ON A THEOREM OF BANACH AND KURATOWSKI AND *K*-LUSIN SETS

Tomek Bartoszyński

Department of Mathematics, Boise State University
Boise, ID 83725 (U.S.A.)
Email: tomek@diamond.boisestate.edu

Lorenz Halbeisen

Department of Pure Mathematics, Queen's University Belfast
Belfast BT7 1NN (Northern Ireland)
Email: halbeis@qub.ac.uk

Abstract

In a paper of 1929, Banach and Kuratowski proved—assuming the continuum hypothesis—a combinatorial theorem which implies that there is no non-vanishing σ-additive finite measure μ on $\mathbb{R}$ which is defined for every set of reals. It will be shown that the combinatorial theorem is equivalent to the existence of a K-Lusin set of size $2^{\aleph_0}$ and that the existence of such sets is independent of $\mathsf{ZFC} + \neg\mathsf{CH}$.

0 Introduction

In [BK29], Stefan Banach and Kazimierz Kuratowski investigated the following problem in measure theory:

Problem: *Does there exist a non-vanishing finite measure μ on $[0,1]$ defined for every $X \subseteq [0,1]$, which is σ-additive and such that for each $x \in [0,1]$, $\mu(\{x\}) = 0$?*

2000 Mathematics Subject Classification: **03E35** 03E17 03E05
Key-words: combinatorial set theory, continuum hypothesis, Lusin sets, consistency results, cardinal characteristics.
First author partially supported by NSF grant DMS 9971282 and Alexander von Humboldt Foundation.

They showed that such a measure does not exist if one assumes the continuum hypothesis, denoted by CH. More precisely, assuming CH, they proved a combinatorial theorem ([BK29, Théorème II]) and showed that this theorem implies the non-existence of such a measure. The combinatorial result is as follows:

BANACH-KURATOWSKI THEOREM Under the assumption of CH, there is an infinite matrix $A^i_k \subseteq [0,1]$ (where $i, k \in \omega$) such that:

(i) For each $i \in \omega$, $[0,1] = \bigcup_{k\in\omega} A^i_k$.

(ii) For each $i \in \omega$, if $k \neq k'$ then $A^i_k \cap A^i_{k'} = \emptyset$.

(iii) For every sequence $k_0, k_1, \dots, k_i, \dots$ of ω, the set $\bigcap_{i\in\omega}(A^i_0 \cup A^i_1 \cup \ldots \cup A^i_{k_i})$ is at most countable.

In the following we call an infinite matrix $A^i_k \subseteq [0,1]$ (where $i, k \in \omega$) for which (i), (ii) and (iii) hold, a **BK-Matrix**.

Wacław Sierpiński proved—assuming CH—in [Si29] and [Si32] two theorems involving sequences of functions on $[0,1]$, and showed in [Si29] and [Si33] that these two theorems are equivalent to the Banach-Kuratowski Theorem, or equivalently, to the existence of a BK-Matrix.

REMARK: Concerning the problem in measure theory mentioned above, we like to recall the well-known theorem of Stanisław Ulam (*cf.* [Ul30] or [Ox80, Theorem 5.6]), who showed that each σ-additive finite measure μ on ω_1, defined for every set $X \subseteq \omega_1$ with $\mu(\{x\}) = 0$ for each $x \in \omega_1$, vanishes identically. This result implies that if CH holds, then there is no non-vanishing σ-additive finite measure on $[0,1]$.

In the sequel we show that even if CH fails, a BK-Matrix—which will be shown to be equivalent to a K-Lusin set of size $2^{\aleph_0}$—may still exist.

Our set-theoretical terminology (including forcing) is standard and may be found in textbooks like [BJ95], [Je78] and [Ku83].

1 The Banach-Kuratowski Theorem revisited

Before we give a slightly modified version of the Banach-Kuratowski proof of their theorem, we introduce some notation.

For two functions $f, g \in {}^\omega\omega$ let $f \preccurlyeq g$ if and only if for each $n \in \omega$, $f(n) \leq g(n)$.

For $\mathscr{F} \subseteq {}^\omega\omega$, let $\lambda(\mathscr{F})$ denote the least cardinality such that for each $g \in {}^\omega\omega$, the cardinality of $\{f \in \mathscr{F} : f \preccurlyeq g\}$ is strictly less than $\lambda(\mathscr{F})$. If $\mathscr{F} \subseteq {}^\omega\omega$ is a family of size $\mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum, then we obviously have $\aleph_1 \leq \lambda(\mathscr{F}) \leq \mathfrak{c}^+$. This leads to the following definition:

$$\mathfrak{l} := \min\{\lambda(\mathscr{F}) : \mathscr{F} \subseteq {}^\omega\omega \wedge |\mathscr{F}| = \mathfrak{c}\}\,.$$

If one assumes CH, then one can easily construct a family $\mathscr{F} \subseteq {}^\omega\omega$ of cardinality $\mathfrak{c}$ such that $\lambda(\mathscr{F}) = \aleph_1$, and therefore, CH implies that $\mathfrak{l} = \aleph_1$.

The crucial point in the Banach-Kuratowski proof of their theorem is [BK29, Théorème II']. In our notation, it reads as follows:

PROPOSITION 1.1 The existence of a BK-Matrix is equivalent to $\mathfrak{l} = \aleph_1$.

For the sake of completeness and for the reader's convenience, we give the Banach-Kuratowski proof of Proposition 1.1.

PROOF: ($\Leftarrow$) Let $\mathscr{F} \subseteq {}^\omega\omega$ be a family of cardinality $\mathfrak{c}$ with $\lambda(\mathscr{F}) = \aleph_1$. In particular, for each $g \in {}^\omega\omega$, the set $\{f \in \mathscr{F} : f \preccurlyeq g\}$ is at most countable. Let f_α ($\alpha < \mathfrak{c}$) be an enumeration of $\mathscr{F}$. Since the interval $[0,1]$ has cardinality $\mathfrak{c}$, there is a one-to-one function Ξ from $[0,1]$ onto $\mathscr{F}$. For $x \in [0,1]$, let $n_i^x := \Xi(x)(i)$. Now, for $i,k \in \omega$, define the sets $A_k^i \subseteq [0,1]$ as follows:

$$x \in A_k^i \text{ if and only if } k = n_i^x\,.$$

It is easy to see that these sets satisfy the conditions (i) and (ii) of a BK-Matrix. For (iii), take any sequence $k_0, k_1, \dots, k_i, \dots$ of ω and pick an arbitrary $x \in \bigcap_{i\in\omega}(A_0^i \cup A_1^i \cup \ldots \cup A_{k_i}^i)$. By definition, for each $i \in \omega$, x is in $A_0^i \cup A_1^i \cup \ldots \cup A_{k_i}^i$. Hence, for each $i \in \omega$ we get $n_i^x \leq k_i$, which implies that for $g \in {}^\omega\omega$ with $g(i) := k_i$ we have $\Xi(x) \preccurlyeq g$. Now, since $\lambda(\mathscr{F}) = \aleph_1$, $\Xi(x) \in \mathscr{F}$ and x was arbitrary, the set $\{x \in [0,1] : \Xi(x) \preccurlyeq g\} = \bigcap_{i\in\omega}(A_0^i \cup A_1^i \cup \ldots \cup A_{k_i}^i)$ is at most countable.
($\Rightarrow$) Let $A_k^i \subseteq [0,1]$ (where $i,k \in \omega$) be a BK-Matrix and let $\mathscr{F} \subseteq {}^\omega\omega$ be the family of all functions $f \in {}^\omega\omega$ such that $\bigcap_{i\in\omega} A_{f(i)}^i$ is non-empty. Is is easy to see that $\mathscr{F}$ has cardinality $\mathfrak{c}$. Now, for any sequence $k_0, k_1, \dots, k_i, \dots$ of ω, the set $\bigcap_{i\in\omega}(A_0^i \cup A_1^i \cup \ldots \cup A_{k_i}^i)$ is at most countable, which implies that for $g \in {}^\omega\omega$ with $g(i) := k_i$, the set $\{f \in \mathscr{F} : f \preccurlyeq g\}$ is at most countable. Hence, $\lambda(\mathscr{F}) = \aleph_1$. $\dashv$

2 *K*-Lusin sets

In this section we show that $\mathfrak{l} = \aleph_1$ is equivalent to the existence of a K-Lusin set of size $\mathfrak{c}$.

We work in the Polish space ${}^{\omega}\omega$.

Fact 2.1 A closed set $K \subseteq {}^{\omega}\omega$ is compact if and only if there is a function $f \in {}^{\omega}\omega$ such that $K \subseteq \{g \in {}^{\omega}\omega : g \preccurlyeq f\}$.

(See [BJ95, Lemma 1.2.3].)

An uncountable set $X \subseteq {}^{\omega}\omega$ is a **Lusin set**, if for each meager set $M \subseteq {}^{\omega}\omega$, $X \cap M$ is countable.

An uncountable set $X \subseteq {}^{\omega}\omega$ is a ***K*-Lusin set**, if for each compact set $K \subseteq {}^{\omega}\omega$, $X \cap K$ is countable.

Lemma 2.2 Every Lusin set is a K-Lusin set.

Proof: By Fact 2.1, every compact set $K \subseteq {}^{\omega}\omega$ is meager (even nowhere dense), and therefore, every Lusin set is a K-Lusin set. ⊣

Lemma 2.3 The following are equivalent:
(a) $\mathfrak{l} = \aleph_1$.
(b) There is a K-Lusin set of cardinality $\mathfrak{c}$.

Proof: This follows immediately from the definitions and Fact 2.1. ⊣

Remark: Concerning Lusin sets we like to mention that Sierpiński gave in [Si37] a combinatorial result which is equivalent to the existence of a Lusin set of cardinality $\mathfrak{c}$.

For $f, g \in {}^{\omega}\omega$, define $f \preccurlyeq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. The cardinal numbers $\mathfrak{b}$ and $\mathfrak{d}$ are defined as follows:

$$\mathfrak{b} := \min\left\{|\mathscr{F}| : \mathscr{F} \subseteq {}^{\omega}\omega \text{ and } \forall g \in {}^{\omega}\omega \exists f \in \mathscr{F}\ (f \not\preccurlyeq^* g)\right\}$$

$$\mathfrak{d} := \min\left\{|\mathscr{F}| : \mathscr{F} \subseteq {}^{\omega}\omega \text{ and } \forall g \in {}^{\omega}\omega \exists f \in \mathscr{F}\ (g \preccurlyeq^* f)\right\}$$

Lemma 2.4 $\mathfrak{l} = \aleph_1$ implies $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$. Moreover, K-Lusin sets are exactly those (uncountable) subsets of ${}^{\omega}\omega$ whose all uncountable subsets are unbounded. (Families like that are also called *strongly unbounded* and they play an important role in preserving unbounded families in iterations, see *e.g.* [BJ95] for details.)

Proof: Assume $\mathfrak{l} = \aleph_1$, then, by Lemma 2.3, there exists a K-Lusin set $X \subseteq {}^{\omega}\omega$ of cardinality $\mathfrak{c}$. It is easy to see that every uncountable subset of X is unbounded, so, $\mathfrak{b} = \aleph_1$. On the other hand, every function $g \in {}^\omega\omega$ dominates only countably many elements of X. Hence, no family $\mathscr{F} \subseteq {}^\omega\omega$ of cardinality strictly less than $\mathfrak{c}$ can dominate all elements of X, and thus, $\mathfrak{d} = \mathfrak{c}$. ⊣

PROPOSITION 2.5 Adding κ many Cohen reals produces a Lusin set of size κ.

(See [BJ95, Lemma 8.2.6].)

THEOREM 2.6 The existence of a K-Lusin set of cardinality $\mathfrak{c}$ is independent of $\mathsf{ZFC} + \neg\mathsf{CH}$.

PROOF: By Proposition 2.5 and Lemma 2.2 it is consistent with ZFC that there is a K-Lusin set of cardinality $\mathfrak{c}$.
On the other hand, it is consistent with ZFC that $\mathfrak{b} > \aleph_1$ or that $\mathfrak{d} < \mathfrak{c}$ (*cf.* [BJ95]). Therefore, by Lemma 2.4, it is consistent with ZFC that there are no K-Lusin sets of cardinality $\mathfrak{c}$. ⊣

By Lemma 2.3 and Proposition 1.1, as a immediate consequence of Theorem 2.6 we get the following.

COROLLARY 2.7 The existence of a BK-Matrix is independent of $\mathsf{ZFC} + \neg\mathsf{CH}$.

3 Odds and Ends

An uncountable set $X \subseteq [0,1]$ is a **Sierpiński set**, if for each measure zero set $N \subseteq [0,1]$, $X \cap N$ is countable.

PROPOSITION 3.1 The following are equivalent:

(a) CH.
(b) There is a Lusin set of cardinality $\mathfrak{c}$ and an uncountable Sierpiński set.
(c) There is a Sierpiński set of cardinality $\mathfrak{c}$ and an uncountable Lusin set.

(See [Ro38, p. 217].)

PROPOSITION 3.2 It is consistent with ZFC that there exists a K-Lusin set of cardinality $\mathfrak{c}$, but there are neither Lusin nor Sierpiński sets.

PROOF: Let $\mathbb{M}_{\omega_2}$ denote the ω_2-iteration of Miller forcing—also called "rational perfect set forcing"—with countable support. Let us start with a model V in which

CH holds and let $G_{\omega_2} = \langle m_\iota : \iota < \omega_2 \rangle$ be the corresponding generic sequence of Miller reals. Then, in $V[G_{\omega_2}]$, G_{ω_2} is a K-Lusin set of cardinality $\mathfrak{c} = \aleph_2$. For this, we have to show the following property:

$$\textit{For all } f \in {}^\omega\omega \cap V[G_{\omega_2}], \textit{ the set } \{\iota : m_\iota \preccurlyeq f\} \textit{ is countable.}$$

Suppose not and let $f \in {}^\omega\omega \cap V[G_{\omega_2}]$ be a witness. Further, let p be an $\mathbb{M}_{\omega_2}$-condition such that

$$p \Vdash_{\mathbb{M}_{\omega_2}} \text{“}\textit{for some } n_0 \in \omega, \textit{ the set } \big\{\iota : \forall k \geq n_0 \big(m_\iota(k) < \dot{f}(k)\big)\big\} \textit{ is uncountable}\text{”}.$$

We can assume that these dominated reals m_ι are among $\{m_\alpha : \alpha < \beta < \omega_2\}$ and that β is minimal. This way, f is added after step β of the iteration. Let $a^* := \mathrm{cl}(\dot{f})$ be the (countable) set of ordinals such that if we know $\{m_\iota : \iota \in a\}$, then we can compute $\dot{f}$. (Notice that a^* is much more than just the support of $\dot{f}$, since it contains also supports of all conditions that are involved in conditions involved in $\dot{f}$, and so on.) Let N be a countable model such that $p, \dot{f} \in N$, $a^* \subseteq N$ and let $\mathbb{M}_{a^*}$ be the iteration of Miller forcing, where we put the empty forcing at stages $\alpha \notin a^*$ (essentially, $\mathbb{M}_{a^*}$ is the same as $\mathbb{M}_{\mathrm{o.t.}(a^*)}$).
The crucial lemma—which is done in [SS00, Lemma 3.1] for Mathias forcing, but also works for Miller forcing—is the following: If $N \models p \in \mathbb{M}_{a^*}$, then there exists a $q \in \mathbb{M}_{\omega_2}$ which is stronger than p such that $\mathrm{cl}(q) = a^*$ and q is $(N, \mathbb{M}_{a^*})$-generic over N. In particular, if $\{m_\iota : \iota < \omega_2\}$ is a generic sequence of Miller reals consistent with q, then $\{m_\iota : \iota \in a^*\}$ is $\mathbb{M}_{a^*}$-generic over N (consistent with p).
So, fix such a q. Now we claim that for $\gamma \in \beta \setminus N$, q forces that $\dot{f}(k) > m_\gamma(k)$ for some $k \geq n_0$: Take any $\gamma \in \beta \setminus N$ and let q^* be a condition stronger than q. Let $q_1^* = q_1|\beta$, and let $q_2^* = q^*|a$. Without loss of generality, we may assume that $q_2^* = q$. Now, first we strengthen q_1^* to determine the length of stem of $q_1^*(\gamma)$, and make it equal to some $k > n_0$. Next we shrink q_2^* to determine the first k digits of $\dot{f}$. Finally, we shrink $q_1^*(\gamma)$ such that $q_1^*(\gamma)(k) > \dot{f}(k)$. Why we can do this? Although f is added after m_γ, from the point of view of model N, it was added before. So, working below condition q_2^* (in $\mathbb{M}_{a^*}$) we can compute as many digits of $\dot{f}$ as we want without making any commitments on m_γ, and vice versa. Even though the computation is in N, it is absolute. This completes the first part of the proof.
On the other hand, it is known (*cf.* [JS94]) that in $V[G_{\omega_2}]$, there are neither Lusin nor Sierpiński sets of any uncountable size, which completes the proof. $\dashv$

PROPOSITION 3.3 It is consistent with ZFC that $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$, but there is no K-Lusin set of cardinality $\mathfrak{c}$.

PROOF: Take a model M in which we have $\mathfrak{c} = \aleph_2$ and in which Martin's Axiom MA holds. Let $G = \langle c_\beta : \beta < \omega_1 \rangle$ be a generic sequence of Cohen reals of length

ω_1. In the resulting model $M[G]$ we have $\mathfrak{b} = \aleph_1$ (since the set of Cohen reals forms an unbounded family) and $\mathfrak{d} = \aleph_2$. On the other hand, there is no K-Lusin set of cardinality $\mathfrak{c}$ in $M[G]$. Why? Suppose $X \subseteq {}^\omega\omega$ has cardinality $\aleph_2$. Take a countable ordinal α and a subset $X' \subseteq X$ of cardinality $\aleph_2$ such that $X' \subseteq M[G_\alpha]$, where $G_\alpha := \langle c_\beta : \beta \leq \alpha \rangle$. Now, $M[G_\alpha] = M[c]$ (for some Cohen real c) and $M[c] \models \mathsf{MA}(\sigma\text{-}centered)$, (*cf.* [Ro79] or [BJ95, Theorem 3.3.8]). In particular, since $\mathsf{MA}(\sigma\text{-}centered)$ implies $\mathfrak{p} = \mathfrak{c}$ and $\mathfrak{p} \leq \mathfrak{b}$, we have $M[c] \models \mathfrak{b} = \aleph_2$. Thus, there is a function which bounds uncountably many elements of X'. Hence, by Lemma 2.4, X cannot be a K-Lusin set. $\dashv$

Let Q be a countable dense subset of the interval $[0,1]$. Then $X \subseteq [0,1]$ is **concentrated on** Q, if every open set of $[0,1]$ containing Q, contains all but countably many elements of X.

PROPOSITION 3.4 The following are equivalent:

(a) There exists a K-Lusin set of cardinality $\mathfrak{c}$.
(b) There exists a concentrated set of cardinality $\mathfrak{c}$.

PROOF: (b)$\rightarrow$(a) Let Q be a countable dense set in $[0,1]$ and let $\varphi : [0,1] \setminus Q \rightarrow {}^\omega\omega$ be a homeomorphism. If $U \subseteq {}^\omega\omega$ is compact, then $\varphi^{-1}[U]$ is compact, so closed in $[0,1]$, and $[0,1] \setminus \varphi^{-1}[U]$ is an open set containing Q. Hence, the image under φ of an uncountable set $X \subseteq [0,1]$ concentrated on Q is a K-Lusin set of the same cardinality as X.
(a)$\rightarrow$(b) The preimage under φ of a K-Lusin set of cardinality $\mathfrak{c}$ is a set concentrated on Q of the same cardinality.
$\dashv$

REMARK: A Lusin set is concentrated on every countable dense set, and concentrated sets have always strong measure zero. However, the existence of a strong measure zero set of size $\mathfrak{c}$ does not imply the existence of a concentrated sets of size $\mathfrak{c}$. In fact, the existence of a strong measure zero set of size $\mathfrak{c}$ is consistent with $\mathfrak{d} = \aleph_1$ (see [BS∞]).

References

[BK29] STEFAN BANACH AND KAZIMIERZ KURATOWSKI: Sur une généralisation du problème de la mesure. *Fundamenta Mathematicae* **14** (1929), 127–131.

[BJ95] TOMEK BARTOSZYŃSKI, HAIM JUDAH: "Set Theory: on the structure of the real line." A. K. Peters, Wellesley, Massachusetts (1995).

[BS∞] TOMEK BARTOSZYŃSKI AND SAHARON SHELAH: Strongly meager and strong measure zero sets, to appear in *Archive for Mathematical Logic.*

[Je78] THOMAS JECH: "Set Theory." Academic Press [Pure and Applied Mathematics], London (1978).

[JS94] HAIM JUDAH AND SAHARON SHELAH: Killing Luzin and Sierpiński sets. *Proceedings of the American Mathematical Society* **120** (1994), 917–920.

[Ku83] KENNETH KUNEN: "Set Theory, an Introduction to Independence Proofs." North Holland [Studies in Logic and the Foundations of Mathematics], Amsterdam (1983).

[Ox80] JOHN C. OXTOBY: "Measure and Category." (2nd Edition) Springer-Verlag [Graduate Texts in Mathematics], New York (1980).

[Ro79] JUDY ROITMAN: Adding a random or Cohen real: topological consequences and the effect on Martin's axiom. *Fundamenta Mathematicae* **103** (1979), 47–60.

[Ro38] FRITZ ROTHBERGER: Eine Äquivalenz zwischen der Kontinuumshypothese und der Existenz der Lusinschen und Sierpińskischen Mengen. *Fundamenta Mathematicae* **30** (1938), 215–217.

[SS00] SAHARON SHELAH AND OTMAR SPINAS: The distributivity numbers of $\mathcal{P}(\omega)/\text{fin}$ and its square. *Transactions of the American Mathematical Society* **352** (2000), 2023–2047.

[Si29] WACŁAW SIERPIŃSKI: Sur un théorème de MM. Banach et Kuratowski. *Fundamenta Mathematicae* **14** (1929), 277–280.

[Si32] WACŁAW SIERPIŃSKI: Remarques sur un théorème de M. Fréchet. *Monatshefte für Mathematik und Physik* **39** (1932), 233–238.

[Si33] WACŁAW SIERPIŃSKI: Sur l'équivalence de deux conséquences de l'hypothèse du continu. *Studia Mathematica* **4** (1933), 15–20.

[Si37] WACŁAW SIERPIŃSKI: Le théorème de M. Lusin comme une proposition de la théorie générale des ensembles. *Fundamenta Mathematicae* **29** (1937), 182–190.

[Ul30] STANISŁAW ULAM: Zur Masstheorie in der allgemeinen Mengenlehre. *Fundamenta Mathematicae* **16** (1930), 140–150.