangle \leq \langle t, Y\rangle$ if and only if $(s,X)^\omega \subseteq (t,Y)^\omega$. It is not hard to see that similar to Mathias forcing, the dual-Mathias forcing can be decomposed as $\mathfrak{W} * \mathrm{P}_{\tilde{\mathfrak{D}}}$, where $\mathfrak{W}$ is defined as above and $\mathrm{P}_{\tilde{\mathfrak{D}}}$ denotes dual-Mathias forcing with conditions only with second coordinate in $\tilde{\mathfrak{D}}$, where $\tilde{\mathfrak{D}}$ is an $\mathfrak{W}$-generic object.
Further, because dual-Mathias forcing has pure decision (cf. [CS]), it is proper and has the Laver property and therefore adds no Cohen reals.

If we make an ω_2-iteration of dual-Mathias forcing with countable support, starting from a model in which the continuum hypothesis holds, we get a model in which the dual-shattering cardinal $\mathfrak{H}$ is equal to ω_2.
Let V be a model of CH and let $P_{\omega_2} := \langle P_\alpha, \dot{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ be a countable support iteration of dual-Mathias forcing, i.e., for all $\alpha < \omega_2$ we have $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is dual-Mathias forcing".
In the sequel we will not distinguish between a member of $\mathfrak{W}$ and its representative. In the proof of the following theorem, a set $C \subseteq \omega_2$ is called ω_1-club if C is unbounded in ω_2 and closed under increasing sequences of length ω_1.

THEOREM 2.9 *If G is P_{ω_2}-generic over V, where $V \models$ CH, then $V[G] \models \mathfrak{H} = \omega_2$.*

Proof: In $V[G]$ let $\langle D_\nu : \nu < \omega_1\rangle$ be a family of open dense subsets of $\mathfrak{W}$. Because dual-Mathias forcing is proper and by a standard Löwenheim-Skolem argument, we find a ω_1-club $C \subseteq \omega_2$ such that for each $\alpha \in C$ and every $\nu < \omega_1$ the set $D_\nu \cap V[G_\alpha]$ belongs to $V[G_\alpha]$ and is open dense in $\mathfrak{W}^{\mathfrak{V}[\mathfrak{G}_\alpha]}$. Let $A \in \mathfrak{W}^{\mathfrak{V}[\mathfrak{G}]}$ be arbitrary. By properness and genericity and because P_{ω_2} has countable support, we may assume that $A \in G(\alpha)'$ for an $\alpha \in C$, where $G(\alpha)'$ is the first component according to the decomposition of Mathias forcing of the $\dot{Q}_\alpha[G_\alpha]$-generic object determined by G. As $\alpha \in C$, $G(\alpha)'$ clearly meets every D_ν $(\nu < \omega_1)$. But now X_α, the $\dot{Q}_\alpha$-generic partition (determined by $G(\alpha)''$) is below each member of $G(\alpha)'$, hence below A and in $\bigcap_{\nu<\omega_1} D_\nu$. Because A was arbitrary, this proves that $\bigcap_{\nu<\omega_1} D_\nu$ is dense in $\mathfrak{W}$ and therefore $\mathbf{d}(\mathfrak{W}) > \omega_1$. Again by properness of dual-Mathias forcing $V[G] \models 2^{\omega_0} = \omega_2$ and we finally have $V[G] \models \mathfrak{H} = \omega_2$. ⊣

In the model constructed in the proof of Theorem 2.9 we have $\mathfrak{H} > \mathfrak{t}$, where $\mathfrak{t}$ is the well-known tower number (for a definition of $\mathfrak{t}$ cf. [vDo]). Moreover, we can show

Corollary 2.10 *The statement $\mathfrak{H} > \mathbf{cov}(\mathcal{M})$ is relatively consistent with* ZFC, *(where $\mathcal{M}$ denotes the ideal of meager sets).*

Proof: Because dual-Mathias forcing is proper and does not add Cohen reals, also forcing with P_{ω_2} does not add Cohen reals. Further it is known that $\mathfrak{t} \leq \mathbf{cov}(\mathcal{M})$ (cf. [PV] or [BJ]). Now because forcing with P_{ω_2} does not add Cohen reals, in $V[G]$ the covering number $\mathbf{cov}(\mathcal{M})$ is still ω_1 (because each real in $V[G]$ is in a meager set with code in V). This completes the proof. ⊣

Remark 2: In [vDo] Theorem 3.1.(c) it is shown that $\omega \leq \kappa < \mathfrak{t}$ implies that $2^\kappa = 2^{\omega_0}$. We do not have a similar result for the dual-shattering cardinal $\mathfrak{H}$. If we start our forcing construction P_{ω_2} with a model $V \models \mathrm{CH} + 2^{\omega_1} = \omega_3$, then (again by properness of dual-Mathias forcing) $V[G] \models \mathfrak{H} = \omega_2 = 2^{\omega_0} < 2^{\omega_1} = \omega_3$ (where G is P_{ω_2}-generic over V).

Remark: Recently Spinas showed in [Sp], that $\mathfrak{H} < \mathfrak{h}$ is consistent with ZFC. But it is still open if MA+¬CH implies that $\omega_1 < \mathfrak{H}$.

3 On the dual-splitting cardinals $\mathfrak{S}$ and $\mathfrak{S}'$

Let X_1, X_2 be two partitions. We say X_1 *splits* X_2 if $X_1 \parallel X_2$ and it exists a partition $Y \sqsubseteq X_2$, such that $X_1 \perp Y$. A family $\mathcal{S} \subseteq (\omega)^\omega$ is called *splitting* if for each non-trivial $X \in (\omega)^\omega$ there exists an $S \in \mathcal{S}$ such that S splits X. The dual-splitting cardinal $\mathfrak{S}$ (resp. $\mathfrak{S}'$) is the least cardinal number κ, for which there exists a splitting family $\mathcal{S} \subseteq (\omega)^\omega$ (resp. $\mathcal{S} \subseteq (\omega)^{\underline{\omega}}$) of cardinality κ.

It is obvious that $\mathfrak{S} \leq \mathfrak{S}'$.

First we compare the dual-splitting number $\mathfrak{S}'$ with the well-known bounding number $\mathfrak{b}$ (a definition of $\mathfrak{b}$ can be found in [vDo]).

Theorem 3.1 $\mathfrak{b} \leq \mathfrak{S}'$.

PROOF: Assume there exists a family $\mathcal{S} = \{S_\iota : \iota < \kappa < \mathfrak{b}\} \subseteq (\omega)^{\underline{\omega}}$ which is splitting. Let $B = \{b_\iota : \iota < \kappa\} \subseteq [\omega]^\omega$ a set of infinite subsets of ω such that $b_\iota \in S_\iota$ (for all $\iota < \kappa$). Let $f_{b_\iota} \in \omega^\omega$ be the (unique) increasing function such that range(f_{b_ι})=b_ι. Because $\kappa < \mathfrak{b}$, the set $\{f_{b_\iota} : \iota < \kappa\}$ is not unbounded. Therefore there exists a function $d \in \omega^\omega$ such that $f_{b_\iota} <^* d$ (for all $\iota < \kappa$). Now with the function d we construct an infinite partition D. First we define an infinite set of pairwise disjoint finite sets p_i $(i \in \omega)$:

$$p_i := [d^i(0), d^{i+1})$$

where d^i denote the i-fold composition of d.

Now the blocks of D are defined as follows:

$$n \text{ is in the } k\text{th block of } D \textit{ iff } n \in p_i \wedge i - \max\{\frac{l}{2}(l+1) < i : l \in \omega\} = k.$$

Because d dominates B, for all $b_\iota \in B$ there exists a natural number m_ι, such that for all $i > m_\iota$: $d^i(0) \leq b_\iota(d^i(0)) < d^{i+1}(i)$ (cf. [vDo] p. 121). So, for all $i > m_\iota$, $p_i \cap b_\iota \neq \emptyset$ and therefore by the construction of the blocks of D, b_ι intersects each block of D. But this implies, that D is not compatible with any element of $\mathcal{S}$ and so $\mathcal{S}$ can not be a splitting family. ⊣

COROLLARY 3.2 *It is consistent with* ZFC, *that* $\mathfrak{s} < \mathfrak{S}'$.

PROOF: Because $\mathfrak{b} \leq \mathfrak{S}'$ is provable in ZFC, it is enough to prove that $\mathfrak{s} < \mathfrak{b}$ is consistent with ZFC, which is proved in [Sh]. ⊣

Now we show that $\mathbf{cov}(\mathcal{M}) \leq \mathfrak{S}$ (where $\mathcal{M}$ denotes the ideal of meager sets). In [CMW] it is shown that if $\kappa < \mathbf{cov}(\mathcal{M})$ and $\{X_\alpha : \alpha < \kappa\} \subseteq (\omega)^\omega$ is a family of partitions, then there exists $Y \in (\omega)^\omega$ such that $Y \perp X_\alpha$ for each $\alpha < \kappa$. This implies the following

COROLLARY 3.3 $\mathbf{cov}(\mathcal{M}) \leq \mathfrak{S}$.

PROOF: Let $S, Y \in (\omega)^\omega$. If $S \perp Y$, then S does not split Y and therefore a family of cardinality less than $\mathbf{cov}(\mathcal{M})$ can not be splitting. ⊣

As a corollary we get again a consistency result.

COROLLARY 3.4 *It is consistent with* ZFC, *that* $\mathfrak{s} < \mathfrak{S}$.

PROOF: If we make an ω_1-iteration of Cohen forcing with finite support starting from a model $V \models \mathbf{cov}(\mathcal{M}) = \omega_2 = \mathfrak{c}$, we get a model in which $\omega_1 = \mathfrak{s} < \mathbf{cov}(\mathcal{M}) = \omega_2 = \mathfrak{c}$ holds. Hence, by Corollary 3.3, this is a model for $\omega_1 = \mathfrak{s} < \mathfrak{S} = \omega_2$. ⊣

Until now we have $\mathbf{cov}(\mathcal{M}), \mathfrak{b} \leq \mathfrak{S}'$, which would be trivial if one could show that $\mathfrak{S}' = \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of $\mathcal{P}(\omega)$. But this is not the case (cf. [CMW]). To construct a model in which $\mathfrak{S}' < \mathfrak{c}$ we will use a modified version of a forcing notion introduced in [CMW].

Let $\mathcal{F}$ be an arbitrary but fixed ultrafilter over ω. Let $\mathbf{Q}$ the notion of forcing defined as follows. The conditions of $\mathbf{Q}$ are pairs $\langle s, A\rangle$ such that $s \in (\mathbb{N})$,

$A \in (\omega)^{<\omega}$, $A(0) \in \mathcal{F}$ and $s \preceq A$, stipulating $\langle s, A\rangle \leq \langle t, B\rangle$ if and only if $t \preceq s$ and $B \sqsubseteq A$. (s is called the stem of the condition.) If $\langle s, A_1\rangle, \langle s, A_2\rangle$ are two $\mathbf{Q}$-conditions, then $\langle s, A_1 \sqcup A_2\rangle \leq \langle s, A_1\rangle, \langle s, A_2\rangle$. Hence, two $\mathbf{Q}$-conditions with the same stem are compatible and because there are only countably many stems, the forcing notion $\mathbf{Q}$ is σ-centered.
Now we will see, that forcing with $\mathbf{Q}$ adds an infinite partition which is compatible with all old infinite partitions but is not contained in any old partition. (So, the forcing notion $\mathbf{Q}$ is in a sense like the dualization of Cohen forcing.)

LEMMA 3.5 *If G is $\mathbf{Q}$-generic over V, then $G \in (\omega)^{\underline{\omega}}$ and $V[G] \models \forall X \in (\omega)^\omega \cap V(G \parallel X \wedge \neg(X \sqsubseteq^* G))$.*

PROOF: Let $X \in V$ be an arbitrary, infinite partition. The set D_n of $\mathbf{Q}$-conditions $\langle s, A\rangle$, such that

(i) $s(0)$ has more than n elements,

(ii) at least n blocks of X are each the union of blocks of A,

(iii) there are at least n different blocks $b_i \in X$, such that $\bigcup b_i \in s \sqcap X$,

is dense in $\mathbf{Q}$ for each $n \in \omega$. Therefore, at least one block of G is infinite (because of (i)), G is compatible with X (because of (ii)) and X is not coarser* than G (because of (iii)). Now, because X was arbitrary, the $\mathbf{Q}$-generic partition G has the desired properties. ⊣

Because the forcing notion $\mathbf{Q}$ is σ-centered and each $\mathbf{Q}$-condition can be encoded by a real number, forcing with $\mathbf{Q}$ does neither collapse any cardinals nor change the cardinality of the continuum and we can prove the following

LEMMA 3.6 *It is consistent with* ZFC *that $\mathfrak{S}' < \mathfrak{c}$.*

PROOF: [CMW] If we make an ω_1-iteration of $\mathbf{Q}$ with finite support, starting from a model in which we have $\mathfrak{c} = \omega_2$, then the ω_1 generic objects form a splitting family. ⊣

Even if a partition does not have a complement, for each non-trivial partition X we can define a non-trivial partition Y, such that $X \perp Y$.
Let $X = \{b_i : i \in \omega\} \in (\omega)^\omega$ and assume that the blocks b_i are ordered by their least element and that each block is ordered by the natural order. A block is called trivial, if it is a singleton. With respect to this ordering define for each non-trivial partition X the partition $X^{\angle}$ as follows.
If $X \in (\omega)^{\underline{\omega}}$ then

n is in the ith block of $X^{\angle}$

iff

n is the ith element of a block of X,

otherwise

$$n, m \text{ are in the same block of } X^{\angle}$$
$$\textit{iff}$$
$$n, m \text{ are both least elements of blocks of } X.$$

It is not hard to see that for each non-trivial $X \in (\omega)^\omega$, $X \perp X^{\angle}$.

A family $\mathcal{W} \subseteq (\omega)^{\underline{\omega}}$ is called *weak splitting*, if for each partition $X \in (\omega)^\omega$, there is a $W \in \mathcal{W}$ such that W splits X or W splits $X^{\angle}$. The cardinal number $w\mathfrak{S}$ is the least cardinal number κ, for which there exists a weak splitting family of cardinality κ. (It is obvious that $w\mathfrak{S} \leq \mathfrak{S}'$.)

A family U is called a *π-base* for a free ultra-filter $\mathcal{F}$ over ω provided for every $x \in \mathcal{F}$ there exists $u \in U$ such that $u \subseteq x$. Define

$$\pi\mathfrak{u} := \min\{|\mathfrak{U}| : \mathfrak{U} \subseteq [\omega]^\omega \text{ is a } \pi\text{-base for a free ultra-filter over } \omega\}.$$

In [BS] it is proved, that $\pi\mathfrak{u} = \mathfrak{r}$ (see also [Va] for more results concerning $\mathfrak{r}$).

Now we can give an upper and a lower bound for the size of $w\mathfrak{S}$.

THEOREM 3.7 *$w\mathfrak{S} \leq \mathfrak{r}$.*

PROOF: We will show that $w\mathfrak{S} \leq \pi\mathfrak{u}$. Let $U := \{u_\iota \in [\omega]^\omega : \iota < \pi\mathfrak{u}\}$ be a π-basis for a free ultra-filter $\mathcal{F}$ over ω. W.l.o.g. we may assume, that all the $u_\iota \in U$ are co-infinite. Let $\mathcal{U} = \{Y_u \in (\omega)^\omega : u \in U \wedge Y_u = \{u_i : u_i = u \vee (u_i = \{n\} \wedge n \notin u)\}\}$. Now we take an arbitrary $X = \{b_i : i \in \omega\} \in (\omega)^\omega$ and define for every $u \in U$ the sets $I_u := \{i : b_i \cap u \neq \emptyset\}$ and $J_u := \{j : b_j \cap u = \emptyset\}$. It is clear that $I_u \cup J_u = \omega$ for every u.

If we find a $u \in U$ such that $|I_u| = |J_u| = \omega$, then Y_u splits X. To see this, define the two infinite partitions

$$Z_1 := \{a_k : a_k = \bigcup_{i \in I_u} b_i \ \vee \ \exists j \in J_u a_k = b_j\}$$

and

$$Z_2 := \{a_k : a_k = \bigcup_{j \in J_u} b_j \ \vee \ \exists i \in I_u a_k = b_i\}.$$

Now we have $X \sqcap Y_u = Z_1$ (therefore $Z_1 \sqsubseteq X, Y_u$) and $Z_2 \sqsubseteq X$ but $Z_2 \perp Y_u$.
(If each block of b_i is finite, then we are always in this case.)

If we find an $x \in \mathcal{F}$ such that $|I_x| < \omega$ (and therefore $|J_x| = \omega$), then we find an $x' \subseteq x$, such that $|I_x| = 1$ and for this $i \in I_x$, $|b_i \setminus x'| = \omega$. (This is because $\mathcal{F}$ is a free ultra-filter.) Now take a $u \in U$ such that $u \subseteq x'$ and we are in the former case for $X^{\angle}$. Therefore, Y_u splits $X^{\angle}$.

If we find an $x \in \mathcal{F}$ such that $|J_x| < \omega$ (and therefore $|I_x| = \omega$), let $I(n)$ be an enumeration of I_x and define $y := x \cap \bigcup_{k \in \omega} b_{I(2k)}$. Then $y \subseteq x$ and $|x \setminus y| = \omega$. Hence, either y or $\omega \setminus y$ is a superset of some $u \in U$. But now $|J_u| = \omega$ and we are in a former case. ⊣

A lower bound for $w\mathfrak{S}$ is $\mathbf{cov}(\mathcal{M})$.

THEOREM 3.8 *$\mathbf{cov}(\mathcal{M}) \leq w\mathfrak{S}$.*

PROOF: Let $\kappa < \mathbf{cov}(\mathcal{M})$ and $\mathcal{W} = \{W_\iota : \iota < \kappa\} \subseteq (\omega)^{\underline{\omega}}$. Assume for each $W_\iota \in \mathcal{W}$ the blocks are ordered by their least element and each block is ordered by the natural order. Further assume that $b_{i(\iota)}$ is the first block of W_ι which is infinite. Now for each $\iota < \kappa$ the set D_ι of functions $f \in \omega^\omega$ such that

$$\begin{aligned}\forall n, m, k \in \omega \quad & \exists h \in \omega t_1 \in b_n, t_2 \in b_m, t_3, t_4 \in b_h \exists s \in b_{i(\iota)} \\ & f(t_1) = f(t_3) \wedge f(t_2) = f(t_4) \wedge |\{s' \leq s : f(s') = f(s)\}| = k + 1.\end{aligned}$$

is the intersection of countably many open dense sets and therefore the complement of a meager set. Because $\kappa < \mathbf{cov}(\mathcal{M})$, we find an unbounded function $g \in \omega^\omega$ such that $g \in \bigcap_{\iota<\kappa} D_\iota$. The partition $G = \{g^{-1}(n) : n \in \omega\} \in (\omega)^{\underline{\omega}}$ is orthogonal with each member of $\mathcal{W}$ and for each $W_\iota \in \mathcal{W}$ and each $k \in \omega$, there exists an $s \in b_{i(\iota)}$, such that s is the kth element of a block of G. Hence, $\mathcal{W}$ can not be a weak splitting family. $\dashv$

4 On the dual-reaping cardinals $\mathfrak{R}$ and $\mathfrak{R}'$

A family $\mathcal{R} \subseteq (\omega)^\omega$ is called *reaping* (resp. *reaping'*), if for each partition $X \in (\omega)^\omega$ (resp. $X \in (\omega)^{\underline{\omega}}$) there exists a partition $R \in \mathcal{R}$ such that $R \perp X$ or $R \sqsubseteq^* X$. The dual-reaping cardinal $\mathfrak{R}$ (resp. $\mathfrak{R}'$) is the least cardinal number κ, for which there exists a reaping (resp. reaping$'$) family of cardinality κ.

It is clear that $\mathfrak{R}' \leq \mathfrak{R}$. Further by finite modifications of the elements of a reaping family, we may replace $\sqsubseteq^*$ by $\sqsubseteq$ in the definition above.

If we cancel in the definition of the reaping number the expression "$R \sqsubseteq^* X$", we get the definition of an orthogonal family.

A family $\mathcal{O} \subseteq (\omega)^\omega$ is called *orthogonal* (resp. *orthogonal'*), if for each non-trivial partition $X \in (\omega)^\omega$ (resp. for each partition $X \in (\omega)^{\underline{\omega}}$) there exists a partition $O \in \mathcal{O}$ such that $O \perp X$. The dual-orthogonal cardinal $\mathfrak{O}$ (resp. $\mathfrak{O}'$) is the least cardinal number κ, for which there exists a orthogonal (resp. orthogonal$'$) family of cardinality κ. (It is obvious that $\mathfrak{O}' \leq \mathfrak{O}$.) Note, that $\mathfrak{o} = \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of $\mathcal{P}(\omega)$ and $\mathfrak{o}$ is defined like $\mathfrak{O}$ but for infinite subsets of ω instead of infinite partitions. (Take the complements of a maximal antichain in $[\omega]^\omega$ of cardinality $\mathfrak{c}$. Because an orthogonal family must avoid all this complements, it has at least the cardinality of this maximal antichain.)

It is also clear that each orthogonal$^{(\prime)}$ family is also a reaping$^{(\prime)}$ family and therefore $\mathfrak{R}^{(\prime)} \leq \mathfrak{O}^{(\prime)}$. Further one can show that $\mathfrak{R}'$ is uncountable (cf. [CMW]). Now we show that $\mathfrak{O}' \leq \mathfrak{d}$, where $\mathfrak{d}$ is the well-known dominating number (for a definition cf. [vDo]), and that $\mathbf{cov}(\mathcal{M}) \leq \mathfrak{O}'$.

LEMMA 4.1 $\mathfrak{O}' \leq \mathfrak{d}$.

PROOF: Let $\{d_\iota \in \omega^\omega : \iota < \mathfrak{d}\}$ be a dominating family. Then it is not hard to see that the family $\{D_\iota : \iota < \kappa\} \subseteq (\omega)^\omega$, where each D_ι is constructed from d_ι like D from d in the proof of Theorem 3.1, is an orthogonal family. $\dashv$

Let $\mathfrak{i}$ be the least cardinality of an independent family (a definition and some results can be found in [Ku]), then

LEMMA 4.2 $\mathfrak{O} \leq \mathfrak{i}$.

PROOF: Let $I \subseteq [\omega]^\omega$ be an independent family of cardinality $\mathfrak{i}$. Let $I' := \{r \in [\omega]^\omega : r \stackrel{*}{=} \bigcap \mathcal{A} \setminus \bigcup \mathcal{B}\}$, where $\mathcal{A}, \mathcal{B} \in [I]^{<\omega}$, $\mathcal{A} \neq \emptyset$, $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $r \stackrel{*}{=} x$ means $|(r \setminus x) \cup (x \setminus r)| < \omega$. It is not hard to see that $|I'| = |I| = \mathfrak{i}$. Now let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ where $\mathcal{I}_1 := \{X_r \in (\omega)^\omega : r \in I' \wedge X_r = \{b_i : b_i = r \vee (b_i = \{n\} \wedge n \notin r)\}\}$ and $\mathcal{I}_2 := \{Y_r : \exists X_r \in \mathcal{I}_1 (Y_r = X_r^{\angle})\}$. We see, that $\mathcal{I} \subseteq (\omega)^\omega$ and $|\mathcal{I}| = \mathfrak{i}$. It leave to show that $\mathcal{I}$ is an orthogonal family.
Let $Z \in (\omega)^\omega$ be arbitrary and let $r := \mathrm{Min}(Z)$. If $r \in I'$, then $X_r \perp Z$ (where $X_r \in \mathcal{I}_1$). And if $r \notin I'$, then there exists an $r' \in I'$ such that $r \cap r' = \emptyset$. But then $Y_{r'} \perp Z$ (where $Y_{r'} \in \mathcal{I}_2$). ⊣

Because $\mathfrak{R} \leq \mathfrak{O}$, the cardinal number $\mathfrak{i}$ is also an upper bound for $\mathfrak{R}$. But for $\mathfrak{R}$, we also find another upper bound.

LEMMA 4.3 $\mathfrak{R} \leq \mathfrak{r}$.

PROOF: Like in Theorem 3.7 we show that $\mathfrak{R} \leq \pi\mathfrak{u}$. Let $U := \{u_\iota \in [\omega]^\omega : \iota < \pi\mathfrak{u}\}$ be as in the proof of Theorem 3.7 and let $\mathcal{U} = \{Y_u \in (\omega)^\omega : u \in U \wedge Y_u = \{u_i : u_i = \omega \setminus u \vee (u_i = \{n\} \wedge n \in u)\}\}$. Take an arbitrary partition $X \in (\omega)^\omega$. Let $r := \mathrm{Min}(X)$ and $r_1 := \{n \in r : \{n\} \in X\}$. If we find a $u \in U$ such that $u \subseteq r_1$, then $Y_u \sqsubseteq X$. Otherwise, we find a $u \in U$ such that either $u \subseteq \omega \setminus r$ or $u \subseteq r \setminus r_1$ and in both cases $Y_u \perp X$. ⊣

Now we will show, that it is consistent with ZFC that $\mathfrak{O}$ can be small. For this we first show, that a Cohen real encode an infinite partition which is orthogonal to each old non-trivial infinite partition. (This result is in fact a corollary of Lemma 5 of [CMW].)

LEMMA 4.4 *If $c \in \omega^\omega$ is a Cohen real over V, then $C := \{c^{-1}(n) : n \in \omega\} \in (\omega)^{\underline{\omega}} \cap V[c]$ and $\forall X \in (\omega)^\omega \cap V(\neg(X \stackrel{*}{=} \{\omega\}) \to C \perp X)$.*

PROOF: We will consider the Cohen-conditions as finite sequences of natural numbers, $s = \{s(i) : i < n < \omega\}$. Let $X = \{b_i : i \in \omega\} \in V$ be an arbitrary, non-trivial infinite partition. The set $D_{n,m}$ of Cohen-conditions s, such that

(i) $|\{i : s(i) = 0\}| \geq n$,

(ii) $\exists k > n \exists i (s(i) = k)$,

(iii) $\exists a_n \in b_n \exists a_m \in b_m \exists l \exists a_1, a_2 \in b_l (s(a_n) = s(a_1) \wedge s(a_m) = s(a_2))$,

is a dense set for each $n, m \in \omega$. Now, because X was arbitrary, the infinite partition C is orthogonal to each infinite partition which is in V. (Note that because of (i), $C \in (\omega)^{\underline{\omega}}$.) ⊣

Now we can show, that $\mathfrak{O}$ can be small.

LEMMA 4.5 *It is consistent with* ZFC *that $\mathfrak{O} < \mathbf{cov}(\mathcal{M})$.*

Proof: If make an ω_1-iteration of Cohen forcing with finite support, starting from a model in which we have $\mathfrak{c} = \omega_2 = \mathbf{cov}(\mathcal{M})$, then the ω_1 generic objects form an orthogonal family. Now because this ω_1-iteration of Cohen forcing does not change the cardinality of $\mathbf{cov}(\mathcal{M})$, we have a model in $\omega_1 = \mathfrak{O} < \mathbf{cov}(\mathcal{M}) = \omega_2$ holds. $\dashv$

Because $\mathfrak{R} \leq \mathfrak{O}$ we also get the relative consistency of $\mathfrak{R} < \mathbf{cov}(\mathcal{M})$. Note that this is not true for $\mathfrak{r}$.

As a lower bound for $\mathfrak{R}'$ we find $\mathfrak{p}$, where $\mathfrak{p}$ is the pseudo-intersection number (a definition of $\mathfrak{p}$ can be found in [vDo]).

Lemma 4.6 $\mathfrak{p} \leq \mathfrak{R}'$.

Proof: In [Be] it is proved that $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$, where

$$\mathfrak{m}_{\sigma\text{-centered}} = \min\{\kappa : \text{“}\mathbf{MA}(\kappa) \text{ for } \sigma\text{-centered posets” fails}\}.$$

Let $\mathcal{R} = \{R_\iota : \iota < \kappa < \mathfrak{p}\}$ be a set of infinite partitions. Now remember that the forcing notion $\mathbf{Q}$ (defined in section 3) is σ-centered and because $\kappa < \mathfrak{p}$ we find an $X \in (\omega)^{\underline{\omega}}$ such that $\mathcal{R}$ does not reap X. $\dashv$

As a corollary we get

Corollary 4.7 *If we assume* $\mathbf{MA}$, *then* $\mathfrak{R}' = \mathfrak{c}$.

Proof: If we assume $\mathbf{MA}$, then $\mathfrak{p} = \mathfrak{c}$. $\dashv$

5 What's about towers and maximal (almost) orthogonal families?

Let κ_{mao} be the least cardinal number κ, for which there exists an infinite *mao* family of cardinality κ. And let κ_{tower} be the least cardinal number κ, for which there exists a family $\mathcal{F} \subseteq (\omega)^\omega$ of cardinality κ, such that $\mathcal{F}$ is well-ordered by $\sqsubseteq^*$ and $\neg\exists Y \in (\omega)^\omega \forall X \in \mathcal{F}(Y \sqsubseteq^* X)$.

Now Krawczyk proved that $\kappa_{mao} = \mathfrak{c}$ (cf. [CMW]) and Carlson proved that $\kappa_{tower} = \omega_1$ (cf. [Ma]). So, these cardinals do not look interesting. But what happens if we cancel the word “almost” in the definition of κ_{mao}?

A family $\mathcal{F} \subseteq (\omega)^\omega$ (resp. $\mathcal{F} \subseteq (\omega)^{\underline{\omega}}$) is a *maximal anti-chain* in $(\omega)^\omega$ (resp. $(\omega)^{\underline{\omega}}$), if $\mathcal{F}$ is a maximal infinite family of pairwise orthogonal partitions. Let κ_A (resp. $\kappa_{A'}$) be the least cardinality of a maximal anti-chain in $(\omega)^\omega$ (resp. $(\omega)^{\underline{\omega}}$).

Note that the corresponding cardinal for infinite subsets of ω would be equal to ω.

First we know that $\mathbf{cov}(\mathcal{M}) \leq \kappa_A, \kappa_{A'}$ (which is proved in [CMW]) and $\mathfrak{b} \leq \kappa_{\mathfrak{A}'}$ (which one can prove like Theorem 3.1). Further it is not hard to see that $\kappa_A \leq \kappa_{A'}$.

But these results concerning κ_A and $\kappa_{A'}$ are also not interesting, because Spinas showed in [Sp] that $\kappa_A = \kappa_{A'} = \mathfrak{c}$.

6 The diagram of the results

Now we summarize the results proved in this article together with other known results.

splitting:

$$
\begin{tikzcd}
\mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{S}' \arrow[rr, dash] \arrow[dd, dash] & & \mathfrak{c} \arrow[ddd, dash] \\
\mathfrak{h} \arrow[d, dash] & & & \\
\mathfrak{H} \arrow[r, dash] \arrow[d, dash] & \mathfrak{S} \arrow[d, dash] & & \\
\omega_1 \arrow[r, dash] & \mathbf{cov}(\mathcal{M}) \arrow[r, dash] & w\mathfrak{S} \arrow[r, dash] & \mathfrak{r}
\end{tikzcd}
$$

reaping:

$$
\begin{tikzcd}
& \mathfrak{d} \arrow[r, dash] \arrow[d, dash] & \mathfrak{i} \arrow[r, dash] \arrow[d, dash] & \mathfrak{c} \arrow[dd, dash] \\
& \mathfrak{O}' \arrow[r, dash] \arrow[d, dash] & \mathfrak{O} \arrow[d, dash] & \\
\mathfrak{p} \arrow[r, dash] & \mathfrak{R}' \arrow[r, dash] & \mathfrak{R} \arrow[r, dash] & \mathfrak{r}
\end{tikzcd}
$$

(In the diagrams, the invariants grow larger, as one moves up or to the right.)

Consistency results:

- $\mathbf{cov}(\mathcal{M}) < \mathfrak{H}$; $\mathfrak{H} < \mathfrak{h}$; $\mathfrak{H} < \mathbf{cov}(\mathcal{M})$ (this is because $\mathfrak{h} < \mathbf{cov}(\mathcal{M})$ is consistent with ZFC)
- $\mathfrak{s} < \mathfrak{S}$; $\mathfrak{S}' < \mathfrak{c}$
- $\mathfrak{O} < \mathbf{cov}(\mathcal{M})$

Note added in proof: Recently, Jörg Brendle informed me that he has proved, that $\mathbf{MA} + \mathfrak{H} < 2^{\aleph_0}$ is consistent with ZFC.

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