# A RESULT IN DUAL RAMSEY THEORY

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Abstract

We present a result which is obtained by combining a result of Carlson with the Finitary Dual Ramsey Theorem of Graham-Rothschild.

We start by introducing some notation.

We conform to the usual practice of identifying the least infinite ordinal $\omega$ with the set of non-negative integers.

Given $\alpha, \beta \leq \omega$, a **partition of $\boldsymbol{\alpha}$ into $\boldsymbol{\beta}$ blocks** is an onto function $X : \alpha \to \beta$ such that $\min\left(X^{-1}(\{n\})\right) < \min\left(X^{-1}(\{m\})\right)$ whenever $n < m < \beta$. Thus, the blocks of $X$ are ordered as their **leaders** (*i.e.*, their least elements).

The **leader function** $\ell : (\alpha)^\beta \times \beta \to \alpha$ is defined by $\ell(X, m) := \min\left(X^{-1}(\{m\})\right)$. Hence, the function $m \mapsto \ell(X, m)$ enumerates the leaders of $X$ in increasing order.

Given $X \in (\alpha)^\beta$ and $Y \in (\alpha)^\gamma$, where $\alpha, \beta, \gamma \leq \omega$, we let $Y \leqslant X$ if $Y$ is **coarser** than $X$, *i.e.*, each block of $Y$ is a union of blocks of $X$.

Given $\alpha, \beta, \gamma \leq \omega$ and $X \in (\alpha)^\beta$, $(X)^\gamma := \{Y \in (\alpha)^\gamma : Y \leqslant X\}$.

Given $\alpha, \beta \leq \omega$ and $k < \omega$, $(\alpha)^\beta_k$ denotes the set of all $X \in (\alpha)^\beta$ such that
 (a) $X^{-1}(\{n\})$ is finite if $k \leq n < \beta$, and
 (b) $\max\left(X^{-1}(\{n\})\right) < \ell(X, n+1)$ if $k \leq n$ and $n + 1 < \beta$.

Given $\alpha, \beta, \gamma \leq \omega$, $X \in (\alpha)^\beta$ and $k, m < \omega$ such that $k \leq \gamma$ and $m \leq \beta$, $(k, m, X)^\gamma$ is the set of all $Y \in (X)^\gamma$ such that

$$\left\{\ell(Y, i) : i < k\right\} \subseteq \left\{\ell(X, j) : j < m\right\}.$$

Note that $(0, m, X)^\gamma = (1, m, X)^\gamma = (X)^\gamma$ for all $m \leq \beta$.

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The **amalgamation function** $\mathcal{A}$ is defined as follows: Given $X \in (\omega)^\omega$ and $t \in (p)^m$, where $0 < m \leq p < \omega$, $\mathcal{A}(t, X)$ is the partition of $\omega$ whose blocks are

$$\bigcup_{i \in t^{-1}(\{0\})} X^{-1}(\{i\}), \ldots \bigcup_{i \in t^{-1}(\{m-1\})} X^{-1}(\{i\}), X^{-1}(\{p\}), X^{-1}(\{p+1\}), \ldots$$

For $t \in (p)^m$, where $m \leq p < \omega$, let $O_t := \{X \in (\omega)^\omega : X \restriction p = t\}$.

We topologize $(\omega)^\omega$ by taking as basic open sets $\emptyset$ and $O_t$ for $t \in \bigcup_{m \leq p < \omega} (p)^m$.

A function $F : (\omega)^\omega \to r$, where $1 \leq r < \omega$, is **clopen** if $F^{-1}(\{i\})$ is a clopen subset of $(\omega)^\omega$ for each $i < r$.

Our starting point is the following immediate consequence of the Dual Ellentuck Theorem (Theorem 4.1 in [1]) of Carlson-Simpson.

Proposition 1. Given $X \in (\omega)^\omega$ and a clopen $F : (\omega)^\omega \to r$, where $1 \leq r < \omega$, there is $Y \in (X)^\omega$ such that $F$ is constant on $(Y)^\omega$.

Even if every block of $X$ is finite, there may not be any homogeneous $Y$ having infinitely many finite blocks.

Proposition 2. There is a clopen $F : (\omega)^\omega \to 2$ with the property that there is no $Y \in (\omega)^\omega$ such that $F$ is constant on $(Y)^\omega$ and $Y$ has infinitely many finite blocks.

*Proof.* Define $F : (\omega)^\omega \to 2$ by stipulating that $F(X) = 0$ if and only if $X^{-1}(\{1\}) \cap \ell(X, 3) \subseteq \ell(X, 2)$. Obviously, $F$ is clopen. Now suppose that there is $Y \in (\omega)^\omega$ such that $Y$ has infinitely many finite blocks and $F$ is constant on $(Y)^\omega$. Pick $Z \in (\omega)^\omega_1$ with $Z \leqslant Y$. Then $F$ is constant on $(Z)^\omega$, which is clearly impossible. $\dashv$

Carlson established a "specialized" version (Theorem 6.9 of [1], which follows from Theorem 2 of [2]) of the Dual Ellentuck Theorem that deals with partitions of $\omega$ having finitely many infinite blocks. Carlson's result immediately implies the following.

Proposition 3. Given $k < \omega$, $X \in (\omega)^\omega_k$ and a clopen $F : (\omega)^\omega \to r$, where $1 \leq r < \omega$, there is $Y \in (\omega)^\omega_k \cap (k, k, X)^\omega$ such that $F$ is constant on $(k, k, Y)^\omega$.

The purpose of this paper is to present the combinatorial result which is obtained by combining Proposition 3 with the Finitary Dual Ramsey theorem of Graham-Rothschild [3]. This last reads as follows.

Proposition 4. Suppose that $1 \leq k \leq m < \omega$ and $1 \leq r < \omega$. Then there is $p < \omega$ such that $p \geq m$ and the following holds: Given $f : (p)^k \to r$, there is $s \in (p)^m$ such that $f$ is constant on $(s)^k$.

We now state our result.

THEOREM. Given $1 < k < m < \omega$, $X \in (\omega)^\omega_k$ and a clopen $F : (\omega)^\omega \to r$, where $1 \leq r < \omega$, there is $Y \in (\omega)^\omega_m \cap (X)^\omega$ such that $F$ is constant on $(k, m, Y)^\omega$.

*Proof.* Using Proposition 4, select $p \geq m$ so that every $f : (p)^k \to r$ is constant on $(s)^k$ for some $s \in (p)^m$. First we define $g : \bigcup_{i \leq p-k} (k-1+i)^{k-1} \to r$ and $Y_0, Y_1, \dots, Y_{p-k}$ so that

(0) $Y_0 \in (\omega)^\omega_k \cap (k, k, X)^\omega$ and $F$ takes the constant value $g(u)$ on $\big(k, k, \mathcal{A}(u, Y_0)\big)^\omega$, where $u$ is the unique element of $(k-1)^{k-1}$ (hence, $\mathcal{A}(u, Y_0) = Y_0$).

(1) $Y_1 \in (\omega)^\omega_{k+1} \cap (k+1, k+1, Y_0)^\omega$ and $F$ takes the constant value $g(t)$ on $\big(k, k, \mathcal{A}(t, Y_1)\big)^\omega$ for every $t \in (k)^{k-1}$.

(2) $Y_2 \in (\omega)^\omega_{k+2} \cap (k+2, k+2, Y_1)^\omega$ and $F$ takes the constant value $g(t)$ on $\big(k, k, \mathcal{A}(t, Y_2)\big)^\omega$ for every $t \in (k+1)^{k-1}$.

$$\vdots$$

$(p-k)$ $Y_{p-k} \in (\omega)^\omega_p \cap (p, p, Y_{p-k-1})^\omega$ and $F$ takes the constant value $g(t)$ on $\big(k, k, \mathcal{A}(t, Y_{p-k})\big)^\omega$ for every $t \in (p-1)^{k-1}$.

For example, to define $Y_3$ and $g \restriction (k+2)^{k-1}$, proceed as follows. Let $t_0, t_1, \dots, t_q$ be an enumeration of the elements of $(k+2)^{k-1}$. Applying Proposition 3 repeatedly, define $T_j$, $Z_j$ and $c_j$ for $j \leq q$ so that

(i) $T_j \in (\omega)^\omega_k$.

(ii) If $j = 0$, $T_j \in \big(k, k, \mathcal{A}(t_j, Y_2)\big)^\omega$ and $Z_j \in (k+3, k+3, Y_2)^\omega$.

(iii) If $j > 0$, $T_j \in \big(k, k, \mathcal{A}(t_j, Z_{j-1})\big)^\omega$ and $Z_j \in (k+3, k+3, Z_{j-1})^\omega$.

(iv) $F$ takes the constant value $c_j$ on $(k, k, T_j)^\omega$.

(v) $\mathcal{A}(t_j, Z_j) = T_j$.

Then set $Y_3 = Z_q$ and $g(t_j) = c_j$ for every $j \leq q$.

Define $f : (p)^k \to r$ by $f(w) = g\big(w \restriction \ell(w, k-1)\big)$. Set $W = Y_{p-k}$. Obviously, $W \in (\omega)^\omega_p \cap (X)^\omega$. Moreover, $F$ takes the constant value $f(w)$ on $\big(k, k, \mathcal{A}(w \restriction \ell(w, k-1), W)\big)^\omega$ for every $w \in (p)^k$. Let $s \in (p)^m$ be such that $f$ is constant on $(s)^k$. Then $Y = \mathcal{A}(s, W)$ is as desired. $\dashv$

The referee pointed out that the theorem and similar results can be derived from Theorem 10 and Theorem 11 of [2].

The theorem is optimal in the following sense:

Proposition 5. Suppose that $1 < k < m < \omega$. Then there is $F : (\omega)^\omega \to 2$ such that $F^{-1}(\{0\})$ is open and there is no $Y \in (\omega)^\omega_m$ with the property that $F$ is constant on $(k, m, Y)^\omega$.

*Proof.* Let $F(Y) = 0$ exactly when $Y(m) \nsubseteq \ell(Y, m+1)$. ⊣

The theorem has the following finitary version, which is proved by arguing as for 3.2 in [1].

Proposition 6. Suppose that $n \leq q \leq m < \omega$, $1 \leq k \leq m$, $n \leq k$ and $1 \leq r < \omega$. Then there is $p < \omega$ such that $p \geq m$ and the following holds: Given $f : (p)^k \to r$, there is $s \in (p)^m_q$ such that $f$ is constant on $(n, q, s)^k$.

*Proof.* Assume that for every $p \geq m$ there is $f_p : (p)^k \to r$ such that for every $s \in (p)^m_q$, $f_p$ is not constant on $(n, q, s)^k$. Define $F : (\omega)^\omega \to r$ by stipulating that $F(T) = f_{\ell(T,k)}\big(T \restriction \ell(T, k)\big)$. Using the theorem (for $1 < n < q$) or Proposition 3 (otherwise), we find $Y \in (\omega)^\omega_q$ such that $F$ is constant on $(n, q, Y)^\omega$. Set $p = \ell(Y, m)$ and $s = Y \restriction m$. Then $p \geq m$ and $s \in (p)^m_q$. Moreover, $f_p$ is constant on $(n, q, s)^k$. Contradiction! ⊣

When $n \in \{0, 1\}$ and $q \in \{m-1, m\}$, Proposition 6 reduces to the Finitary Dual Ramsey Theorem. When $n = k$ and $q \in \{m-1, m\}$, it reduces to the $n$-parameter set theorem of Graham-Rothschild [3], which generalizes the Finitary Dual Ramsey Theorem.

## References


[1] Timothy J. Carlson and Steve G. Simpson: *A dual form of Ramsey's Theorem*, ***Advances in Mathematics***, vol. 53 (1984), 265–290.

[2] Timothy J. Carlson: *Some unifying principles in Ramsey theory*, ***Discrete Mathematics***, vol. 68 (1988), 117–169.

[3] Ronald L. Graham and Bruce L. Rothschild: *Ramsey's theorem for n-parameter sets*, ***Transactions of the American Mathematical Society***, vol. 159 (1971), 257–292.



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