Ultrafilter Spaces on the Semilattice of Partitions

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Abstract

The Stone-Čech compactification of the natural numbers $\beta\omega$ (or equivalently, the space of ultrafilters on the subsets of ω) is a well-studied space with interesting properties. Replacing the subsets of ω by partitions of ω in the construction of the ultrafilter space gives non-homeomorphic spaces of partition ultrafilters corresponding to $\beta\omega$. We develop a general framework for spaces of this type and show that the spaces of partition ultrafilters still have some of the nice properties of $\beta\omega$, even though none of them is homeomorphic to $\beta\omega$. Further, in a particular space, the minimal height of a tree π-base and P-points are investigated.

Keywords: Compactness; Cosubobjects; Partitions; P-Points; Separability; Shattering families; Stone topologies on semilattices; Ultrafilter spaces

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1 Introduction

This paper is a glance at a generalization of the space of ultrafilters over ω, the Stone-Čech compactification of the natural numbers, or just $\beta\omega$. The space $\beta\omega$ and its remainder, the space $\beta\omega \setminus \omega$, are well-studied spaces with a lot of interesting properties. For example both spaces are quasicompact and Hausdorff and therefore, $C(\beta\omega)$ and $C(\beta\omega \setminus \omega)$ are Banach algebras.

We shall provide the reader with a very general approach to ultrafilter spaces on arbitrary semilattices, but the main focus of this paper is a particular class of ultrafilter spaces because of their intimate category theoretical connection with the Stone-Čech compactification: spaces of ultrafilters on the semilattice of partitions.

These objects are indeed the dualization of $\beta\omega$ in the following category theoretical sense:

Look at the category of sets **Set**. As usual for any object $M \in \mathrm{Obj}_{\mathbf{Set}}$ we define the subobjects of M to be the equivalence classes of monos[2] (*i.e.*, in the category of sets, injections) with codomain M. The collection $\mathrm{Sub}_{\mathbf{Set}}(M)$ of subobjects of M is partially ordered by

$$[f] \leq [g] :\iff \exists h : A \to B (f = gh),$$

when $f : A \to M$ and $g : B \to M$.

Indeed, in this case $\mathrm{Sub}_{\mathbf{Set}}(M)$ is a Boolean algebra with greatest element $[\mathrm{id}_M]$ and least element $[o_M]$ where o_M is the unique morphism with codomain M whose domain is the initial object of the category **Set** (the empty set).

The Stone-Čech compactification $\beta\omega$ is the space of ultrafilters in the Boolean algebra $\mathrm{Sub}_{\mathbf{Set}}(\omega)$.

We can now dualize by reversing all occurring arrows: The dualization of

[2] Two monos $f : A \to M$ and $g : B \to M$ are defined to be equivalent iff there is an isomorphism $h : A \to B$ such that $f = gh$.

$\mathrm{Sub}_{\mathbf{Set}}(M)$ is the collection $\mathrm{Cosub}_{\mathbf{Set}}(M)$ of all equivalence classes of epis [3] (*i.e.*, in the category of sets, surjections) with domain M, which we will call the **cosubobjects** of M. Again, this collection is partially ordered by

$$[f] \leq [g] :\iff \exists h : B \to A (f = hg),$$

when $f : M \to A$ and $g : M \to B$.

Moving from the abstract to the concrete, in the category of sets, a cosubobject of ω is just an equivalence class of surjective functions with domain ω modulo permutation of their ranges. This gives us a partition of ω by looking at the preimages of singletons of elements of the range of such functions. Note that following this translation, the relation $\leq$ defined on $\mathrm{Cosub}_{\mathbf{Set}}(M)$ just gives us the "is coarser than" relation on partitions.

Thus $\langle \mathrm{Cosub}_{\mathbf{Set}}(\omega), \leq \rangle$ is again a partially ordered structure and its space of ultrafilters is in this sense the dualization of the Stone-Čech compactification $\beta\omega$.

The important distinction between infinite sets and finite sets in the Stone-Čech compactification that allows to distinguish between principal ultrafilters (*i.e.*, the representants of the countable dense subset ω of $\beta\omega$) and non-principal ultrafilters becomes dualized to the distinction between partitions into infinitely many classes and partitions into finitely many classes. [4]

But the move from subobjects to cosubobjects changes quite a lot: We will see in this paper that $\langle \mathrm{Cosub}_{\mathbf{Set}}(\omega), \leq \rangle$ cannot be a Boolean algebra. An immediate consequence of this lack of a complementation function is that some variations of techniques, that were merely different viewpoints in the case of the Stone-Čech compactification, now actually give different topological spaces. We will show that spaces that are homeomorphic in the classical case fall apart in the dual case, especially two consequences of compactness, countable compactness and the Hausdorff separation property, belong to two different

[3] Two epis $f : M \to A$ and $g : M \to B$ are defined to be equivalent iff there is an isomorphism $h : B \to A$ such that $f = hg$.

[4] Note that the notion of finiteness of a subobject of ω can be categorially expressed by use of the Dedekind formalization of finiteness (using the Axiom of Choice) : $[f]$ is finite if and only if every mono from the domain of f to itself is an isomorphism.

spaces in the dual case — so none of the possible dualization of the Stone-Čech compactification is compact anymore.

This alone should be enough motivation to delve deeper into that subject matter to get more information about these spaces and find the most natural dualization of $\beta\omega$. We close our paper with an extensive list of projects and open problems that result from these non-homeomorphicity results. Many areas of application for the Stone-Čech compactification that are nowadays very well understood deserve to be explored in our dual case.

We will restrict our attention in this paper to partitions which consist solely of infinite blocks. There is no innate category theoretical reason behind this, but we believe that the additional information that the size of a block might convey could add unwanted combinatorial phenomena to the theory of spaces of partition ultrafilters. After all, set-theoretically speaking, the elements of a subset of ω also do not carry an additional information about their size, so we try to avoid hidden information by restricting our attention to partitions with large blocks, and thus receive some sort of homogeneity.

For readers interested in other approaches, we mention this restriction (and a possible lifting of it) in Section 5.

2 Ultrafilter Spaces on Semilattices

In this section we define topologies on the set of ultrafilters on semilattices.

2.1 Semilattices and Partitions

A **semilattice** $\mathbb{L} = \langle L, \preccurlyeq, \mathbf{0} \rangle$ consists of a set L, a least element $\mathbf{0}$ and a partial ordering $\preccurlyeq$ on L such that for all $x, y \in L$ we have the following: There is a $z \in L$ with $z \preccurlyeq x$ and $z \preccurlyeq y$, and for every w with $w \preccurlyeq x$ and $w \preccurlyeq y$ we have $w \preccurlyeq z$ (the **infimum** of x and y) which we denote as usual by $x \wedge y$. Furthermore, for each $x \in L$, the least element $\mathbf{0} \in L$ should satisfy $\mathbf{0} \wedge x = \mathbf{0}$. A semilattice without a least element can easily be supplemented by one.

Furthermore, an element $x \in L$ is called an **atom** if for all $y \preccurlyeq x$ we have either $y = x$ or $y = \mathbf{0}$. A semilattice is said to be **downward splitting** if below each $x \in L$ which is not an atom there are $y_0 \in L$ and $y_1 \in L$ such that $\mathbf{0} \neq y_0 \preccurlyeq x$, $\mathbf{0} \neq y_1 \preccurlyeq x$, and $y_0 \wedge y_1 = \mathbf{0}$.

Let $\mathbb{L} = \langle L, \preccurlyeq \rangle$ be a semilattice. Two elements $x, y \in L$ are called **orthogonal**, and we write $x \perp y$ if $x \wedge y = \mathbf{0}$. Otherwise, they are called **compatible**. If we want to stress the connection between the relations $\perp$ and $\preccurlyeq$, we write $\perp_{\preccurlyeq}$.

A semilattice $\mathbb{L}$ is called **complemented** if there is a function $\sim : L \to L$ satisfying

(C1) $\forall x \in L({\sim}{\sim}x = x)$,
(C2) $\forall x, y \in L(y \wedge x = \mathbf{0} \leftrightarrow y \preccurlyeq {\sim}x)$.

Complemented semilattices are extremely well-behaved: We can define the reverse relation by stipulating $x \succcurlyeq y$ iff $({\sim}x \preccurlyeq {\sim}y)$. Then $\mathbb{L}^{\ddagger} := \langle L, \succcurlyeq \rangle$ is a semilattice with least element ${\sim}\mathbf{0}$ (where $\mathbf{0}$ is the least element of $\mathbb{L}$) and the semilattice $\mathbb{L}^{\ddagger}$ is isomorphic to $\mathbb{L}$ via the map $\sim$.

The semilattice we are mainly interested in is the semilattice of partitions of ω.

A **partition** X (of ω) consisting of pairwise disjoint, non-empty sets such that $\bigcup X = \omega$. The elements of a partition are called the **blocks**.

We elaborated in Section 1 on the possibility of a categorial definition as cosubobjects of ω and why we will only consider partitions of ω all of whose blocks are infinite sets. So, in the following the word "partition" by convention always refers to partitions of ω all of whose blocks are infinite. We also consider finite partitions, this means partitions containing finitely many blocks, and the partition containing only one block is denoted by $\{\omega\}$. The set of all partitions is denoted by (ω), the set of all partitions containing infinitely (resp. finitely) many blocks is denoted by $(\omega)^{\omega}$ (resp. $(\omega)^{<\omega}$).

Let X and Y be two partitions. We say X is **coarser** than Y, or that Y is **finer** than X (and write $X \sqsubseteq Y$) if each block of X is the union of blocks of Y. Let $X \sqcap Y$ denote the finest partition which is coarser than X and Y.

Similarly, $X \sqcup Y$ denotes the coarsest partition which is finer than X and Y.

In the following we investigate the semilattices $(\omega)_{\sqsubseteq} := \langle (\omega), \sqsubseteq \rangle$ and $(\omega)_{\sqsupseteq} := \langle (\omega) \cup \{\mathbf{0}\}, \sqsupseteq \rangle$, and in Section 4 we will investigate $(\omega)^{\omega}_{\sqsubseteq} := \langle (\omega)^{\omega}, \sqsubseteq \rangle$. Notice that the least element $\mathbf{0}$ in $(\omega)_{\sqsubseteq}$ is $\{\omega\} \in (\omega)$, whereas the set $(\omega)_{\sqsupseteq}$ does not have a least element on its own, so we have to add one. [5]

Because they figure prominently in the lattice theoretical description of $\beta\omega$, we also mention the two well-known semilattices $\mathcal{P}(\omega)_{\subseteq} := \langle \mathcal{P}(\omega), \subseteq \rangle$, where $\mathcal{P}(\omega)$ is the power-set of ω, and $[\omega]^{\omega}_{\subseteq} := \langle [\omega]^{\omega}, \subseteq \rangle$, where $[\omega]^{\omega}$ is the set of all infinite subsets of ω. As we noted in Section 1, $\mathcal{P}(\omega)_{\subseteq}$ is not just a semilattice but a Boolean algebra.

2.2 Ultrafilters on semilattices

Let $\mathbb{L} = \langle L, \preccurlyeq, \mathbf{0} \rangle$ be an arbitrary semilattice.

A family $\mathcal{B} \subseteq L$ is called a **filter base on** $\mathbb{L}$ if the following holds: For any $x, y \in \mathcal{B}$ we have $x \wedge y \in \mathcal{B}$, and $\mathbf{0} \notin \mathcal{B}$. If $\mathcal{B}$ is a filter base, we shall call $[\mathcal{B}] := \{y : \exists x \in \mathcal{B}(x \preccurlyeq y)\}$ the **filter generated by** $\mathcal{B}$. A filter base $\mathcal{F}$ is called a **filter** if $[\mathcal{F}] = \mathcal{F}$. A filter $\mathcal{F}$ is called an **ultrafilter** if $\mathcal{F}$ is not properly contained in any other filter on $\mathbb{L}$. A filter $\mathcal{F}$ is **principal** if there is an $x \in L$ such that $\mathcal{F} = [\{x\}]$, otherwise it is called **non-principal**. As easy consequences of the definition of ultrafilters (and, in the case of Fact 2.2, Zorn's Lemma) we get the following facts:

Fact 2.1 $\mathcal{F}$ is an ultrafilter on $\mathbb{L}$ if and only if for any $x \in L$ either $x \in \mathcal{F}$ or there is a $y \in \mathcal{F}$ such that $y \wedge x = \mathbf{0}$.

Fact 2.2 If X is a family of elements of L with the **finite intersection property** (*i.e.*, for any finite subfamily $\{x_0, ..., x_n\} \subseteq X$ we have $x_0 \wedge ... \wedge x_n \neq \mathbf{0}$), then there is an ultrafilter $\mathcal{F}$ on L with $X \subseteq \mathcal{F}$.

Let Ю$(\mathbb{L})$ denote the **set of all ultrafilters on** $\mathbb{L}$. The Cyrillic letter "Ю" for the sound "yu" should remind the reader of the "u" in "ultrafilter". Note

[5] Every partition can be properly refined because all blocks are infinite, so there is no finest partition.

that we make use of the assumption that the semilattice has a least element: Although we could get rid of the mention of $\mathbf{0}$ in the definition of filter by postulating $\mathcal{F} \neq L$ instead of $\mathbf{0} \notin \mathcal{F}$, we cannot prove Fact 2.2 without the least element. To see this, look at an arbitrary linear order $\mathbb{L} = \langle L, \preccurlyeq \rangle$ without least element. Filters are just endsegments of $\mathbb{L}$, but there can be no maximal proper endsegment. Thus, on this semilattice, there is no ultrafilter at all.

2.3 Topologies on $\mathrm{IO}(\mathbb{L})$

We can define topologies on $\mathrm{IO}(\mathbb{L})$ in two different ways:

First define for each $x \in L$ two sets $(x)^+ := \{p \in \mathrm{IO}(\mathbb{L}) : x \in p\}$ and $(x)^- := \{p \in \mathrm{IO}(\mathbb{L}) : x \notin p\} = \mathrm{IO}(\mathbb{L}) \setminus (x)^+$. Set $\mathcal{O}^+ := \{(x)^+ : x \in L\}$ and $\mathcal{O}^- := \{(x)^- : x \in L\}$ and call the topology generated by $\mathcal{O}^+$ the **positive topology** τ^+ and the topology generated by $\mathcal{O}^-$ the **negative topology** τ^- on $\mathrm{IO}(\mathbb{L})$. (Note that $\mathcal{O}^+$ is a base for τ^+, but $\mathcal{O}^-$ is not necessarily a base for τ^-. This difference accounts for some of the asymmetries.)

In the following we shall use the notation $\mathrm{IO}^+(\mathbb{L}) := \langle \mathrm{IO}(\mathbb{L}), \tau^+ \rangle$ and $\mathrm{IO}^-(\mathbb{L}) := \langle \mathrm{IO}(\mathbb{L}), \tau^- \rangle$.

An immediate consequence of Fact 2.1 is that $\tau^- \subseteq \tau^+$, since

$$(x)^- = \bigcup \{(y)^+ : y \wedge x = \mathbf{0}\}.$$

In the case of complemented semilattices, these two topologies coincide: To see this, just note that (C2) implies that ultrafilters contain either x or $\sim x$ for each $x \in L$, and that (C1) implies that $\sim$ is a surjective function. Thus, if $\mathbb{L}$ is complemented, then for each basic open set $O \in \mathcal{O}^+$ there is an open set $\tilde{O} \in \mathcal{O}^-$ such that $O = \tilde{O}$, whence $\mathrm{IO}^+(\mathbb{L}) = \mathrm{IO}^-(\mathbb{L})$.

It is easy to see that $\mathrm{IO}^+(\mathcal{P}(\omega)_\subseteq)$ is just $\beta\omega$ and that $\mathrm{IO}^+([\omega]^\omega_\subseteq)$ is homeomorphic to $\beta\omega \setminus \omega$. Further, since both semilattices are complemented (for $[\omega]^\omega_\subseteq$, just take the complement if it's infinite and $\mathbf{0}$ if the set is cofinite), we get that $\mathrm{IO}^+(\mathcal{P}(\omega)_\subseteq)$ is homeomorphic to each of the spaces $\mathrm{IO}^-(\mathcal{P}(\omega)_\subseteq)$, $\mathrm{IO}^+(\mathcal{P}(\omega)_\supseteq)$ and $\mathrm{IO}^-(\mathcal{P}(\omega)_\supseteq)$, and that $\mathrm{IO}^+([\omega]^\omega_\subseteq)$ is homeomorphic to $\mathrm{IO}^-([\omega]^\omega_\subseteq)$, $\mathrm{IO}^+([\omega]^\omega_\supseteq)$ and $\mathrm{IO}^-([\omega]^\omega_\supseteq)$.

We shall call a topological space **principal** if it contains an open set with just one element. (Proposition 2.3 will explain the choice of the name "principal" for this property.) Being principal is obviously a property preserved under homeomorphisms, so it is a topological invariant.

Proposition 2.3 Let $\mathbb{L}$ be a semilattice which splits downward. Then the following are equivalent:

(i) $\mathrm{IO}^+(\mathbb{L})$ is a principal space, and
(ii) $\mathrm{IO}(\mathbb{L})$ contains a principal ultrafilter.

PROOF. "(ii) $\Rightarrow$ (i)": Let $p := [\{x\}] \in \mathrm{IO}(\mathbb{L})$. Take any $q \in \mathrm{IO}(\mathbb{L})$ with $x \in q$. Then $q \supseteq p$ and hence by maximality of q and p (both are ultrafilters), we have $p = q$. Thus we have $(x)^+ = \{p\}$ and this is our open set with one element.

"(i) $\Rightarrow$ (ii)": Now let $\{p\}$ be an open set with one element. Obviously, such a set must be a basic open set. Let $(x)^+ = \{p\}$.

Case I: If x is an atom then $q := [\{x\}]$ is an ultrafilter and with $x \in p$ we have $q \subseteq p$. Since both p and q are ultrafilters, we have $p = q$. Thus p is principal and we are done.

Case II: If x is not an atom, we can (by the property of downward splitting) pick elements $\mathbf{0} \neq y_0 \preccurlyeq x$ and $\mathbf{0} \neq y_1 \preccurlyeq x$ such that $y_0 \wedge y_1 = \mathbf{0}$. By Fact 2.2, the sets $\{x, y_0\}$ and $\{x, y_1\}$ can be extended to ultrafilters p_1 and p_2. Obviously, both p_1 and p_2 are elements of $(x)^+$ and $p_1 \neq p_2$, contradicting the assumption that $(x)^+$ is a singleton. **q.e.d.**

First of all, note that you can't drop the assumption of downward splitting: Take any dense linear order $\mathbb{L} = \langle L, \preccurlyeq \rangle$ with a least element $\mathbf{0}$. Then $\mathrm{IO}(\mathbb{L})$ contains just one element (the ultrafilter of all elements $x \neq \mathbf{0}$) and this element is non-principal, but the space $\mathrm{IO}^+(\mathbb{L})$ is principal (since it is a point).

The nice characterization of Proposition 2.3 does not work in the case of the negative topologies, since the existence of closed singletons (which would be the analogue of being a principal space for the negative topologies) is provable in general regardless of the existence of principal ultrafilters:

Fact 2.4 For any semilattice $\mathbb{L}$, the spaces $\mathrm{IO}^+(\mathbb{L})$ and $\mathrm{IO}^-(\mathbb{L})$ are T_1 spaces (*i.e.*, all singletons are closed).

PROOF. For any singleton $\{p\}$ look at $\bigcup_{x\notin p}(x)^+$ for the positive topology and $\bigcup_{x\in p}(x)^-$ for the negative topology. A simple argument using the maximality of ultrafilters shows that these sets are just the complement of $\{p\}$. But since they are open in the respective topologies, $\{p\}$ is closed in either topology.
q.e.d.

Later on (in Proposition 3.7 and Proposition 3.12) we shall show that the separation property T_1 is in general as good as it gets: There are examples of semilattices $\mathbb{L}$ with non-Hausdorff spaces $\mathrm{IO}^-(\mathbb{L})$.

For the positive topologies, the property of principality has another application: In a more special case, we can deduce for principal spaces that the set of principal ultrafilters is dense in the positive topology. For this, we shall call a semilattice $\mathbb{L}$ **principally generated** if for each $x \in \mathbb{L}$, where $x \neq \mathbf{0}$, there is a $y \preccurlyeq x$ such that $[\{y\}]$ is a principal ultrafilter on $\mathbb{L}$. Note that if $\mathbb{L}$ is principally generated, then $\mathrm{IO}(\mathbb{L})$ contains principal ultrafilters.

Observation 2.5 If $\mathbb{L}$ is a principally generated semilattice, then the set of principal ultrafilters is dense in $\mathrm{IO}^+(\mathbb{L})$.

PROOF. Let $(x)^+$ be an arbitrary, non-empty basic open set. By the assumption, there is $y \preccurlyeq x$ such that $p := [\{y\}]$ is an ultrafilter. Thus, $p \in (x)^+$ and hence the set of principal ultrafilters intersects any open set. **q.e.d.**

3 The Ultrafilter Spaces on the Set of Partitions

In order to prove the following results we introduce first some notation.

In the following, for an arbitrary set x, let $|x|$ denote the cardinality of x. We always identify a natural number $n \in \omega$ with the set $n = \{m \in \omega : m < n\}$. For $x \subseteq \omega$ let $\min(x) := \bigcap x$. If X is a partition, then $\mathrm{Min}(X) := \{\min(x) : x \in X\}$;

and for $n \in \omega$ and $X \in (\omega)^\omega$, $X(n)$ denotes the unique block $x \in X$ such that $|\min(x) \cap \mathrm{Min}(X)| = n+1$. ($X(n)$ is just the nth block of X in the order of increasing minimal elements.) Finally, a partition is called **trivial** if it contains only one block.

Concerning $\mathrm{IO}((\omega)_\sqsubseteq)$, we like to mention the following

Fact 3.1 If p is an ultrafilter on (ω) and p contains a finite partition, then there is a 2-block partition X such that $p = [\{X\}]$, and hence, p is principal.

PROOF. Let $m := \min\{n : \exists Y \in p(|Y| = n)\}$. This minimum exists by assumption. Let $X \in p$ be such that $|X| = m$.

First we show that for all $Y \in p$ we have $X \sqsubseteq Y$. Suppose this is not the case for some $Y \in p$, then we have $X \neq X \sqcap Y \in p$ (since p is a filter), which implies $|X \sqcap Y| < |X| = m$ and contradicts the definition of m. On the other hand, there is a 2-block partition Z with $Z \sqsubseteq X$, and because $Z \sqsubseteq X$ we get $Z \sqsubseteq Y$ for any $Y \in p$. Therefore, since p is an ultrafilter, we get $Z = X$, which implies $[\{X\}] = p$ and $m = 2$. **q.e.d.**

This leads to the following observations:

Fact 3.2 The space $\mathrm{IO}^+((\omega)_\sqsubseteq)$ is a principal topological space, whereas the space $\mathrm{IO}^+((\omega)_\sqsupseteq)$ is non-principal.

PROOF. That $\mathrm{IO}^+((\omega)_\sqsubseteq)$ is principal follows directly from Fact 3.1 and Proposition 2.3. For the second assertion we note that for every partition $Y \in (\omega)$ we find $Z_1, Z_2 \in (\omega)$ such that $Y \sqsubseteq Z_1$, $Y \sqsubseteq Z_2$ and $Z_1 \sqcup Z_2 = \mathbf{0}$, and therefore, we find $p_1, p_2 \in \mathrm{IO}((\omega)_\sqsupseteq)$ with $Z_1 \in p_1$ and $Z_2 \in p_2$, which implies that p_1 and p_2 both belong to $(Y)^+$. So, for each $Y \in (\omega)$, the set $(Y)^+$ is not a singleton. (In fact, by this argument, $\mathrm{IO}^+((\omega)_\sqsupseteq)$ doesn't have any finite open sets.) **q.e.d.**

As in the space $\beta\omega$, the principal ultrafilters in $\mathrm{IO}((\omega)_\sqsubseteq)$ form a dense set in $\mathrm{IO}^+((\omega)_\sqsubseteq)$ by Observation 2.5, but since there are continuum many 2-block partitions (one for each subset of ω) in $\mathrm{IO}((\omega)_\sqsubseteq)$, they cannot witness that the space $\mathrm{IO}^+((\omega)_\sqsubseteq)$ is separable. Moreover, we get the following

Observation 3.3 The space $\mathrm{IO}^+((\omega)_\sqsubseteq)$ is not separable.

PROOF. Spinas proved in [Sp97] that there is an uncountable set $\{X_\iota : \iota \in I\} \subseteq (\omega)^\omega$ of infinite partitions such that $X_\iota \sqcap X_{\iota'} = \{\omega\}$ whenever $\iota \neq \iota'$. Thus, $(X_\iota)^+ \cap (X_{\iota'})^+ = \emptyset$ (for $\iota \neq \iota'$), which implies that there is no countably dense set in the space $\mathrm{IO}^+((\omega)_\sqsubseteq)$. **q.e.d.**

Proposition 3.4 The space $\mathrm{IO}^+((\omega)_\sqsubseteq)$ is a Hausdorff space.

PROOF. Let p and q be two distinct ultrafilters $\mathrm{IO}((\omega)_\sqsubseteq)$. Because $p \neq q$ and both are maximal filters, we find partitions $X \in p$ and $Y \in q$ such that $X \sqcap Y = \mathbf{0}$. So we get $p \in (X)^+$, $q \in (Y)^+$ and $(X)^+ \cap (Y)^+ = \emptyset$. **q.e.d.**

Before we prove the next proposition, we state the following useful

Lemma 3.5 If $X_0, \ldots, X_n \in (\omega)$ is a finite set of non-trivial partitions, then there is a non-trivial partition $Y \in (\omega)$ such that $Y \perp_\sqsubseteq X_i$ for all $i \leq n$.

PROOF. Let $Z_0 := \mathrm{Min}(X_0)$. If Z_i is such that $Z_i \cap X_{i+1}(k) \neq \emptyset$ for every $k \leq |X_{i+1}|$, then $Z_{i+1} = Z_i$. Otherwise, we define $Z_{i+1} \supseteq Z_i$ as follows: If $Z_i \cap X_{i+1}(k) \neq \emptyset$, then $Z_{i+1} \cap X_{i+1}(k) = Z_i \cap X_{i+1}(k)$; and if $Z_i \cap X_{i+1}(k) = \emptyset$, then $Z_{i+1} \cap X_{i+1}(k) = \min(X_{i+1}(k))$. It is easy to see that $\omega \setminus Z_i$ is infinite for every $i \leq n$. Finally, let $Y = \{Y(0), Y(1)\} \in (\omega)$ be such that $Z_n \subseteq Y(0)$ and by construction we get $Y \perp_\sqsubseteq X_i$ for all $i \leq n$. **q.e.d.**

Proposition 3.6 The space $\mathrm{IO}^+((\omega)_\sqsubseteq)$ is not quasicompact.

PROOF. Let $\mathcal{A} = \{(X)^+ : X \in (\omega)^\omega\}$, then it is easy to see that $\bigcup \mathcal{A} = \text{IO}((\omega)_\sqsubseteq)$. We will show that A is a cover with no finite subcovers. Assume to the contrary that there are finitely many infinite partitions $X_0, \ldots, X_n \in (\omega)^\omega$ such that $(X_0)^+ \cup \ldots \cup (X_n)^+ = \text{IO}((\omega)_\sqsubseteq)$. By Lemma 3.5 we find a $Y \in (\omega)$ such that $Y \perp_\sqsubseteq X_i$ (for all $i \leq n$). Let $p \in \text{IO}((\omega)_\sqsubseteq)$ be such that $Y \in p$, then $X_i \notin p$ (for all $i \leq n$), which contradicts the assumption. **q.e.d.**

3.2 The space $\text{IO}^-((\omega)_\sqsubseteq)$

Proposition 3.7 The space $\text{IO}^-((\omega)_\sqsubseteq)$ is not a Hausdorff space.

PROOF. Let p and q be two distinct ultrafilters in $\text{IO}((\omega)_\sqsubseteq)$. Take any non-trivial partitions $X_0, \ldots, X_k$, $Y_0, \ldots, Y_\ell \in (\omega)$ such that $p \in (X_0)^- \cap \ldots \cap (X_k)^-$ and $q \in (Y_0)^- \cap \ldots \cap (Y_\ell)^-$. Now, by Lemma 3.5, there is a non-trivial partition Z such that $Z \perp_\sqsubseteq X_i$ (for $i \leq k$) and $Z \perp_\sqsubseteq Y_j$ (for $j \leq \ell$), which implies $Z \in \bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq \ell}(Y_j)^-$. Hence, $\bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq \ell}(Y_j)^-$ is not empty. **q.e.d.**

Proposition 3.8 The space $\text{IO}^-((\omega)_\sqsubseteq)$ is countably compact.

PROOF. Let $\mathcal{A} = \{\bigcap A_i : i \in \omega\}$ be such that $\bigcup \mathcal{A} = \bigcup_{i \in \omega}(\bigcap A_i) = \text{IO}((\omega)_\sqsubseteq)$, where each A_i is a finite set of open sets of the form $(X)^-$ for some $X \in (\omega)$. Assume $\bigcup_{i \in I}(\bigcap A_i) \neq \text{IO}((\omega)_\sqsubseteq)$ for every finite set $I \subseteq \omega$. If $A_i = \{(X_0^i)^-, \ldots, (X_n^i)^-\}$ and $A_j = \{(X_0^j)^-, \ldots, (X_m^j)^-\}$ and $\bigcap A_i \cup \bigcap A_j \neq \text{IO}((\omega)_\sqsubseteq)$, then we find a $p \in \text{IO}((\omega)_\sqsubseteq)$ such that $p \in \text{IO}((\omega)_\sqsubseteq) \setminus \bigcap A_i \cup \bigcap A_j$. Hence, there are $k \leq n$ and $\ell \leq m$ such that X_k^i and X_ℓ^j are both in p, which implies $X_k^i \sqcap X_\ell^j \neq \mathbf{0}$. We define a tree T as follows: For $n \in \omega$ the sequence $\langle s_0, \ldots, s_n \rangle$ belongs to T if and only if for every $i \leq n$ there is an $(X_k^i)^- \in A_i$ such that $s_i = X_k^i$ and $(s_0 \sqcap \ldots \sqcap s_n) \neq \mathbf{0}$. The tree T, ordered by inclusion, is by construction (and by our assumption) a tree of height ω and each level of T is finite. Therefore, by König's Lemma, the tree T contains an infinite branch. Let $\langle X^i : i \in \omega \rangle$ be an infinite branch of T, where $X^i \in A_i$. By construction of T, for every finite $I = \{\iota_0, \ldots, \iota_n\} \subseteq \omega$ we have $X^{\iota_0} \sqcap \ldots \sqcap X^{\iota_n} \neq \mathbf{0}$. Thus the partitions constituting the branch have the finite intersection property and therefore we find a $p \in \text{IO}((\omega)_\sqsubseteq)$ such that $X^i \in p$ for every $i \in \omega$. Now, $p \notin \bigcup_{i \in \omega}(X^i)^-$ which implies that $p \notin \bigcup \mathcal{A}$, but this contradicts $\bigcup \mathcal{A} = \text{IO}((\omega)_\sqsubseteq)$. **q.e.d.**

Proposition 3.9 The space $\mathrm{IO}^+((\omega)_{\sqsupseteq})$ is a Hausdorff space.

PROOF. Let p and q be two distinct ultrafilters $\mathrm{IO}((\omega)_{\sqsupseteq})$. Because $p \neq q$ and both are maximal filters, we find partitions $X \in p$ and $Y \in q$ such that $X \sqcup Y = \mathbf{0}$. Hence we get $p \in (X)^+$, $q \in (Y)^+$ and $(X)^+ \cap (Y)^+ = \emptyset$. **q.e.d.**

Before we prove the next proposition, we state the following useful

Lemma 3.10 *If $X_0, \ldots, X_n \in (\omega)^{<\omega}$ is a finite set of non-trivial, finite partitions, then there is a finite partition $Y \in (\omega)^{<\omega}$ such that $Y \perp_{\sqsupseteq} X_i$ for all $i \leq n$.*

PROOF. Define an equivalence relation on ω as follows:

$$s \approx t \ :\Longleftrightarrow\ \forall i, k\Big(s \in X_i(k) \leftrightarrow t \in X_i(k)\Big)$$

Because every partition X_i is finite and we only have finitely many partitions X_i, at least one of the equivalence classes must be infinite, say I. Since each block of each partition X_i is infinite and the partitions have been assumed to be non-trivial, we also must have $\omega \setminus I$ is infinite. Let $I_{-1} := I$ and define $I_{i+1} := I_i \dot\cup \{s_{i+1}\}$ in such a way that for any $t \in I$ we have $s_{i+1} \in X_{i+1}(k) \rightarrow t \notin X_{i+1}(k)$. Let $Y := \{I_n, \omega \setminus I_n\}$, then $Y \in (\omega)$ and for every $i \leq n$, $Y \sqcup X_i$ contains a finite block and therefore, $Y \perp_{\sqsupseteq} X_i$ (for all $i \leq n$). **q.e.d.**

Proposition 3.11 The space $\mathrm{IO}^+((\omega)_{\sqsupseteq})$ is not quasicompact.

PROOF. Let $\mathcal{A} = \{(X)^+ : X \in (\omega)^{<\omega}\}$, then it is easy to see that $\bigcup \mathcal{A} = \mathrm{IO}((\omega)_{\sqsupseteq})$. Assume to the contrary that there are finitely many finite partitions $X_0, \ldots, X_n \in (\omega)^{<\omega}$ such that $(X_0)^+ \cup \ldots \cup (X_n)^+ = \mathrm{IO}((\omega)_{\sqsupseteq})$. By Lemma 3.10 we find a $Y \in (\omega)^{<\omega}$ such that $Y \perp_{\sqsupseteq} X_i$ (for all $i \leq n$). Let $p \in \mathrm{IO}((\omega)_{\sqsupseteq})$ be such that $Y \in p$, then $X_i \notin p$ (for all $i \leq n$), which contradicts the assumption. **q.e.d.**

3.4 The space $\mathrm{IO}^-((\omega)_{\sqsupseteq})$

Proposition 3.12 The space $\mathrm{IO}^-((\omega)_{\sqsupseteq})$ is not a Hausdorff space.

PROOF. We first show that if $p \in (X)^-$ for some $X \in (\omega)^\omega$, then there is an $X' \in (\omega)^{<\omega}$ such that $X' \sqsubseteq X$ (and therefore $(X')^- \subseteq (X)^-$) and $p \in (X')^-$. Since $p \in (X)^-$, there is a $Y \in p$ such that $Y \sqcup X = \mathbf{0}$, which is equivalent to the statement (because we only allowed infinite blocks): There are $y \in Y$ and $x \in X$ such that $x \cap y$ is a non-empty, finite set. Now, for $X' := \{x, \omega \setminus x\}$ we obviously have $X' \sqsubseteq X$ and $p \in (X')^-$.

Let p and q be two distinct ultrafilters in $\mathrm{IO}((\omega)_{\sqsupseteq})$. Take any partitions $X_0, \ldots, X_k$, $Y_0, \ldots, Y_l \in (\omega)$ such that $p \in (X_0)^- \cap \ldots \cap (X_k)^-$ and $q \in (Y_0)^- \cap \ldots \cap (Y_l)^-$. By the fact mentioned above we may assume that the X_i's as well as the Y_i's are finite partitions. Now, by Lemma 3.10, there is a finite partition Z such that $Z \perp_{\sqsupseteq} X_i$ (for $i \leq k$) and $Z \perp_{\sqsupseteq} Y_j$ (for $j \leq l$), which implies $Z \in \bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq l}(Y_j)^-$. Hence, $\bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq l}(Y_j)^-$ is not empty.

q.e.d.

Proposition 3.13 The space $\mathrm{IO}^-((\omega)_{\sqsupseteq})$ is countably compact.

PROOF. Replacing "$\sqcap$" by "$\sqcup$" and "$\sqsubseteq$" by "$\sqsupseteq$", one can simply copy the proof of Proposition 3.8. **q.e.d.**

3.5 Conclusion

Now we are ready to state the main result of this paper.

Theorem 3.14 None of the spaces $\mathrm{IO}^+((\omega)_{\sqsubseteq})$, $\mathrm{IO}^-((\omega)_{\sqsubseteq})$, $\mathrm{IO}^+((\omega)_{\sqsupseteq})$ and $\mathrm{IO}^-((\omega)_{\sqsupseteq})$ is homeomorphic to $\beta\omega$ or $\beta\omega \setminus \omega$. Moreover, no two of the spaces $\beta\omega$, $\beta\omega \setminus \omega$, $\mathrm{IO}^+((\omega)_{\sqsubseteq})$, $\mathrm{IO}^-((\omega)_{\sqsubseteq})$ and $\mathrm{IO}^+((\omega)_{\sqsupseteq})$ are homeomorphic.

PROOF. The proof is given in the following table which is just the compilation of the results from Sections 3 and 2. The separation property T_1 holds for all spaces and thus does not help to discern any two spaces; it is just included for completeness.

	$\beta\omega$	$\beta\omega\setminus\omega$	$\mathbb{O}^+((\omega)_{\sqsubseteq})$	$\mathbb{O}^-((\omega)_{\sqsubseteq})$	$\mathbb{O}^+((\omega)_{\sqsupseteq})$	$\mathbb{O}^-((\omega)_{\sqsupseteq})$
principal	YES	NO	YES		NO	
T_1	YES	YES	YES	YES	YES	YES
Hausdorff	YES	YES	YES	NO	YES	NO
ctb. compact	YES	YES		YES		YES
quasicompact	YES	YES	NO		NO	

q.e.d.

Note that in the language of Section 1, this immediately implies that the partial order $\langle \mathrm{Cosub}_{\mathbf{Set}}(\omega), \leq\rangle$ of cosubobjects of ω is not a Boolean algebra (not even a complemented semilattice) since otherwise we would have $(\omega)_{\sqsubseteq} \cong (\omega)_{\sqsupseteq}$ and hence $\mathbb{O}^+((\omega)_{\sqsubseteq})$ and $\mathbb{O}^+((\omega)_{\sqsupseteq})$ would be homeomorphic.

4 About the space $\mathbb{O}^+((\omega)^{\omega}_{\sqsubseteq})$

To investigate the space $\mathbb{O}^+((\omega)^{\omega}_{\sqsubseteq})$ we first introduce some notations.

For $X \in (\omega)$ and $n \in \omega$ let $X \sqcap \{n\}$ be the partition we get, if we glue all blocks of X together which contain a member of n. If $X, Y \in (\omega)^{\omega}$, then we write $X \sqsubseteq^* Y$ if there is an $n \in \omega$ such that $(X \sqcap \{n\}) \sqsubseteq Y$. For $X, Y \in (\omega)^{\omega}$ it is not hard to see that in the space $\mathbb{O}^+((\omega)^{\omega}_{\sqsubseteq})$ we have $(X)^+ \subseteq (Y)^+$ if and only if $X \sqsubseteq^* Y$.

4.1 The height of tree π-bases of $\mathbb{O}^+((\omega)^{\omega}_{\sqsubseteq})$

We first give the definition of the dual-shattering cardinal $\mathfrak{H}$.

A family $\mathcal{A} \subseteq (\omega)^{\omega}$ is called **maximal orthogonal** (*m.o.*) if $\mathcal{A}$ is a maximal family of pairwise orthogonal partitions. A family $\mathcal{H}$ of *m.o.* families of partitions **shatters** a partition $X \in (\omega)^\omega$, if there are $H \in \mathcal{H}$ and two distinct partitions in H which are both compatible with X. A family of *m.o.* families of partitions is **shattering** if it shatters each member of $(\omega)^\omega$. The dual-shattering cardinal $\mathfrak{H}$ is the least cardinal number κ, for which there is a shattering family of cardinality κ.

The dual-shattering cardinal $\mathfrak{H}$ is a dualization of the well-known shattering cardinal $\mathfrak{h}$ introduced by Balcar, Pelant and Simon in [BaPeSi80] where the letter $\mathfrak{h}$ comes from the word "height". In [BaPeSi80] it is proved that

$$\mathfrak{h} = \min\{\kappa : \text{there is a tree } \pi\text{-base for } \beta\omega \setminus \omega \text{ of } \mathfrak{h}\text{eight } \kappa\}$$

where a family $\mathcal{B}$ of non-empty open sets is called a $\boldsymbol{\pi}$**-base** for a space S provided every non-empty open set contains a member of $\mathcal{B}$, and a **tree** $\boldsymbol{\pi}$**-base** T is a π-base which is a tree when considered as a partially ordered set under reverse inclusion (*i.e.*, for every $t \in T$ the set $\{s \in T : s \supseteq t\}$ is well-ordered by $\supseteq$). The height of an element $t \in T$ is the ordinal α such that $\{s \in T : s \supsetneqq t\}$ is of order type α, and the **height** of a tree T is the smallest ordinal α such that no element of T has height α.

One can show that $\mathfrak{H} \leq \mathfrak{h}$ and $\mathfrak{H} \leq \mathfrak{S}$, where $\mathfrak{S}$ is the dual-splitting cardinal (cf. [CiKrMaWę∞]).

It is consistent with the axioms of set theory (denoted by ZFC) that $\mathfrak{H} = \aleph_2 = 2^{\aleph_0}$ (*cf.* [Ha98]) and also that $\mathfrak{H} = \aleph_1 < \mathfrak{h} = \aleph_2$ (*cf.* [Sp97]). Further it is consistent with $\mathsf{ZFC} + \mathsf{MA} + 2^{\aleph_0} = \aleph_2$ that $\mathfrak{H} = \aleph_1 < \mathfrak{h} = \aleph_2$, where MA denotes Martin's Axiom (*cf.* [Br∞]).

Following Balcar, Pelant and Simon, it is not hard to prove the following

Proposition 4.1 Let $\mathfrak{H}$ be the dual-shattering cardinal defined as above, then

$$\mathfrak{H} = \min\left\{\kappa : \text{there is a tree } \pi\text{-base for } \mathrm{IO}^+((\omega)^\omega_{\sqsubseteq}) \text{ of height } \kappa\right\}.$$

PROOF. Having in mind that for every countable decreasing sequence of basic open sets $(X_0)^+ \supseteq (X_1)^+ \supseteq \ldots \supseteq (X_n)^+ \supseteq \ldots$ there is a basic open set $(Y)^+$ such that for all $i \in \omega$ we have $(Y)^+ \subseteq (X_i)^+$ (*cf.* [Ma86, Prop. 4.2]), one can follow the proof of the Base Matrix Lemma 2.11 of [BaPeSi80]. As a matter of fact we like to mention that every infinite *m.o.* family has the cardinality of the continuum (cf. [CiKrMaWę∞] or [Sp97]). **q.e.d.**

Because the shattering cardinal and the dual-shattering cardinal can be different, this gives us an asymmetry between the two spaces $\beta\omega \setminus \omega$ and $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$.

4.2 On P-points in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$

In this section we give a sketch of the proof that P-points exist under the assumption of the Continuum Hypothesis, and in general both existence and non-existence of P-points are consistent with the axioms of set theory. To do this, we will use the technique of forcing (*cf.* [Je78]).

An ultrafilter $p \in \mathrm{IO}^+(\mathbb{L})$ is a ***P*-point** if the intersection of any family of countably many neighbourhoods of p is a (not necessarily open) neighbourhood of p.

First we show that a P-point in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$ induces in a canonical way a P-point in $\beta\omega \setminus \omega$.

Lemma 4.2 If there is a P-point in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$, then there is a P-point in $\beta\omega \setminus \omega$ as well.

PROOF. Let p be a P-point in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$, then it is not hard to see that the filter generated by $\{\mathrm{Min}(X) : X \in p\}$ is a P-point in $\beta\omega \setminus \omega$. **q.e.d.**

Proposition 4.3 It is consistent with ZFC that there are no P-points in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$.

PROOF. Shelah proved (*cf.* [Sh98, Chapter VI, §4]) that it is consistent with ZFC that there are no P-points in $\beta\omega \setminus \omega$. But in a model of ZFC in which there are no P-points in $\beta\omega \setminus \omega$, there are also no P-points in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$ by Lemma 4.2. **q.e.d.**

Let $\mathbb{U} = \langle (\omega)^\omega, \leq \rangle$ be the partial order defined as follows:

$$X \leq Y \ \Leftrightarrow\ X \sqsubseteq^* Y\,.$$

The forcing notion $\mathbb{U}$ is a natural dualization of $\mathcal{P}(\omega)/\text{fin}$.

Lemma 4.4 If G_p is $\mathbb{U}$-generic over $\mathbf{V}$, then G_p is a P-point in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$ in the model $\mathbf{V}[G_p]$.

PROOF. First notice that the forcing notion $\mathbb{U}$ is σ-closed (*cf.* [Ma86, Proposition 4.2]) and hence, $\mathbb{U}$ does not add new reals. For every countable set of neighbourhoods $\{N_i : i \in \omega\}$ of the filter G_p we find a countable set of partitions $\{X_i : i \in \omega\} \subseteq G_p$ such that $(X_i)^+ \subseteq N_i$ and $X_i \sqsubseteq^* X_j$ for $i \geq j$. Now, since every partition $X \in (\omega)^\omega$ can be encoded by a real number and $\mathbb{U}$ does not add new reals, there is a $\mathbb{U}$-condition Y which forces that the sequence $X_0 \,{}^*\!\sqsupseteq X_1 \,{}^*\!\sqsupseteq \ldots$ belongs to $\mathbf{V}$, and since $\mathbb{U}$ is σ-closed we find an infinite partition $Z \sqsubseteq Y$ such that $Z \sqsubseteq^* X_i$ for every $i \in \omega$. Hence, Z forces that $(Z)^+$ belongs to $\bigcap_{i\in\omega} N_i$ and that Z belongs to G_p. **q.e.d.**

Proposition 4.5 Assume CH, then there is a P-point in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$.

PROOF. Assume $\mathbf{V} \models \mathsf{CH}$. Let χ be large enough such that $\mathcal{P}((\omega)^\omega) \in H(\chi)$, *i.e.*, the power-set of $(\omega)^\omega$ (in $\mathbf{V}$) is hereditarily of size $< \chi$. Let $\mathbf{N}$ be an elementary submodel of $\langle H(\chi), \in\rangle$ containing all the reals of $\mathbf{V}$ such that $|\mathbf{N}| = 2^{\aleph_0}$. We consider the forcing notion $\mathbb{U}$ in the model $\mathbf{N}$. Since $|\mathbf{N}| = 2^{\aleph_0}$, in $\mathbf{V}$ there is an enumeration $\{D_\alpha \subseteq (\omega)^\omega : \alpha < 2^{\aleph_0}\}$ of all dense sets of $\mathbb{U}$ which lie in $\mathbf{N}$. Since $\mathbb{U}$ is σ-closed and because $\mathbf{V} \models \mathsf{CH}$, $\mathbb{U}$ is $2^{\aleph_0}$-closed in $\mathbf{V}$ and therefore we can construct a descending sequence $\{X_\alpha : \alpha < 2^{\aleph_0}\}$ in $\mathbf{V}$ such that $X_\alpha \in D_\alpha$ for each $\alpha < 2^{\aleph_0}$. Let $G_p := \{X \in (\omega)^\omega : X_\alpha \sqsubseteq X \text{ for some } X_\alpha\}$, then G_p is $\mathbb{U}$-generic over $\mathbf{N}$. By Lemma 4.4 we have $\mathbf{N}[G_p] \models$ "there is a P-point in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$" and because $\mathbf{N}$ contains all reals of $\mathbf{V}$ and every countable descending sequence of basic open sets $(Y_i)^+$ can be encoded by a real number, the P-point G_p in the model $\mathbf{N}[G_p]$ is also a P-point in $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$ in the model $\mathbf{V}$, which completes the proof. **q.e.d.**

5 Open Questions

As we already mentioned, we consider our present work more as a teaser. This paper leaves many questions open, and one of our goals is to awaken the interest in delving deeper into this subject matter. The range of questions reaches from deeper inquiries about the spaces explored in this paper to completely different spaces derived by the same general method from semilattices.

We would like to remark that de Groote mentions in [dG00] a non-compactness result similar to Proposition 3.11 for the Stone space of a Hilbert lattice $\mathbb{L}(\mathcal{H})$ (*i.e.*, the lattice of closed subspaces of a given Hilbert space $\mathcal{H}$). This shows that the general framework (topological properties of ultrafilter spaces over a semilattice) is of interest for a broader community.

5.1 Semilattices from Recursion Theory

Many semilattices with an enormous amount of structure results occur in Recursion Theory: We could take the semilattice $\langle \mathcal{D}, \leq_T \rangle$ of Turing degrees (*cf.* [Co93]), the semilattice $\langle \mathcal{H}, \leq_{\Delta^1_1} \rangle$ of hyperdegrees (*cf.* [Hi78]), or any of the multifarious degree structures derived from reducibility relations on the reals. Note that none of these structures can be complemented semilattices: We mentioned that in any complemented semilattice $\langle L, \preccurlyeq \rangle$ the semilattice is homomorphic to $\langle L, \succcurlyeq \rangle$. But for all degree structures derived from reducibility relations we know that the set $\{d : d \preccurlyeq e\}$ is countable for each e, whereas in most cases the sets $\{d : d \succcurlyeq e\}$ are uncountable.

Therefore the investigation of the ultrafilter spaces on these semilattices seems to be promising. In addition to these full degree structures, Recursion Theory has to offer the semilattice of recursively enumerable sets (*cf.* [So87]) and similar structures. For a general overview about known results from Recursion Theory, we refer the reader to [Sl91].

5.2 Other Partition Semilattices

But we don't have to leave the realm of partitions to find new mathematically interesting semilattices. In Section 4, we already started to inquire into the properties of the space $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$. This space can be considered as the dualization of $\beta\omega \backslash \omega$ and its properties are similar to those of the space $\mathrm{IO}^+((\omega)_\sqsupseteq)$. Because $\sqcap$ and $\sqcup$ are not inverse to each other, it is unlikely that the spaces $\mathrm{IO}^+((\omega)^\omega_\sqsubseteq)$ and $\mathrm{IO}^+((\omega)_\sqsupseteq)$ are homeomorphic. Further, a dual-shattering cardinal $\mathfrak{H}^*$ can also be defined in the space $\mathrm{IO}^+((\omega)_\sqsupseteq)$ (*cf.* [CiKrMaWę∞]). What is the relation between $\mathfrak{H}^*$ and $\mathfrak{H}$ and between $\mathfrak{H}^*$ and $\mathfrak{h}$?

We had restricted our attention to partitions which have only infinite blocks. What happens if we consider *all* partitions of ω? Let $(\omega)^\star$ denote the set of all possible partitions of ω. What are the topological properties of the spaces $\mathrm{IO}^+((\omega)^\star_\sqsubseteq)$, $\mathrm{IO}^-((\omega)^\star_\sqsubseteq)$, $\mathrm{IO}^+((\omega)^\star_\sqsupseteq)$ and $\mathrm{IO}^-((\omega)^\star_\sqsupseteq)$? Are they all different and is one of them homeomorphic to $\beta\omega$ or $\beta\omega \setminus \omega$? What is the relation (if there is any) between $\mathrm{IO}^+((\omega)_\sqsubseteq)$ and $\mathrm{IO}^+((\omega)^\star_\sqsubseteq)$ (and likewise for all other pairs)?

We can boldly step forward along the ordinals and look at partitions of larger sets than ω. We could compare spaces of ultrafilters of partitions on ω_1 with our spaces. This comparison is to be seen in connection with the open question whether $\beta\omega \setminus \omega$ can be homeomorphic to $\beta\omega_1 \setminus \omega_1$ (*cf.* [vM90, §5, Problem 1]).

5.3 Deeper Knowledge about the Spaces presently under Investigation

After listing more natural candidates for the underlying semilattice, we now proceed to ask deeper questions about the four spaces scrutinized in this paper.

In this paper our main focus was to show that the spaces are different from each other and from $\beta\omega$ and $\beta\omega \setminus \omega$. One small question is left open by Theorem 3.14, though: Is there any topological property which we could use to distinguish between $\mathrm{IO}^-((\omega)_\sqsubseteq)$ and $\mathrm{IO}^-((\omega)_\sqsupseteq)$? Regardless of the answer to this more technical question, the non-homeomorphicity results are supposed to be a starting point from which one could now move on to derive more properties of the spaces.

At first, one could investigate properties related to the ones that we checked in this paper (*e.g.*, separability or the Lindelöf property). For the noncompact positive spaces $\mathrm{IO}^+((\omega)_\sqsubseteq)$ and $\mathrm{IO}^+((\omega)_\sqsupseteq)$ it is not even obvious that they are compactifiable. To this end, we'd have to show that they are $\mathrm{T}_{3\mathrm{a}}$ (*cf.* [En89, Theorem 3.5.1]). Then, and probably more interesting, one could go back to the original motivation: the spaces of partition ultrafilters were constructed as a dualization of the Stone-Čech compactification of the natural numbers. From this point on, one could try to find similarities and differences between our spaces and the Stone-Čech compactification.

To name but a few:

(i) One of the most important characterizations of the Stone-Čech compactification uses the **extension of mappings** concept:

$\beta\omega$ is the unique space X such that every continuous map from ω to a compact space can be uniquely extended to a continuous map defined on X.

$$\begin{tikzcd} \omega \arrow[r, "\iota"] \arrow[dr, "f"'] & X = \beta\omega \arrow[d, "\hat{f}"] \\ & K \end{tikzcd}$$

Can we show or refute anything analogous for any of our spaces of partition ultrafilters?

This question seems to be connected to the categorial dualization process by which we moved from subsets to partitions in Section 1. It could lead to more inquiries by looking at other categoral characterizations and properties of $\beta\omega$ (*cf.* [Wa74, §10]).

(ii) Building on the previous point, if we have some extension principle for continuous maps, then we could introduce the notion of **ultrafilter types** as usual for $\beta\omega$ (*cf.* [Wa74, 3.41]). What can be said about types of ultrafilters in this sense over partition semilattices? Even if the extension of mappings does not work, a classification of the points of the ultrafilter spaces according to some measure of complexity of the underlying partitions should be possible.

(iii) Deeply connected with ultrafilter types is the **Rudin-Keisler order** of ultrafilters (*cf.* [CoNe74, §9]. In this field we would also expect that independently from the success or failure to get a extension of mappings principle, we should be able to stratify the ultrafilters according to complexity. Note that the Rudin-Keisler order provides us with a possibility of characterizing special ultrafilters on ω. Could there be a topological description of the Ramsey* ultrafilters of [HaLö∞] via a dualized Rudin-Keisler ordering?

(iv) Especially interesting seems the question of autohomeomorphisms of ultrafilter spaces. For $\beta\omega$ we know that under CH there are many autohomeomorphisms ([vM90, Lemma 1.6.1]), but Shelah has shown (*cf.* [vM90, Section 2.6]) that consistently, every autohomeomorphism is induced by a permutation of ω. Results like this become a different feel when we work in ultrafilter spaces which are not separable like $\mathrm{IO}^{+}((\omega)_{\sqsubseteq})$.

(v) Basically, one can take any part of the abundant literature on $\beta\omega$ and try to understand the dualized version.

We feel that we have illustrated how little is known about the fascinating field of ultrafilters on semilattices, and we would like to see more results along these lines.

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