ON ASYMPTOTIC MODELS IN BANACH SPACES

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ABSTRACT

A well known application of Ramsey's Theorem to Banach Space Theory is the notion of a spreading model $(\tilde{e}_i)$ of a normalized basic sequence (x_i) in a Banach space X. We show how to generalize the construction to define a new creature (e_i), which we call an asymptotic model of X. Every spreading model of X is an asymptotic model of X and in most settings, such as if X is reflexive, every normalized block basis of an asymptotic model is itself an asymptotic model. We also show how to use the Hindman-Milliken Theorem—a strengthened form of Ramsey's Theorem—to generate asymptotic models with a stronger form of convergence.

1. Introduction

Ramsey Theory, and especially Ramsey's Theorem, is a very powerful tool in infinitary combinatorics and has many interesting (and

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sometimes unexpected) applications in various fields of Mathematics. Generally speaking, theorems in Ramsey Theory are of the type that a function into a finite set can be restricted to some sort of infinite substructure, on which it is constant. In applications to analysis we successively apply Ramsey's Theorem to certain ε-nets to obtain infinite substructures on which certain Lipschitz functions are nearly constant in an asymptotic sense (cf. e.g., [Od80] or [HKO, Part III]).

A well known application of Ramsey's Theorem ([Ra29, Theorem A]) to Banach Space Theory is due to A. Brunel and L. Sucheston (cf. [BS73]). Roughly speaking, it says that every normalized basic sequence in a Banach space has a subsequence which is "asymptotically" subsymmetric, ultimately yielding a spreading model.

There are two main directions to generalize Ramsey's Theorem. One is in terms of partitions and another one leads to the so-called Ramsey property. (Some results concerning the symmetries between the combination of these two directions can be found in [Ha98].) Both directions are already used in Banach Space Theory. For example the fact that Borel sets have the Ramsey property is used in Farahat's proof of Rosenthal's Theorem, which says that a normalized sequence has a subsequence which is either equivalent to the unit vector basis of ℓ_1 or is weakly Cauchy. Further, a combination of both directions is used by W.T. Gowers in the proof of his famous Dichotomy Theorem.

In the sequel, we prove a generalized version of the Brunel-Sucheston Theorem by using Ramsey's Theorem. We apply this to basic arrays, namely certain sequences of basic sequences in X. Also we show how a generalization of Ramsey's Theorem, the Hindman-Milliken Theorem, can be used to construct asymptotic models with a stronger form of convergence.

The object we obtain, a basis $(e_i)_{i\in\omega}$ for some infinite dimensional Banach space E, we call an asymptotic model of X. Asymptotic models include not only all spreading models of X, and even in many cases all normalized block bases of such, but more general sequences as well. If the sequences in the generating basic array are all block bases of a fixed basis or are all weakly null then the notion lies somewhere between that of spreading models and asymptotic structure (see [MMT95]), although it is closer in flavor to the theory of spreading models. The construction we use to get an asymptotic model has been used in the past by several authors to study spreading models and the behavior of sequences over X (e.g., [Ro83], [Ma83] and [AOST]). In particular in [Ro83] the concept of an ∞-type over a Banach space is introduced and this actually contains within it the notion of an asymptotic model. But our more restricted viewpoint in this paper is the first study of what we have chosen to call "asymptotic models" themselves.

In Section 3 we recall the Hindman-Milliken Theorem. In Section 4 we define and construct asymptotic models. In addition we make a number of observations about asymptotic models and their relation with spreading models and asymptotic structure. Section 5 generalizes some results of [$\mathrm{OS98}_2$] to the setting of asymptotic models. Section 6 concerns some stronger versions one might hope to have, but as we show one cannot achieve in general. In this section we also raise some open problems.

For the reader's convenience, we recall some set theoretic terminology we will use frequently. A natural number n is considered as the set of all natural numbers less than n, in particular, $0 = \emptyset$. Let $\omega = \{0, 1, 2, \ldots\}$ denote the set of all natural numbers. By the way, we always start counting by 0. Some more set theoretic terminology will be introduced in the following section.

The notation concerning sequence spaces is standard and can be found in textbooks like [Di84], [Gu92] and [LT77]. However, for the sake of the non-expert, we recall some definitions.

A sequence $(x_i)_{i\in\omega}$ in a normed space is **normalized** if for all $i \in \omega$, $\|x_i\| = 1$, and it is **seminormalized** if there exists an M with $0 < M < \infty$ such that all $i \in \omega$, $\frac{1}{M} \leq \|x_i\| \leq M$. If $(x_i)_{i\in\omega}$ is a sequence of non-zero vectors in a Banach space X, then $(x_i)_{i\in\omega}$ is **basic** iff there exists $C < \infty$ so that for all $n < m$ and $(a_i)_{i\in m} \subseteq \mathbb{R}$, $\|\sum_{i\in n} a_i x_i\| \leq C\|\sum_{i\in m} a_i x_i\|$. The smallest such C is called the **basis constant** of $(x_i)_{i\in\omega}$ and $(x_i)_{i\in\omega}$ is then called C-**basic**. The basic sequence $(x_i)_{i\in\omega}$ is **monotone basic** if it is 1-basic, and it is **bimonotone** if it is monotone and the tail projections are monotone as well (i.e.,

$I - P_n$ has norm one if P_n is the n^{th} initial projection). If $(x_i)_{i\in\omega}$ is basic, then every x in the closed linear span of $(x_i)_{i\in\omega}$ can be uniquely expressed as $\sum_{i\in\omega} a_i x_i$ for some $(a_i)_{i\in\omega} \subseteq \mathbb{R}$. Basic sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are C-**equivalent** if there exist constants A and B with $AB \leq C$ so that for all $n \in \omega$ and scalars $(a_i)_{i\in n}$

$$A^{-1}\Big\| \sum_{i\in n} a_i x_i \Big\| \leq \Big\| \sum_{i\in n} a_i y_i \Big\| \leq B \Big\| \sum_{i\in n} a_i x_i \Big\| \ .$$

For a basic sequence $(x_i)_{i\in\omega}$ and scalars $(b_l)_{l\in\omega}$, a sequence of non-zero vectors $(y_j)_{j\in\omega}$ of the form

$$y_j = \sum_{l=p_k}^{p_{k+1}-1} b_l x_l \ ,$$

where $p_0 < p_1 < \ldots < p_k < \ldots$ is an increasing sequence of natural numbers, is called a **block basic sequence** or just a **block basis**.

A basic sequence $(x_i)_{i\in\omega}$ is called **boundedly complete** if, for every sequence of scalars $(a_i)_{i\in\omega}$ such that $\sup_n \| \sum_{i\in n} a_i x_i \| < \infty$, the series $\sum_{i\in\omega} a_i x_i$ converges. A basic sequence $(x_i)_{i\in\omega}$ is **unconditional** if for any sequence $(a_i)_{i\in\omega}$ of scalars and for any permutation π of ω, i.e., for any bijection $\pi : \omega \to \omega$, $\sum_{i\in\omega} a_i x_i$ converges if and only if $\sum_{i\in\omega} a_{\pi(i)} x_{\pi(i)}$ converges. A non-zero sequence of vectors $(x_i)_{i\in\omega}$ is **unconditional basic** iff there exists $C < \infty$ so that for all $n \in \omega$, $\varepsilon_i = \pm 1$ and $(a_i)_{i\in n} \subseteq \mathbb{R}$, $\| \sum_{i\in n} \varepsilon_i a_i x_i \| \leq C \| \sum_{i\in n} a_i x_i \|$. The smallest such C is the **unconditional basis constant** of (x_i).

A normalized basic sequence $(x_i)_{i\in\omega}$ is C-**subsymmetric** if $(x_i)_{i\in\omega}$ is C-equivalent to each of its subsequences (notice that we do not require it to be unconditional which differs from the terminology of [LT77]).

For a set of vectors A, $\langle A \rangle$ denotes the linear span of A and $[A]$ denotes the **closure of the linear span** of A. Note that if the normalized basic sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are C-equivalent, then the spaces $[(x_i)_{i\in\omega}]$ and $[(y_i)_{i\in\omega}]$ are C-isomorphic.

The **dual space** of a Banach space X is denoted by X^*.

Suppose that $(x_i)_{i\in\omega}$ is a basic sequence. For each x^* in $[(x_i)_{i\in\omega}]^*$ and each $n \in \omega$, let $\|x^*\|_{(n)}$ be the norm of the restriction of x^* to

$\big[\{x_i : i > n\}\big]$. Then $(x_i)_{i\in\omega}$ is **shrinking** if for each $x^* \in [(x_i)_{i\in\omega}]^*$, $\lim_{n\to\infty} \|x^*\|_{(n)} = 0$.

If Y is a normed linear space, B_Y denotes the **closed unit ball** of Y and S_Y is the **unit sphere**. In the sequel, X will always denote a separable infinite dimensional real Banach space.

2. Special Partitions

Let $\omega + 1 := \omega \cup \{\omega\}$, so if $\eta \in \omega + 1$, then η is either a natural number or $\eta = \omega$. If x is a set, we write $|x|$ for the cardinality of x. We will use ω also as a cardinal number, namely $\omega = |\omega|$. If x is a set and $\eta \in \omega + 1$, then

$$[x]^\eta := \{y \subseteq x : |y| = \eta\}$$

and

$$[x]^{<\eta} := \{y \subseteq x : |y| < \eta\}\,.$$

If $a, b \subseteq \omega$, we write $a < b$ in place of "for all $n \in a$ and $m \in b$, $n < m$". Note that $a < b$ implies $a \in [\omega]^{<\omega}$.

A **partition** P of set S is a set of non-empty, pairwise disjoint subsets of S such that $\bigcup P = S$. For a partition P, the sets $b \in P$ are called the **blocks** of P.

In the following we consider "special" partitions of subsets of ω.

If P is a partition of some subset of ω, then P is called a **special partition**, if for all blocks $a, b \in P$ we have either $a < b$, or $a = b$, or $a > b$.

Notice that if P is a special partition with infinitely many blocks, then all of its blocks are finite.

For $\eta \in \omega + 1$, let $\langle\omega\rangle^\eta$ denote the set of all special partitions of subsets of ω such that $|P| = \eta$. In particular, $\langle\omega\rangle^\omega$ is the set of all special partitions with infinitely many blocks.

Let P_1, P_2 be two special partitions. We say that P_1 is *coarser* than P_2, or that P_2 is **finer** than P_1, and write $P_1 \sqsubseteq P_2$, if each block of P_1 is the union of blocks of P_2.

For a special partition P and $\eta \in \omega + 1$ let

$$\langle P\rangle^\eta := \{Q : Q \sqsubseteq P \wedge |Q| = \eta\}\,.$$

If P is a special partition and $b \in P$, then $\min(b) := \bigcap b$ denotes the minimum of the set b. If we order the blocks of P by their minimum, then $P(n)$ denotes the nth block with respect to this ordering.

If P_1, P_2 are two special partitions, then we write $P_1 \sqsubseteq^* P_2$ if there is an $n \in \omega$ such that

$$\big(P_1 \setminus \{P_1(i) : i \in n\}\big) \sqsubseteq P_2 \,.$$

In other words, $P_1 \sqsubseteq^* P_2$ if all but finitely many blocks of P_1 are unions of blocks of P_2.

Fact 1. If $P_0 \,{}^*\!\sqsupseteq\, P_1 \,{}^*\!\sqsupseteq\, P_2 \,{}^*\!\sqsupseteq \ldots {}^*\!\sqsupseteq\, P_i \,{}^*\!\sqsupseteq \ldots$ where $P_i \in \langle\omega\rangle^\omega$ (for each $i \in \omega$), then there is a special partition $P \in \langle\omega\rangle^\omega$ such that for each $i \in \omega$, $P \sqsubseteq^* P_i$.

(The proof is similar to the proof of Fact 2.3 of [Ha98].)

3. The Hindman-Milliken Theorem

First, we recall the well-known Hindman Theorem, and then we give Milliken's generalization of Hindman's Theorem.

If $A \in [\omega]^{<\omega}$, then we write $\sum A$ for $\sum_{a\in A} a$, where we define $\sum \emptyset := 0$.

In [Hi74], N. Hindman proved the following.

Theorem 3.1 (Hindman's Theorem)**.** *If m is a positive natural number and $f : \omega \to m$ is a function, then there exist $r \in m$ and $x \in [\omega]^\omega$ such that whenever $A \in [x]^{<\omega}$ is non-empty, we have $f(\sum A) = r$.*

R. Graham and B. Rothschild noted that Hindman's Theorem can be formulated in terms of finite sets and their unions instead of natural numbers and their sums. This yields the following.

Theorem 3.2 (Hindman's Theorem (Set Version))**.** *If m is a positive natural number, $I \in [\omega]^\omega$ and $f : [I]^{<\omega} \to m$ is a function, then there exist $r \in m$ and an infinite set $H \subseteq [I]^{<\omega}$ such that $a \cap b = \emptyset$ for all distinct sets $a, b \in H$, and whenever $A \in [H]^{<\omega}$ is non-empty, we have $f(\bigcup A) = r$.*

Using Hindman's Theorem as a strong pigeonhole principle, K. Milliken proved a strengthened version of Ramsey's Theorem, which we will call the Hindman-Milliken Theorem (cf. [Mi75, Theorem 2.2]). The Hind-man-Milliken Theorem in terms of unions can be stated as follows:

Theorem 3.3 (Hindman-Milliken Theorem (Set Version))**.** *Let m, n be positive natural numbers, $Q \in \langle\omega\rangle^\omega$ and $f : \langle Q\rangle^n \to m$ a function, then there is an $P \in \langle Q\rangle^\omega$ such that f is constant on $\langle P\rangle^n$.*

As consequences of the Hindman-Milliken Theorem one gets Ramsey's Theorem (Theorem A of [Ra29]) as well as Hindman's Theorem (cf. [Mi75]).

4. Asymptotic Models

First we recall the notion of a spreading model. If $(x_i)_{i\in\omega}$ is a normalized basic sequence in a Banach space X and $\varepsilon_n \downarrow 0$ (a sequence of positive real numbers which tends to 0), then one can find a subsequence $(y_i)_{i\in\omega}$ of $(x_i)_{i\in\omega}$ such that the following holds: For any positive $n \in \omega$, any sequence $(a_k)_{k\in n} \in [-1,1]^n$ and any natural numbers $n \le i_0 < \ldots < i_{n-1}$ and $n \le j_0 < \ldots < j_{n-1}$ we have

$$\left| \Big\| \sum_{k\in n} a_k y_{i_k} \Big\| - \Big\| \sum_{k\in n} a_k y_{j_k} \Big\| \right| < \varepsilon_n \ .$$

This is proved by using Ramsey's Theorem iteratively for a finite δ_n-net in the unit ball of ℓ_∞^n (δ_n depends upon ε_n) to stabilize, up to δ_n, the functions $f(i_0,\ldots,i_{n-1}) \equiv \|\sum_{i\in n} a_i x_i\|$ over a subsequence $(y_i)_{i\in\omega}$ of $(x_i)_{i\in\omega}$ for each $(a_i)_{i\in n}$ in the δ_n-net. Thus, one obtains a limit, $\|\sum_{i\in n} a_i \tilde{e}_i\|$, for each finite sequence $(a_i)_{i\in n}$ of scalars. The sequence $(\tilde{e}_i)_{i\in\omega}$ is called a **spreading model** of $(y_i)_{i\in\omega}$; $(\tilde{e}_i)_{i\in\omega}$ is a normalized 1-subsymmetric basis for $\tilde{E}$, the closed linear span of the $\tilde{e}_i$'s, and $\tilde{E}$ is called a **spreading model** of X generated by $(\tilde{e}_i)_{i\in\omega}$. Hence, for any natural numbers $j_0 < \ldots < j_{n-1}$ we have $\|\sum_{i\in n} a_i \tilde{e}_i\| = \|\sum_{i\in n} a_i \tilde{e}_{j_i}\|$. If $(y_i)_{i\in\omega}$ is weakly null, $(\tilde{e}_i)_{i\in\omega}$ is suppression-1 unconditional: $\|\sum_{i\in F} a_i \tilde{e}_i\| \le \|\sum_{i\in\omega} a_i \tilde{e}_i\|$ for all $F \subseteq \omega$ and each sequence $(a_i)_{i\in\omega}$ of scalars. These facts can be found in [BL84] or [Od80].

Before presenting our extension we set some notation.

We shall call $(x_i^n)_{n,i\in\omega}$ a **K-basic array** in X, if for all $n \in \omega$, $(x_i^n)_{i\in\omega}$ is a K-basic normalized sequence in X and moreover if for all $m \in \omega$ and all integers $m \le i_0 < \ldots < i_{m-1}$, every sequence $(x_{i_j}^j)_{j\in m}$ is K-basic. Furthermore, $(x_i^n)_{n,i\in\omega}$ is a **basic array** in X if it is a K-basic array for some $K < \infty$.

If X has a basis $(x_i)_{i\in\omega}$ then $(x_i^n)_{n,i\in\omega}$ is a **block basic array** in X (with respect to $(x_i)_{i\in\omega}$) if in addition each row $(x_i^n)_{i\in\omega}$ is a block basis of $(x_i)_{i\in\omega}$ and all sequences $(x_{i_j}^j)_{j\in m}$ as described above are also block bases of $(x_i)_{i\in\omega}$.

In what we present, the only important part of the array is the upper triangular part: $\{x_i^n : n \in \omega \text{ and } i \ge n\}$. The lower triangular part can be ignored or omitted and we shall often do so.

Proposition 4.1. *Let $(x_i^n)_{n,i\in\omega}$ be a K-basic array in some Banach space X. Then given $\varepsilon_n \downarrow 0$, there exists a sequence $(k_h)_{h\in\omega}$ of ω so that for all $n \in \omega$, all $(b_i)_{i\in n} \in [-1,1]^n$, all $n \le i_0 < \cdots < i_{n-1}$ and all $n \le \ell_0 < \cdots < \ell_{n-1}$,*

$$\left| \left\| \sum_{j\in n} b_j x_{k_{i_j}}^j \right\| - \left\| \sum_{j\in n} b_j x_{k_{\ell_j}}^j \right\| \right| < \varepsilon_n \ .$$

Proof. As in the case of spreading models, this follows easily from Ramsey's Theorem and the standard diagonalization argument. One $\frac{\varepsilon_n}{2}$-stabilizes $f(i_0, \ldots, i_{n-1}) := \| \sum_{j\in n} b_j x_{i_j}^j \|$ over all subsequences of length n on some subsequence of ω for each of finitely many $(b_j)_{j\in n} \in [-1,1]^n$ out of some δ_n-net in $B_{\ell_\infty^n}$. $\square$

If the conclusion of the proposition holds for $(y_i^n)_{n,i\in\omega}$, where $y_i^n = x_{k_i}^n$, then the iterated limit, $\lim_{i_0\to\infty} \cdots \lim_{i_{n-1}\to\infty} \| \sum_{j\in n} b_j y_{i_j}^j \|$, defines a norm on c_{00}, the linear space of finitely supported real sequences on ω. We let E be the completion of c_{00} under this norm. The unit vector basis $(e_i)_{i\in\omega}$ thus becomes a K-basis for E. We call $(e_i)_{i\in\omega}$ or E an **asymptotic model** of X *generated by* $(y_i^n)_{n,i\in\omega}$.

If $(x_i^n)_{n,i\in\omega}$ is a basic array and $i_0 < i_1 < \cdots$ then $(y_j^n)_{n,j\in\omega}$, where $y_j^n = x_{i_j}^n$, is called a **subarray** of $(x_i^n)_{n,i\in\omega}$. Proposition 4.1 says that every basic array admits a subarray which generates an asymptotic model. Also clearly if $(y_i^n)_{n,i\in\omega}$ generates $(e_i)_{i\in\omega}$, then every subarray of $(y_i^n)_{n,i\in\omega}$ generates (e_i) as well.

We shall have occasion to use the following simple lemma.

Lemma 4.2. *For each $n \in \omega$ let $(x_i^n)_{i\in\omega}$ be a normalized sequence in a Banach space X. If either*

a) *each $(x_i^n)_{i\in\omega}$ is weakly null or*
b) *each $(x_i^n)_{i\in\omega}$ is a block basis of some basic sequence $(x_i)_{i\in\omega}$ in X,*

then the array $(x_i^n)_{n,i\in\omega}$ admits a basic subarray $(y_i^n)_{n,i\in\omega}$. If a)*, then given $\varepsilon > 0$, $(y_i^n)_{n,i\in\omega}$ can be chosen to be a $1+\varepsilon$-basic array. If* b)*, $(y_i^n)_{n,i\in\omega}$ can be chosen to be a block basic array of $(x_i)_{i\in\omega}$.*

Proof. To prove b) we need just choose the subarray $(y_i^n)_{n,i\in\omega}$ so that for all $n \in \omega$, $j \in n$, $i \in n+1$, $\max\big(\mathrm{supp}(y_{n-1}^j)\big) < \min\big(\mathrm{supp}(y_n^i)\big)$ where if $y = \sum a_i x_i$ then $\mathrm{supp}(y) = \{i : a_i \neq 0\}$. a) is proved by a slight generalization of the proof of the well known fact that a normalized weakly null sequence admits a $1+\varepsilon$-basic subsequence. One takes $\varepsilon_n \downarrow 0$ rapidly and then chooses the column $(y_n^i)_{i\in\omega}$ so that $|f(y_n^i)| < \varepsilon_n$ for $i \in n+1$ and each f in a finite $1+\varepsilon_n$-norming set of functionals of $B_{\langle y_i^j : i,j\in n\rangle}$. $\square$

We will call a basic array (x_i^n) whose rows, $(x_i^n)_{i\in\omega}$, are all weakly null a **weakly null basic array**.

If $(e_i)_{i\in\omega}$ is a spreading model of X generated by the basic sequence $(x_i)_{i\in\omega}$, then clearly $(e_i)_{i\in\omega}$ is an asymptotic model of X as well (generated by $(x_i^n)_{n,i\in\omega}$ where $x_i^n = x_i$ for all $n, i \in \omega$). A block basis of a spreading model need not be a spreading model, however, this is not usually the case for asymptotic models. But first we introduce some new notation and a new stronger way of obtaining asymptotic models.

A basic array is a **strong K-basic array** if in addition to the defining conditions of a K-basic array, for all integers $m \leq i_0 < i_1 < \cdots < i_{m-1}$, every sequence of non-zero vectors $(y_j)_{j\in m}$ is K-basic whenever $y_j \in \langle x_s^j : i_j \leq s < i_{j+1}\rangle$. Note that the proof of

Lemma 4.2 actually yields that one can choose the subarray $(y_i^n)_{n,i\in\omega}$ to be strong basic.

Let $(x_i^n)_{n,i\in\omega}$ be a strong basic array. Given $m\in\omega$, a finite set of positive integers $F=\{i_0,i_1,\dots,i_{n-1}\}$ with $i_0<\cdots<i_{n-1}$, and a (possibly infinite) sequence $\boldsymbol{a}=(a_0,a_1,\dots)$ of scalars of length at least n with $a_i\neq 0$ for some $i\in n$, we define

$$x^m(F,\boldsymbol{a}) := \frac{\sum_{j\in n} a_j x_{i_j}^m}{\left\| \sum_{j\in n} a_j x_{i_j}^m \right\|} \ .$$

Theorem 4.3. *Let X be a Banach space and let $(x_i^n)_{n,i\in\omega}$ be a strong K-basic array in X for some $K<\infty$. For each $i\in\omega$ and each non-empty finite set of integers $F=\{i_0,\dots,i_{n-1}\}$ with $i_0<\cdots<i_{n-1}$, let $\boldsymbol{a}_F^i$ be a (possibly infinite) sequence of scalars of length at least n and not identically zero in the first n coordinates and let $\varepsilon_n\downarrow 0$. Then there exists a special partition $P=\{P(k):k\in\omega\}\in\langle\omega\rangle^\omega$ such that the following holds: For all positive $n\in\omega$ and $(b_i)_{i\in n}\in[-1,1]^n$ and all $s,t\in\langle P\rangle^n$ with $\min(s(0)),\min(t(0))\geq n$ we have*

$$\left| \left\| \sum_{i\in n} b_i\, x^i\big(s(i),\boldsymbol{a}_{s(i)}^i\big) \right\| - \left\| \sum_{i\in n} b_i\, x^i\big(t(i),\boldsymbol{a}_{t(i)}^i\big) \right\| \right| < \varepsilon_n \ .$$

Proof. The theorem follows from the Hindman-Milliken Theorem the same way that one obtains a subsequence of a given basic sequence $(x_i)_{i\in\omega}$ yielding a spreading model via Ramsey's Theorem: Given finitely many sequences $(b_i)_{i\in n}\in[-1,1]^n$, a δ_n-net in $B_{\ell_\infty^n}$ (the unit ball of ℓ_∞^n) for an appropriate δ_n, and a special partition $P\in\langle\omega\rangle^\omega$, then one can find $Q\in\langle P\rangle^\omega$ so that for all $t,r\in\langle Q\rangle^n$ we have

$$(*)\qquad \left| \left\| \sum_{i\in n} b_i\, x^i\big(t(i),\boldsymbol{a}_{t(i)}^i\big) \right\| - \left\| \sum_{i\in n} b_i\, x^i\big(r(i),\boldsymbol{a}_{r(i)}^i\big) \right\| \right| < \delta_n \ .$$

One then uses standard approximation and diagonalization arguments to conclude the proof (see Fact 1).

Indeed, given $(b_i)_{i\in n}$ and a special partition $P\in\langle\omega\rangle^\omega$, we partition the interval $[-n,n]$ into say m disjoint subintervals $(I_i)_{i\in m}$, each of length less than δ_n. Given $t \in \langle P\rangle^n$, we let

$$f(t) := j \text{ if and only if } \Big\| \sum_{i\in n} b_i\, x^i\big(t(i), \boldsymbol{a}^i_{t(i)}\big)\Big\| \in I_j \ .$$

An application of the Hindman-Milliken Theorem yields $Q \in \langle P\rangle^\omega$ so that $(*)$ holds for all $t, r \in \langle Q\rangle^n$. We repeat this for each $(b_i)_{i\in n}$. For an arbitrary $(c_i)_{i\in n} \in [-1,1]^n$ one chooses $(b_i)_{i\in n}$ from this δ_n-net with $|c_i - b_i| < \delta_n$ (for all $i \in n$). Hence, for $t, r \in \langle Q\rangle^n$,

$$\begin{aligned}
&\bigg| \Big\| \sum_{i\in n} c_i\, x^i\big(t(i), \boldsymbol{a}^i_{t(i)}\big)\Big\| - \Big\| \sum_{i\in n} c_i\, x^i\big(r(i), \boldsymbol{a}^i_{r(i)}\big)\Big\| \bigg| = \\
&\quad \bigg| \Big\| \sum_{i\in n} c_i\, x^i\big(t(i), \boldsymbol{a}^i_{t(i)}\big) - \sum_{i\in n} b_i\, x^i\big(t(i), \boldsymbol{a}^i_{t(i)}\big) \\
&\quad + \sum_{i\in n} b_i\, x^i\big(t(i), \boldsymbol{a}^i_{t(i)}\big)\Big\| - \Big\| \sum_{i\in n} c_i\, x^i\big(r(i), \boldsymbol{a}^i_{r(i)}\big) \\
&\quad - \sum_{i\in n} b_i\, x^i\big(r(i), \boldsymbol{a}^i_{r(i)}\big) + \sum_{i\in n} b_i\, x^i\big(r(i), \boldsymbol{a}^i_{r(i)}\big)\Big\| \bigg| \ ,
\end{aligned}$$

which by the triangle inequality is

$$\begin{aligned}
&\le \sum_{i\in n} |c_i - b_i|\, \big\| x^i\big(t(i), \boldsymbol{a}^i_{t(i)}\big)\big\| \\
&\quad + \bigg| \Big\| \sum_{i\in n} b_i\, x^i\big(t(i), \boldsymbol{a}^i_{t(i)}\big)\Big\| - \Big\| \sum_{i\in n} b_i\, x^i\big(r(i), \boldsymbol{a}^i_{r(i)}\big)\Big\| \bigg| \\
&\quad + \sum_{i\in n} |c_i - b_i|\, \big\| x^i\big(r(i), \boldsymbol{a}^i_{r(i)}\big)\big\| \\
&\quad < n\delta_n + \delta_n + n\delta_n < \varepsilon_n \ ,
\end{aligned}$$

provided $\delta_n < \frac{\varepsilon_n}{2n+1}$. □

Remark 4.4. One obtains as a limit a norm on c_{00} (the linear space of finitely supported sequences of scalars), $\big\| \sum_{i\in k} b_i e_i \big\|$, where $(e_i)_{i\in\omega}$ is the unit vector basis for c_{00}.

We say that $(e_i)_{i\in\omega}$ is a **strong asymptotic model** generated by the strong basic array $(x^n_i)_{n,i\in\omega}$, the special partition $P \in \langle\omega\rangle^\omega$ and the set of sequences $\{\boldsymbol{a}^i_F : i \in \omega,\ F \in [\omega]^{<\omega}\}$. In this case, it is also easy to see that $(e_i)_{i\in\omega}$ is an asymptotic model of X generated by the basic array $(y_i^n)_{n,i\in\omega}$, where

$$y_i^n = x^n(P(i), \boldsymbol{a}^n_{P(i)}) \quad \text{for } n, i \in \omega\,.$$

Thus, asymptotic models can be generated by a stronger type of convergence. We do not have an application for this. However, it could prove useful in attacking some of the problems in Section 6; those of the type where the assumption is that every asymptotic model is of a certain type.

We note several special cases of strong asymptotic models $(e_i)_{i\in\omega}$ generated by $(x_i^n)_{n,i\in\omega}$, $P \in \langle\omega\rangle^\omega$ and $\{\boldsymbol{a}^i_F : i \in \omega,\ F \in [\omega]^{<\omega}\}$.

(4.4.1.) Let $(x_i)_{i\in\omega}$ be a normalized basic sequence in X and set $x_i^n = x_i$ for all $n, i \in \omega$. Let $\boldsymbol{a}^i_F = (1, 0, 0, \dots)$ for all $i \in \omega$ and $F \in [\omega]^{<\omega}$. Then $(e_i)_{i\in\omega}$ is a spreading model of a subsequence of $(x_i)_{i\in\omega}$.

(4.4.2.) Let $x_i^n = x_i$ for all $n, i \in \omega$, where again $(x_i)_{i\in\omega}$ is a fixed normalized basic sequence in X. For $i \in \omega$ let $\boldsymbol{a}^i$ be a not identically zero sequence of scalars and set $\boldsymbol{a}^i_F = \boldsymbol{a}^i$ for each $F \in [\omega]^{<\omega}$. (The non-zero condition is technically violated here, but we can assume that for some $Q \in \langle\omega\rangle^\omega$, $\boldsymbol{a}^i_{Q(j)}$ is not identically zero in the first $|Q(j)|$ coordinates if $i \le j$ and use the theorem to choose $P \in \langle Q\rangle^\omega$.) In this case we shall say that $(e_i)_{i\in\omega}$ is a **strong asymptotic model of $(x_i)_{i\in\omega}$ generated by P and $(\boldsymbol{a}^i)_{i\in\omega}$**.

(4.4.3.) Assume that we are in the situation of (4.4.2) with in addition $\boldsymbol{a}^i = \boldsymbol{a}$ for all $i \in \omega$ and some fixed $\boldsymbol{a}$. Then we will say that $(e_i)_{i\in\omega}$ is a **strong asymptotic model of $(x_i)_{i\in\omega}$ generated by P and $\boldsymbol{a}$**. In this case, $(e_i)_{i\in\omega}$ is also a spreading model of a normalized block basis of $(x_i)_{i\in\omega}$.

Indeed, for each $i \in \omega$ let $y_i = x\big(P(i), \boldsymbol{a}\big)$, then $(y_i)_{i\in\omega}$ is a normalized block basis of $(x_i)_{i\in\omega}$. Also from the definitions, given $n \in \omega$ and $(b_i)_{i\in n} \in [-1,1]^n$,

$$\left|\Big\|\sum_{i\in n} b_i\, y_{j_i}\Big\| - \Big\|\sum_{i\in n} b_i\, e_i\Big\|\right| \le \varepsilon_n\,,$$

provided that $n \le j_0 < \cdots < j_{n-1}$. Thus, $(e_i)_{i\in\omega}$ is a spreading model of $(y_i)_{i\in\omega}$.

(4.4.4.) If (e_i) is an asymptotic model generated by the strong basic array $(x_i^n)_{n,i\in\omega}$ then (e_i) is a strong asymptotic model generated by (x_i^n), P and $(\boldsymbol{a}_F^i)$ where $P(i) = \{i\}$ and each $\boldsymbol{a}_F^i = (1,0,0,\dots)$.

Proposition 4.5. *Let $(e_i)_{i\in\omega}$ be an asymptotic model of X generated by the basic array (x_i^n). Suppose (x_i^n) is either a weakly null array or a block basis array (w.r.t. some basic sequence in X). Let $(f_i)_{i\in\omega}$ be a normalized block basis of $(e_i)_{i\in\omega}$. Then $(f_i)_{i\in\omega}$ is also an asymptotic model of X.*

Proof. Let $(x_i^n)_{n,i\in\omega}$ generate $(e_i)_{i\in\omega}$. Choose $Q \in \langle\omega\rangle^\omega$ and $\boldsymbol{a}^i$'s such that for every $i \in \omega$, $|Q(i)|$ is equal to the length of $\boldsymbol{a}^i$ and $f_i = e\big(Q(i), \boldsymbol{a}^i\big)$. We shall define a new K-basic array $(y_i^n)_{n,i\in\omega}$ which asymptotically generates $(f_i)_{i\in\omega}$. For $i \in \omega$ let $\tilde{x}_i$ be the i^{th} diagonal of the array $(x_i^n)_{n,i\in\omega}$, so, $\tilde{x}_i = (x_i^0, x_{i+1}^1, \dots, x_{i+n}^n, \dots)$. As before, let $\tilde{x}_i\big(F, \boldsymbol{a}\big)$ be defined relative to this sequence. For $n, i \in \omega$ let $z_i^n = \tilde{x}_i\big(Q(n), \boldsymbol{a}^n\big)$. By passing to a subarray of $(z_i^n)_{n,i\in\omega}$ we obtain, as in Lemma 4.2, an array $(y_i^n)_{n,i\in\omega}$ which is K-basic and asymptotically generates $(f_i)_{i\in\omega}$. $\square$

Remark 4.6. The proposition is false in the general setting. The problem with the proof is that the rows of $(y_i^n)_{n,i\in\omega}$ need not be uniformly basic. We sketch how to construct a space X admitting an asymptotic model $(x_i)_{i\in\omega}$ for which some normalized block basis $(y_i)_{i\in\omega}$ of $(x_i)_{i\in\omega}$ is not an asymptotic model of X. First we define a norm on $[(x_i)_{i\in\omega}]$ where $(x_i)_{i\in\omega}$ is a linearly independent sequence in some linear space. Let $n_i \uparrow \infty$ rapidly and let $(E(i))_{i\in\omega}$ be a special partition of ω with $|E(i)| = n_i$. Set for $x = \sum a_i x_i$, $\|x\| = \max(\|(a_i)\|_{\ell_2}, (\|E_i x\|_{\ell_1})_{T^*})$ where $E_i x$ is the restriction of x to E_i and T^* is the dual norm to Tsirelson's space T. $(x_i)_{i\in\omega}$ is an unconditional basis for the reflexive space $[(x_i)_{i\in\omega}]$. Let $y_i = \frac{1}{|E_i|}\sum_{j\in E_i} x_j$. Then $(y_i)_{i\in\omega}$ is a normalized block basis of $(x_i)_{i\in\omega}$ which is equivalent to the unit vector basis of T^*.

Let $X = [(x_i)_{i\in\omega}] \oplus_\infty (\sum \ell_1)_{\ell_2}$. Let $x_i^n = x_i + e_i^n$ where $(e_i^n)_{i\in\omega}$ is the unit vector basis of the n^{th} copy of ℓ_1 in $(\sum \ell_1)_{\ell_2}$. Then $(x_i^n)_{n,i\in\omega}$ is a basic array and generates the asymptotic model $(x_i)_{i\in\omega}$. It can be shown however that $(y_i)_{i\in\omega}$ is not an asymptotic model of X. The basis $(x_i)_{i\in\omega} \cup (e_i^n)_{n,i\in\omega}$ for X is boundedly complete and unconditional and thus by passing to a subarray we may assume that $y_i^n = z_n + w_i^n$ where $z_n \in X$ and $(w_i^n)_{i\in\omega}$ is a seminormalized block basis of the basis above, in some order, for X.

If P is the natural projection of X onto $(\sum \ell_1)_{\ell_2}$, there must exist m so that, passing to another subarrary, $\inf_{n\geq m} \inf_{i\geq n} \|P(w_i^n)\| > 0$. Otherwise, a subsequence of $(y_i)_{i\in\omega}$ would be generated by a block basis array of $(x_i)_{i\in\omega}$ which is impossible. It then follows that $(y_i)_{i\in\omega}$ must dominate the unit vector basis of ℓ_2 due to the structure of $(\sum \ell_1)_{\ell_2}$. Again, this is false.

It is always true, however, that a normalized block basis of any spreading model of X is again an asymptotic model of X. The difficulty of choosing $(y_i^n)_{n,i\in\omega}$ to be a basic subarray (in the proof of Proposition 4.5) disappears in this instance.

We next collect together a number of remarks and propositions concerning asymptotic models.

Observation 4.7. (4.7.1.) It is not true in general that an asymptotic model $(e_i)_{i\in\omega}$ of a basic sequence $(x_i)_{i\in\omega}$ (as in (4.4.2.)) will be equivalent to a block basis of some spreading model of X, even if X is reflexive.

Indeed, consider $X = \left(\sum \ell_2\right)_{\ell_p}$, with $2 < p < \infty$. The only spreading models of X are ℓ_p (isometrically) and ℓ_2 (isomorphically). This is well-known and easily verified. Letting $(e_i^n)_{i\in\omega}$ be the unit vector basis of the "n^{th} copy" of ℓ_2 in X, we can order the unconditional basis $(e_i^n)_{n,i\in\omega}$ for X as follows:

$$\left(e_0^0, e_1^0, e_0^1, e_2^0, e_1^1, e_0^2, e_3^0, e_2^1, e_1^2, e_0^3, \dots\right).$$

Take $P(0) = \{0\}$, $P(1) = \{1, 2\}$, $P(2) = \{3, 4, 5\}$, $P(3) = \{6, 7, 8, 9\}$, $\dots$. Then this basis along with $P = \{P(i) : i \in \omega\} \in \langle\omega\rangle^\omega$ generates a strong asymptotic model $(e_i)_{i\in\omega}$ for the sequence of $\boldsymbol{a}^i$'s defined as follows. Let n_i be positive integers increasing to ∞ and take $\boldsymbol{a}^0 = \boldsymbol{a}^1 = \cdots = \boldsymbol{a}^{n_0} = (1, 0, 0, 0, \dots)$, $\boldsymbol{a}^{n_0+1} = \cdots = \boldsymbol{a}^{n_0+n_1} =$

$(0,1,0,0,\dots)$, $\boldsymbol{a}^{n_0+n_1+1} = \cdots = \boldsymbol{a}^{n_0+n_1+n_2} = (0,0,1,0,\dots)$, etc. Then $(e_i)_{i\in\omega}$, as is easily checked, is the unit vector basis of $\left(\sum \ell_2^{n_i}\right)_{\ell_p}$, which is not equivalent to a block basis of any spreading model in X.

(4.7.2.) One can slightly change the space in (4.7.1.) to obtain a reflexive space X and a strong asymptotic model $(e_i)_{i\in\omega}$ which is both not equivalent to a block basis of a spreading model nor does $E = [(e_i)_{i\in\omega}]$ embed into X. The same sort of scheme as presented in (4.7.1.) works for $X = (\sum T)_{\ell_2}$, the ℓ_2 sum of Tsirelson's space T (see [FJ74]). The only spreading models of this space are all isomorphic to ℓ_1 or ℓ_2. For, if P_n is the norm 1 natural projection of X onto the "n^{th} copy" of T in X, and $(x_i)_{i\in\omega}$ is a normalized basic sequence in this reflexive space, then passing to a subsequence we may assume *either*: for all n, $\lim_{i\to\infty} \|P_n x_i\| = 0$, in which case, by a gliding hump argument, $(x_i)_{i\in\omega}$ has ℓ_2 as a spreading model; *or*: for some n, $\lim_{i\to\infty} \|P_n x_i\| > 0$, in which case $(x_i)_{i\in\omega}$ has a subsequence whose spreading model is isomorphic to ℓ_1. Now, if we use the basis ordering of (4.7.1.) and the same $P(i)$'s, and take the $\boldsymbol{a}^i$'s to be such that for each sequence $(0,0,\dots,0,1,0,0,\dots)$, infinitely many $\boldsymbol{a}^i$'s are equal to this sequence, then we obtain $(\sum \ell_1)_{\ell_2}$ as a strong asymptotic model. This does not embed into X.

(4.7.3.) Spreading models join the infinite and arbitrarily spread out and finite dimensional structure of X. Another such joining is the theory of **asymptotic structure** developed by B. Maurey, V. Milman and N. Tomczak-Jaegermann (see [MMT95]). In its simplest form this can be described as follows. Suppose X has a basis $(x_i)_{i\in\omega}$. For a positive $n \in \omega$, a normalized basic sequence $(e_i)_{i\in n}$ belongs to the n^{th}-asymptotic structure of X, denoted $\{X\}_n$, if for all $\varepsilon > 0$, given $m_0 \in \omega$ there exists $y_0 \in S_{\langle (x_i)_{i\in\omega\setminus m_0}\rangle}$, so that for all $m_1 \in \omega$ there exists $y_1 \in S_{\langle (x_i)_{i\in\omega\setminus m_1}\rangle}, \dots,$ so that for all $m_{n-1} \in \omega$ there exists $y_{n-1} \in S_{\langle (x_i)_{i\in\omega\setminus m_{n-1}}\rangle}$, so that $(y_i)_{i\in n}$ is $(1+\varepsilon)$-equivalent to $(e_i)_{i\in n}$. (Here, $S_{\langle (x_i)_{i\in\omega\setminus m_j}\rangle}$ denotes the unit sphere of the linear span of $\{x_i : i \in \omega \setminus m_j\}$.)

One difference between this and spreading models is that spreading models are infinite. However one can paste together the elements of the sets $\{X\}_n$ as follows. $(e_i)_{i=1}^{\infty}$ is an **asymptotic version** of X if for all n, $(e_i)_{i=1}^{n} \in \{X\}_n$ [MMT95]. But certain infinite threads are lost nonetheless. Furthermore, spreading models arise from "every normalized basic sequence has a subsequence...". $\{X\}_n$ can be described in terms of infinitely branching trees of length n. The initial nodes and the successors of any node form a normalized block basis of $(x_i)_{i\in\omega}$. We can label such a tree as $T_n = \{x_{(m_0,\dots,m_k)} : 0 \le m_0 < \dots < m_k,\ k \in n\}$ ordered by $x_\alpha \le x_\beta$ if the sequence α is an initial segment of β. Then $(e_i)_{i\in n} \in \{X_n\}$ iff there exists a tree T_n so that for all $\varepsilon > 0$ there exists n_0 so that if $n_0 \le m_0 < \dots < m_{n-1}$, then $(x_{(m_0,\dots,m_k)})_{k\in n}$ is $1+\varepsilon$-equivalent to $(e_i)_{i\in n}$. This stronger structure yields in some sense a more complete theory than that of spreading models where a number of problems remain open. The theory of asymptotic models generated by block basic arrays, while being closer to that of spreading models, lies somewhere between the two. The theory and open problems of spreading models and asymptotic structure motivate some of our questions and results below.

Further, it is clear that if X has a basis $(x_i)_{i\in\omega}$ and $(e_i)_{i\in\omega}$ is an asymptotic model of X generated by a block basis array (w.r.t. $(x_i)_{i\in\omega}$), then for all n, $(e_i)_{i\in n} \in \{X\}_n$.

(4.7.4.) Suppose that X has a basis and that all spreading models of a normalized block basis are equivalent. *Must all spreading models be equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \le p < \infty$?* This question, due to S. Argyros, remains open. Some partial results are in [AOST]. The analogous question for asymptotic models has a positive answer.

Indeed, suppose that all asymptotic models of all block basis arrays of X are equivalent. If $(\tilde{e}_i)_{i\in\omega}$ is a spreading model of such a space, then all of its normalized block bases, being asymptotic models by Proposition 4.5, must be equivalent and the result follows from Zippin's Theorem (see [Zi66] or [LT77, p. 59]).

(4.7.5.) If X is reflexive and $(e_i)_{i\in\omega}$ is an asymptotic model of X, then $(e_i)_{i\in\omega}$ is suppression-1 unconditional. More generally, this holds if $(e_i)_{i\in\omega}$ is generated by $(x_i^n)_{n,i\in\omega}$ where for each $n \in \omega$, $(x_i^n)_{i\in\omega}$ is weakly null.

The proof is very much the same as the analogous result for spreading models. Let $(b_i)_{i\in n} \in [-1,1]^n$ and $i_0 \in n$. We need only show $\|\sum_{i\in n\setminus\{i_0\}} b_i e_i\| \le \|\sum_{i\in n} b_i e_i\|$.

Let $m \ge n$. Since $(x_j^{i_0})_{j\in\omega}$ is weakly null there exists a convex combination of small norm: $\|\sum_{p\in k} c_p x^{i_0}_{m+i_0+p}\| < \varepsilon_m$. For $p \in k$ we consider the vector

$$y_p = \sum_{i\in i_0} b_i x^i_{m+i} + b_{i_0} x^{i_0}_{m+i_0+p} + \sum_{i=i_0+1}^{n-1} b_i x^i_{m+k+i}\ .$$

$|\,\|\sum_{i\in n} b_i e_i\| - \|y_p\|\,| < \varepsilon_m$ and so

$$\Big\|\sum_{p\in k} c_p y_p\Big\| \le \Big\|\sum_{i\in n} b_i e_i\Big\| + \varepsilon_m\ .$$

but also

$$\Big\|\sum_{p\in k} c_p y_p\Big\| \ge \Big\|\sum_{\substack{i\in n\\ i\ne i_0}} b_i e_i\Big\| - \varepsilon_m - |b_{i_0}|\varepsilon_m$$

and this yields the desired inequality.

(4.7.6.) In general, the n^{th} asymptotic structure $\{X\}_n$ of a Banach space X with a basis $(x_i)_{i\in\omega}$ does not coincide with $\big\{(e_i)_{i\in n} : (e_i)_{i\in\omega}$ is an asymptotic model generated by a block basis array of $(x_i)_{i\in\omega}\big\}$. In fact, these may be vastly different for every subspace of X generated by a block basis of $(x_i)_{i\in\omega}$.

To see this we recall that Th. Schlumprecht and the second named author in [OS99, Section 3] constructed a reflexive X so that $(y_i)_{i\in n} \in \{X\}_n$ for all normalized monotone basic sequences $(y_i)_{i\in n}$. Since this includes the highly unconditional summing basis (of length n) the claim follows from (4.7.5).

(4.7.7.) It is possible for a space X to have ℓ_1 as an asymptotic model yet no spreading model of X is isomorphic to ℓ_1, nor to c_0 or any ℓ_p $(1 < p < \infty)$.

Indeed, the reflexive space X constructed in [AOST] has the property that no spreading model is isomorphic to ℓ_p $(1 \le p < \infty)$ nor c_0. Yet every spreading model of X contains an isomorphic copy of ℓ_1.

(4.7.8.) There exists a reflexive space X for which no asymptotic model contains an isomorphic copy of c_0 or ℓ_p $(1 \le p \le \infty)$.

X is the space constructed by Th. Schlumprecht and the second named author in [OS95]; we recall the example: $\|\cdot\|$ is a norm on c_{00} satisfying the following implicit equation.

$$\|x\| := \max\left\{\|x\|_{c_0}, \Big(\sum_{k\in\omega}\|x\|_{n_k}^2\Big)^{1/2}\right\},$$

where $\|x\|_{n_k} = \sup\left\{\frac{1}{f(n_k)}\sum_{i\in n_k}\|E_i x\| : E_0 < \cdots < E_{n_k-1}\right\}$, $f(n_k) = \log_2(1+n_k)$ and $(n_k)_{k\in\omega}$ is a sequence of positive integers satisfying

$$\sum_{k\in\omega}\frac{1}{f(n_k)} < \frac{1}{10}\,.$$

X is the completion of c_{00} under this norm. The unit vector basis $(u_i)_{i\in\omega}$ of c_{00} is a 1-unconditional basis for X and X is reflexive. The fact that X does not admit an asymptotic model $(e_i)_{i\in\omega}$ equivalent to the unit vector basis of ℓ_1 (and hence, by Proposition 4.5, no asymptotic model E contains ℓ_1) is similar to the proof in [OS95] that no spreading model is isomorphic to ℓ_1, and so we shall only sketch the argument.

Suppose that $(e_i)_{i\in\omega}$ is an asymptotic model of X and is equivalent to the unit vector basis of ℓ_1. We may assume that $(e_i)_{i\in\omega}$ is generated by the basic array $(x_i^n)_{n,i\in\omega}$ where each $(x_i^n)_{i\in\omega}$ is a normalized block basis of $(u_i)_{i\in\omega}$. By iteratively passing to a subsequence of each row $(x_i^n)_{i\in\omega}$ and diagonalizing, we may assume that $\big(\|x_j^n\|_{n_i}\big)_{i\in\omega}$ converges weakly in B_{ℓ_2} as $j \to \infty$ to $\boldsymbol{a}^n \in B_{\ell_2}$. Considering the sequence $(\boldsymbol{a}^n)_{n\in\omega} \subseteq B_{\ell_2}$ and passing to a subsequence of the rows, we may assume that $(\boldsymbol{a}^n)_{n\in\omega}$ converges weakly in B_{ℓ_2} to some $\boldsymbol{a} \in B_{\ell_2}$. This corresponds to passing to a subsequence of $(e_i)_{i\in\omega}$, but that is still equivalent to the unit vector basis of ℓ_1 and so we lose nothing here. Thus, we are in the situation where the limit distribution in ℓ_2 of the n^{th} row $(x_i^n)_{i\in\omega}$ is $\boldsymbol{a}^n$ and therefore we can assume $\big(\|x_i^n\|_{n_j}\big)_{j\in\omega}$ in ℓ_2 is equal to $\boldsymbol{a}^n + \boldsymbol{h}_i^n$, where $(\boldsymbol{h}_i^n)_{i\in\omega}$ is weakly null in ℓ_2. Furthermore, $\boldsymbol{a}^n = \boldsymbol{a}+\boldsymbol{h}^n$, where $\boldsymbol{h}^n$ is weakly null in ℓ_2 and hence, we may assume, a block basis in ℓ_2. In this manner, for any N and $(b_i)_{i\in N} \in [-1,1]^N$

we have $\|\sum_{i\in N} b_i e_i\| \approx \|\sum_{i\in N} b_i x^i_{k_i}\|$, provided $N \le k_0 < \cdots < k_{N-1}$.

Now we can also assume that $\|\sum_{i\in N} b_i e_i\| \ge 0.99 \cdot \sum_{i\in N} |b_i|$. This is because ℓ_1 is not distortable (see [Ja64]) and every block basis of an asymptotic model of X is (by Proposition 4.5) also an asymptotic model. Thus, by carefully choosing the k_i's, we have $0.99\cdot\sum_{i\in N}|b_i| < \|\sum_{i\in N} b_i x^i_{k_i}\|$ where $(\|x^i_{k_i}\|)_{\ell_2} \approx \boldsymbol{a}^i + \boldsymbol{h}^i + \boldsymbol{h}^i_{k_i}$ and the vectors $(\boldsymbol{h}^i + \boldsymbol{h}^i_{k_i})_{i\in N}$ are a block basis in ℓ_2. At this point, we use the argument in Theorem 1.3 of [OS95] to see that, if N is sufficiently large depending upon $\boldsymbol{a}$, this is impossible.

Furthermore, the arguments of [OS95] apply easily to show that it is not possible to have an asymptotic model $(e_i)_{i\in\omega}$ equivalent to the unit vector basis of c_0 or ℓ_p $(1 < p < \infty)$, which completes the proof of (4.7.8.).

(4.7.9.) The proof of (4.7.8) actually reveals that no spreading model of an asymptotic model of X can be isomorphic to ℓ_1 (or c_0 or any ℓ_p). For if $E = [(e_i)_{i\in\omega}]$ is an asymptotic model of X, then any spreading model of E is necessarily a spreading model of a normalized block basis $(f_i)_{i\in\omega}$ of $(e_i)_{i\in\omega}$ and this in itself is an asymptotic model of X. Let $(\tilde{e}_i)_{i\in\omega}$ be the spreading model of $(f_i)_{i\in\omega}$. The proof shows that, for sufficiently large N, we cannot have $\|\sum_{i\in N} b_i f_{k_i}\| \ge 0.99 \cdot \sum_{i\in N}|b_i|$ for all $(b_i)_{i\in N} \in [-1,1]^N$ and any $k_0 < \cdots < k_{N-1}$.

(4.7.10.) G. Androulakis, the second named author, Th. Schlumprecht and N. Tomczak-Jaegermann have constructed [AOST] a reflexive Banach space X for which no spreading model is reflexive, isomorphic to c_0 or isomorphic to ℓ_1. However, every X admits an asymptotic model which is either reflexive or isomorphic to c_0 or ℓ_1.

Indeed, X admits a spreading model $\tilde{E}$ with an unconditional basis and by [Ja64], $\tilde{E}$ is either reflexive or contains an isomorphic copy of c_0 or ℓ_1. So, the result follows by Remark 4.6.

There is a big difference between considering all asymptotic models of X and of those generated by weakly null basic arrays or block basic arrays as our next proposition illustrates. Also it illustrates again the difference between the class of spreading models and asymptotic models: if $(e_i)_{i\in\omega}$ is a spreading model of c_0, then $(e_i)_{i\in\omega}$ is equivalent to either the summing basis or the unit vector basis of c_0.

Proposition 4.8. *Let $(e_i)_{i\in\omega}$ be a normalized bimonotone basic sequence. Then $(e_i)_{i\in\omega}$ is 1-equivalent to an asymptotic model of c_0.*

Proof. Let $\varepsilon_k \downarrow 0$. For all positive integers k there exist $n_k \in \omega$ and vectors $(x_i^k)_{i\in k} \in S_{\ell_\infty^{n_k}}$ so that

$$(1-\varepsilon_k)\Big\|\sum_{i\in k} a_i e_i\Big\| \le \Big\|\sum_{i\in k} a_i x_i^k\Big\| \le \Big\|\sum_{i\in k} a_i e_i\Big\|$$

for all $(a_i)_{i\in k} \in \mathbb{R}^k$. Indeed, we choose $(f_i^k)_{i\in n_k} \subseteq B_{[(e_i)_{i\in\omega}]^*}$ so that $\sup_{i\in n_k} |f_i^k(e)| \ge (1-\varepsilon_k)\|e\|$ for $e \in [(e_i)_{i\in k}]$ and $f_i^k(e_i) = 1$ for $i \in k$, and let $x_i^k = T^k e_i$, where $T^k : [(e_i)_{i\in k}] \to \ell_\infty^{n_k}$ is given by $T^k e = (f_i^k(e))_{i\in n_k}$.

We write $c_0 = (\sum \ell_\infty^{n_k})_{c_0}$ and regard $(x_i^k)_{i\in k}$ as being contained in the indicated copy of $\ell_\infty^{n_k} \subseteq c_0$. Let $(y_i^k)_{k\in\omega, i\ge k}$ be defined by $y_i^0 = x_0^1 + \dots + x_0^{i+1}$ and in general $y_i^k = x_k^{k+1} + \dots + x_k^{i+1}$.

It is easy to check that $(y_i^k)_{k,i\in\omega}$ is a basic array (the rows are equivalent to the summing basis) and this array generates $(e_i)_{i\in\omega}$. □

Remark 4.9. Recall [DLT00] that a basic sequence $(x_i)_{i\in\omega}$ is said to be **asymptotically isometric to** c_0, if for some sequence $\varepsilon_n \downarrow 0$ for all $(a_n)_{n\in\omega} \in c_0$,

$$\sup_n (1-\varepsilon_n)|a_n| \le \Big\|\sum_{n\in\omega} a_n x_n\Big\| \le \sup_n |a_n| \ .$$

In this case the proof of Proposition 4.8 can be adopted to yield that $[(x_i)_{i\in\omega}]$ admits all normalized bimonotone basic sequences as asymptotic models. In general, using that c_0 is not distortable [Ja64], one has that if X is isomorphic to c_0 then for all $K > 1$ there exists $C(K)$ so that if $(e_i)_{i\in\omega}$ is a normalized K-basic sequence, then X admits an asymptotic model $C(K)$-equivalent to $(e_i)_{i\in\omega}$. We do not know if the conclusion to Proposition 4.8 holds in this case. We also do not know if this property characterizes spaces containing c_0 (see the open problems in Section 6). By way of contrast it is easy to see that all asymptotic models of ℓ_p $(1 < p < \infty)$ are 1-equivalent to the unit vector basis of ℓ_p. Moreover we have

Proposition 4.10. *If $(e_i)_{i\in\omega}$ is an asymptotic model of ℓ_1 then $(e_i)_{i\in\omega}$ is equivalent to the unit vector basis of ℓ_1.*

Proof. Let $(x_i^n)_{n,i\in\omega}$ be a K-basic array generating $(e_i)_{i\in\omega}$. Since each row is K-basic there exists $\delta > 0$ so that for all $n, m \in \omega$ there exists $k \in \omega$ with $\|P^m(x_i^n)\| > \delta$ for $i \geq k$ where P^m is the tail projection of ℓ_1, $P^m(a_i) = (0, \ldots, 0, a_m, a_{m+1}, \ldots)$. Using that the unit vector basis of ℓ_1 is boundedly complete we can find a subsequence $(y_i^n)_{i\in\omega}$ of each row $(x_i^n)_{i\in\omega}$ of the form $y_i^n = y_n + h_i^n$ where $h_i^n \to 0$ weak* in ℓ_1 as $i \to \infty$ and $\|h_i^n\| \geq \delta$. Thus, up to arbitrarily small perturbations, we may assume h_i^n and h_j^n are disjointly supported for $i \neq j$. And doing all this by a diagonal process we can assume that $(y_i^n)_{n,i\in\omega}$ is a subarray of $(x_i^n)_{n,i\in\omega}$. It follows easily that

$$\Big\| \sum a_i e_i \Big\| \geq \delta \sum |a_i| \ . \quad \square$$

From Proposition 4.8 we see that ℓ_1 can be an asymptotic model of a space X with a basis without being an asymptotic model generated by a block basic array. But this cannot happen in a boundedly complete situation:

Proposition 4.11. *Let $(d_i)_{i\in\omega}$ be a boundedly complete basis for Y and let $X \subseteq Y$ be a weak* closed subspace. If ℓ_1 is an asymptotic model of X, then ℓ_1 is an asymptotic model generated by a basic array $(x_i^n)_{n,i\in\omega}$ where for each n, $(x_i^n)_{i\in\omega}$ is weak* null.*

In this proposition the weak* topology on Y is the natural one generated by regarding Y as the dual space of $[(d_i^*)_{i\in\omega}]$, where the d_i^*'s are the biorthogonal functionals of the d_i's (this is, for all i, j, $d_i^* d_j = \delta_j^i$). Thus, $d_n = \sum a_i^n d_i \to d = \sum a_i d_i$ weak* if $(d_n)_{n\in\omega}$ is bounded and $a_i^n \to a_i$ for each $n \in \omega$.

Proof of Proposition 4.11. Let $(y_i^n)_{n,i\in\omega} \subseteq X$ generate the asymptotic model $(e_i)_{i\in\omega}$ which is equivalent to the unit vector basis of ℓ_1. As in the preceding proposition by passing to a subarray we may assume $y_i^n = f^n + x_i^n$ where for each n, $(x_i^n)_{i\in\omega}$ is weak* null and $(f^n)_{n\in\omega} \subseteq X$. If $(f^n)_{n\in\omega\setminus k}$ is not equivalent to the unit vector basis of ℓ_1 for some k, then some block sequence of absolute convex combinations of the f^n's is norm null. We use this (as in the proof of Proposition 4.5) to generate a new basic array of the same form where $\|f^n\| < \varepsilon_n$ for $\varepsilon_n \downarrow 0$ rapidly, and so, a subarray of $(x_i^n/\|x_i^n\|)_{n,i\in\omega}$ generates the unit vector basis of ℓ_1. $\square$

The asymptotic models of L_p ($1 < p < \infty$) are necessarily unconditional and in fact every normalized unconditional basic sequence in L_p is equivalent to an asymptotic model.

Proposition 4.12. *Let $1 < p < \infty$. There exists $K_p < \infty$ so that if $(x_i)_{i\in\omega}$ is a normalized K-unconditional basic sequence in L_p then $(x_i)_{i\in\omega}$ is KK_p-equivalent to some asymptotic model of L_p.*

Proof. This follows easily from arguments of G. Schechtman [S74]. There exists $K_p < \infty$ so that $(x_i)_{i\in\omega}$ is KK_p-equivalent to a normalized block basis $(y_i)_{i\in\omega}$ of the Haar basis $(h_i)_{i\in\omega}$ for L_p. Furthermore if $(z_i)_{i\in\omega}$ is a block basis of $(h_i)_{i\in\omega}$ with $|z_i| = |y_i|$ for all i, then $(x_i)_{i\in\omega}$ is KK_p-equivalent to $(z_i)_{i\in\omega}$. For $n \in \omega$, let $(y_i^n)_{i\in\omega}$ be a normalized block basis of $(h_i)_{i\in\omega}$ with $|y_i^n| = |y_n|$ for all i. By Lemma 4.2, some subarray of $(y_i^n)_{n,i\in\omega}$ is thus a block basis array of $(h_i)_{i\in\omega}$. By our above remarks and Proposition 4.1, some subarray of $(y_i^n)_{n,i\in\omega}$ generates an asymptotic model KK_p-equivalent to $(x_i)_{i\in\omega}$. $\square$

Another natural question is if X has Y as an asymptotic model and Y has Z as an asymptotic model, does X have an asymptotic model isomorphic to Z? If one replaces "asymptotic model" in the question with "spreading model", the answer is negative (see [BM79]). In the following, we present an example that shows the answer also to be negative in a strong way for asymptotic models.

Example 4.13. There exist reflexive Banach spaces X and Y so that Y is a spreading model of X, ℓ_1 is a spreading model of Y and ℓ_1 is not isomorphic to any asymptotic model of X.

Proof. X and Y will both be completions of c_{00} under certain norms which will make the unit vector basis of c_{00} an unconditional basis for each space. We will denote these bases by $(v_i)_{i\in\omega}$ for X and $(u_i)_{i\in\omega}$ for Y. Both spaces will be reflexive.

First we construct the spaces Y and X. The construction bears some similarity with those in [MR77] and [LT77, p. 123]. To begin, let $(m_j)_{j\in\omega}$ be an increasing sequence of integers with $m_0 = 1$ and for any $k \in \omega$: $m_0 + \cdots + m_k < 2m_k$, $\sum_{n\in\omega\setminus\{0\}} \frac{1}{\sqrt{m_n}} < 1$ and $\frac{(2m_k)^2}{\sqrt{m_{k+1}}} < 1$. Let $\mathcal{F}$ be the subset of c_{00} given as follows:

$$\mathcal{F} := \left\{ f = \sum_{j\in n} \frac{1_{E_{i_j}}}{\sqrt{m_{i_j}}} : n \in \omega,\ |E_{i_j}| \leq m_{i_j},\ n \leq i_0 < \cdots < i_{n-1} \right.$$
$$\left. \text{and} \quad E_{i_k} \cap E_{i_l} = \emptyset \text{ whenever } k \neq l \right\},$$

where $1_{E_{i_j}} \in c_{00}$ is the indicator function,

$$1_{E_{i_j}}(k) = \begin{cases} 1 & \text{if } k \in E_{i_j}, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in c_{00}$, let

$$\|x\|_Y := \sup \left\{ \Big(\sum_{k\in m} \langle f_k, x\rangle^3 \Big)^{1/3} : m \in \omega,\ (f_k)_{k\in m} \subseteq \mathcal{F} \text{ and} \right.$$
$$\left. \text{the } f_k\text{'s are disjointly supported} \right\},$$

where $\langle f_k, x\rangle$ is the scalar product of f_k and x. We say $E \in [\omega]^{<\omega}$ is **admissible** if $\min(E) \geq |E|$ and $g \in c_{00}$ is admissible if $\operatorname{supp}(g)$ (the support of g) is admissible. Set $\mathcal{G} := \{f|_E : E \text{ is admissible and } f \in \mathcal{F}\} = \{f \in \mathcal{F} : f \text{ is admissible}\}$, and for $x \in c_{00}$, let

$$\|x\|_X := \sup \left\{ \Big(\sum_{k\in m} \langle g_k, x\rangle^3 \Big)^{1/3} : m \in \omega,\ (g_k)_{k\in m} \subseteq \mathcal{G} \text{ and} \right.$$
$$\left. \text{the } g_k\text{'s are disjointly supported} \right\}.$$

We will also write $g(x)$ for $\langle g, x\rangle$. It is clear that $(v_j)_{j\in\omega}$ and $(u_j)_{j\in\omega}$ are each suppression-1 unconditional bases for X and Y, respectively. Because each basis admits a lower ℓ_3 estimate on disjointly supported vectors, neither space contains ℓ_∞^n's uniformly (see [Jo76]). Thus, both bases are boundedly complete. Also both bases are shrinking and hence, X and Y are reflexive. To see this for Y (the proof for X is similar) suppose $(y_i)_{i\in\omega}$ is a normalized block basis of $(u_j)_{j\in\omega}$ which is not weakly null. By the definition of the norm in Y, and passing to a subsequence of $(y_i)_{i\in\omega}$, we obtain $f \in \mathcal{F}$ and $\varepsilon > 0$ with $|\langle f, y_j\rangle| > \varepsilon$ for all j, which is clearly impossible.

The sequence $(u_j)_{j\in\omega}$ is 1-symmetric and is the spreading model of $(v_j)_{j\in\omega}$ (since if one moves a vector far enough to the right in c_{00}, then the Y norm expressions all become allowable).

Let $E_0 < \cdots < E_j < \cdots$ be sets of natural numbers with $|E_j| = m_j$ and let $y_j = \frac{1_{E_j}}{\sqrt{m_j}}$ (for $j \in \omega$). Then $\|y_j\|_Y \geq 1$ and $\sup_{j\in\omega} \|y_j\|_Y < \infty$. Indeed, for some fixed $q \in \omega$, let $y = \frac{1_{E_q}}{\sqrt{m_q}}$. First suppose $f \in \mathcal{F}$, and therefore, f is of the form $f = \sum_{j\in n} \frac{1_{E_{i_j}}}{\sqrt{m_{i_j}}}$ (for some disjoint collection $(E_{i_j}) \subseteq [\omega]^{<\omega}$ with $|E_{i_j}| \leq m_{i_j}$ and $n \leq i_0 < \cdots < i_{n-1}$). We shall estimate $\langle f, y\rangle$ from above, and thus we may assume $\mathrm{supp}(f) \subseteq E_q$. Write $f = f^1 + f^2 + f^3$, where

$$f^1 = \sum_{\substack{j\in n\\ i_j<q}} \frac{1_{E_{i_j}}}{\sqrt{m_{i_j}}}, \quad f^2 = \begin{cases} \dfrac{1_{E_q}}{\sqrt{m_q}} & \text{if some } i_j = q, \\ 0 & \text{otherwise,} \end{cases} \quad f^3 = \sum_{\substack{j\in n\\ i_j>q}} \frac{1_{E_{i_j}}}{\sqrt{m_{i_j}}}.$$

By the properties of the sequence $(m_j)_{j\in\omega}$ we have

$$\langle f^1, y\rangle = \sum_{\substack{j\in n\\ i_j<q}} \frac{|E_{i_j}|}{\sqrt{m_{i_j}}\sqrt{m_q}} \leq \frac{2m_{q-1}}{\sqrt{m_q}}, \quad \langle f^2, y\rangle \leq \frac{m_q}{\sqrt{m_q}\sqrt{m_q}} = 1$$

and

$$\langle f^3, y\rangle = \sum_{\substack{j\in n\\ i_j>q}} \frac{|E_{i_j}|}{\sqrt{m_{i_j}}\sqrt{m_q}} \leq \frac{\sqrt{m_q}}{\sqrt{m_{q+1}}}.$$

Now suppose that $f_k = \sum_{j\in n_k} \frac{1_{E_{i^k_j}}}{\sqrt{m_{i^k_j}}} \in \mathcal{F}$ and the $(f_k)_{k\in m}$ are disjointly supported with $\mathrm{supp}(f_k) \subseteq E_q$ for each $k \in m$. As above, each f_k is of the form $f_k = f^1_k + f^2_k + f^3_k$. Thus by the triangle inequality in ℓ_3,

$$\Big(\sum_{k\in m}\langle f_k,y\rangle^3\Big)^{1/3} \le \Big(\sum_{k\in m}\langle f_k^1,y\rangle^3\Big)^{1/3} + \Big(\sum_{k\in m}\langle f_k^2,y\rangle^3\Big)^{1/3} + \Big(\sum_{k\in m}\langle f_k^3,y\rangle^3\Big)^{1/3} .$$

The first term is

$$\Big(\sum_{k\in m}\sum_{j\in n_k,\ i_j^k<q}\Big(\frac{|E_{i_j^k}|}{\sqrt{m_{i_j^k}}\sqrt{m_q}}\Big)^3\Big)^{1/3} ,$$

and since (by the earlier calculation)

$$\sum_{j\in n_k,\ i_j^k<q}\frac{|E_{i_j^k}|}{\sqrt{m_{i_j^k}}\sqrt{m_q}} \le \frac{2m_{q-1}}{\sqrt{m_q}} ,$$

the first term is

$$\le \Big(\Big(\frac{2m_{q-1}}{\sqrt{m_q}}\Big)^2\Big(\sum_{k\in m}\sum_{\substack{j\in n_k\\ i_j^k<q}}\frac{|E_{i_j^k}|}{\sqrt{m_{i_j^k}}\sqrt{m_q}}\Big)\Big)^{1/3}$$

$$\le \Big(\Big(\frac{2m_{q-1}}{\sqrt{m_q}}\Big)^2\frac{m_q}{\sqrt{m_q}}\Big)^{1/3} = \Big(\frac{(2m_{q-1})^2}{\sqrt{m_q}}\Big)^{1/3} < 1 .$$

The second term is of the form

$$\Big(\sum_{k\in m}\Big(\frac{l_k}{\sqrt{m_q}\sqrt{m_q}}\Big)^3\Big)^{1/3} ,$$

where $\sum_{k\in m} l_k \le m_q$, and therefore, it is $\le \sum_{k\in m}\frac{l_k}{m_q} \le 1$.

The third term is

$$\Big(\sum_{k\in m}\Big(\sum_{\substack{j\in n_k\\ i_j^k>q}}\frac{|E_{i_j^k}|}{\sqrt{m_{i_j^k}}\sqrt{m_q}}\Big)^3\Big)^{1/3} \le \sum_{k\in m}\sum_{\substack{j\in n_k\\ i_j^k>q}}\frac{|E_{i_j^k}|}{\sqrt{m_{i_j^k}}\sqrt{m_q}} \le \frac{m_q}{\sqrt{m_{q+1}}\sqrt{m_q}} = \frac{\sqrt{m_q}}{\sqrt{m_{q+1}}} < 1 .$$

Thus, $(y_j)_{j\in\omega}$ is a seminormalized block basis of $(u_j)_{j\in\omega}$ in Y. Moreover, from the definition of the norm, namely $\mathcal{F}$, if $n \le i_0 < \cdots < i_{n-1}$ and $(b_i)_{i\in n}$ are scalars, then $\|\sum_{j\in n} b_i y_{i_j}\| \ge |\sum_{i\in n} b_i|$, and hence, if we pass to a subsequence of $(y_j)_{j\in\omega}$ having a spreading model, then this spreading model is equivalent to the unit vector basis of ℓ_1.

It remains to show that ℓ_1 is not isomorphic to an asymptotic model of X.

By the uniform convexity of ℓ_3 we have:

$$(*) \qquad \text{for any } \varepsilon > 0 \text{ there exists } \lambda < 1 \text{ such that}$$
$$\text{if } x, y \in B_{\ell_3} \text{ with } \|x+y\|_{\ell_3} > 2\lambda, \text{ then } \|x-y\| < \varepsilon\,.$$

We shall now fix parameters $1 > \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > 0.9$, $0 < \varepsilon_1 < \varepsilon_3 < \varepsilon_4 < 1/4$, $\delta_4 = 1-\lambda_4$, $\delta_1 = 1-\lambda_1$ as follows. We use $(*)$ to obtain λ_4 from ε_4, where we require ε_4 (and λ_4) to satisfy $1-2\delta_4-2\varepsilon_4 > \lambda_5$. λ_3 and ε_3 are chosen so that for any normalized basic sequence $(x_i)_{i\in\omega}$ with a λ_3-lower ℓ_1 estimate, if $\|y_i - x_i\| < \varepsilon_3$ for all $i\in\omega$, then $(y_i/\|y_i\|)_{i\in\omega}$ admits a λ_4-lower ℓ_1 estimate. Then choose λ_2 so that $\lambda_2^3 + \varepsilon_3^3 > 1$. Take $\varepsilon_1 > 0$ to determine λ_1 by $(*)$ so that $1-2\delta_1-\varepsilon_1 > \lambda_2$. If ℓ_1 is an asymptotic model of X, then, since X is reflexive, by the proof that ℓ_1 is not distortable (cf. [Ja64]), we may assume that X admits a block basis array $(x_i^n)_{n,i\in\omega}$ which asymptotically generates $(e_i)_{i\in\omega}$, where $\|\sum_{i\in n} b_i e_i\| > \lambda_1 \sum_{i\in n}|b_i|$ for all scalars $(b_i)_{i\in n}$ not identically zero.

Claim. *For $n \ge 1$ there exists $K_n \in \omega$ and $i_n \in \omega$ so that if $i \ge i_n$ there exists $F_i \subseteq \operatorname{supp} x_i^n$ with $|F_i| \le K_n$ and $\|x_i^n|_{\omega\setminus F_i}\| < \varepsilon_3$.*

To see this, fix $n \ge 1$. Since $\|e^0 + e^n\| > 2\lambda_1$, there exists $k\in\omega$ so that if $i > k$, then $\|x_k^0 + x_i^n\| > 2\lambda_1$. Let $i > k$ be fixed and choose disjointly supported $(g_j)_{j\in m} \subseteq \mathcal{G}$ so that

$$(1) \qquad \Big(\sum_{j\in m} \big(g_j(x_k^0) + g_j(x_i^n)\big)^3\Big)^{1/3} > 2\lambda_1\ .$$

Thus, by our choice of ε_1 using $(*)$,

$$(2) \qquad \Big\|\big(g_j(x_k^0)\big)_{j\in m} - \big(g_j(x_i^n)\big)_{j\in m}\Big\|_{\ell_3} < \varepsilon_1\ .$$

We reorder the g_j's and choose $\bar{m} \leq m$ so that for $j \in \bar{m}$, $\operatorname{supp}(g_j) \cap \operatorname{supp}(x_k^0) \neq \emptyset$, and for $j \in m \setminus \bar{m}$, $\operatorname{supp}(g_j) \cap \operatorname{supp}(x_k^0) = \emptyset$. From (1) and the triangle inequality in ℓ_3, $\left(\sum_{j \in m} g_j(x_i^n)^3\right)^{1/3} > 1 - 2\delta_1$, and from (2) and the choice of $\bar{m}$ we obtain $(\sum_{j \in m \setminus \bar{m}} g_j(x_i^n)^3)^{1/3} < \varepsilon_1$. Thus, by the triangle inequality,

$$(3) \qquad \Big(\sum_{j \in \bar{m}} g_j(x_i^n)^3\Big)^{1/3} > 1 - 2\delta_1 - \varepsilon_1 > \lambda_2 \ .$$

By admissibility restrictions for $j \in \bar{m}$, $|\operatorname{supp}(g_j)| \leq \max(\operatorname{supp}(x_k^0))$ and thus, since $\bar{m} \leq \max(\operatorname{supp}(x_k^0))$,

$$|\bigcup_{j \in \bar{m}} \operatorname{supp}(g_j)| \leq (\max(\operatorname{supp}(x_k^0)))^2 =: K_n \ .$$

Let $F_i = \bigcup_{j \in \bar{m}}(\operatorname{supp}(g_j) \cap \operatorname{supp}(x_i^n))$, so $|F_i| \leq K_n$. By (3), $1 = \|x_i^n\| > (\lambda_2^3 + \|x_i^n|_{\omega \setminus F_i}\|^3)^{1/3}$ and so, by our choice of $\lambda_2^3 + \varepsilon_3^3 > 1$ we obtain $\|x_i^n|_{\omega \setminus F_i}\| < \varepsilon_3$, which proves the claim.

Using the claim for $n \geq 1$, let $y_i^n = x_i^n|_{\omega \setminus F_i} / \|x_i^n|_{\omega \setminus F_i}\|$ for $i > i_n$ and $y_i^n = x_i^n$ for $i \leq i_n$. By Proposition 4.1, we pass to a subarray asymptotically generating $(f_i)_{i \in \omega}$. By our choice of ε_3 and the claim, for all not identically zero scalars $(b_i)_{1 \leq i \leq n}$, $\|\sum_{i=1}^n b_i f_i\| > \lambda_4 \cdot \sum_{i=1}^n |b_i|$. Since $|\operatorname{supp}(y_i^n)| \leq K_n$ for $n \geq 1$, by passing to another subarray we may assume that for $n \geq 1$ there exists $x^n \in c_{00}$ so that if $i \geq n$ and $y_i^n = (0, \ldots, 0, a_1^n, , \ldots, 0, a_2^n, 0, \ldots, 0, a_{p_n}^n, 0, \ldots)$ where the a_k^n's are the non-zero coordinates of x_i^n, then $x^n = (a_1^n, \ldots, a_{p_n}^n, 0, 0, \ldots)$. Of course, $p_n \leq K_n$. In short, the y_i^n's are an identically distributed normalized block basis of $(u_j)_{j \in \omega}$ and $(v_j)_{j \in \omega}$, i.e., in both X and Y norms. This is done by passing to a subsequence in each row, iteratively, so that the distributions converge to that of x^n. We then diagonalize. This array still asymptotically generates $(f_i)_{i \in \omega}$. Of course, we lost our 0^{th} row, so, let us relabel every thing as $(x_i^n)_{n,i \in \omega}$ asymptotically generating $(f_i)_{i \in \omega}$ with the λ_4-lower ℓ_1 estimates and the fact the x_i^n equals x^n in distribution for $i \geq n$. And our old K_n becomes K_{n-1} in the new labeling.

From this point on we work in Y (when computing $\|\sum_{i \in m} b_i x_{i_n}^n\|$ for i_0 large, the X and Y norms coincide). For $x = (a_0, \ldots, a_{n-1}, 0,$

$0, \dots) \in c_{00}$, let $x^* := (a_{\pi(0)}, \dots, a_{\pi(n-1)}, 0, 0, \dots)$, where π is a permutation of n such that $|a_{\pi(0)}| \geq \cdots \geq |a_{\pi(n-1)}|$. By passing to a subsequence of the rows (the new array still asymptotically generates ℓ_1 with lower estimate λ_4; indeed, it generates a subsequence of $(f_i)_{i\in\omega}$) we may assume that $x^{n*} \to x \in c_0$ coordinatewise, where $x = (a_0, a_1, \dots)$ with $|a_0| \geq |a_1| \geq \cdots$. Also, since Y is reflexive, $x \in Y$ and $\|x\|_Y \leq 1$. Choose $p \in \omega$ so that $\|(a_p, a_{p+1}, \dots)\|_Y < \varepsilon_4$; choose $M \in \omega$ so that $\frac{1}{\sqrt{m_M}} K_0^2 < \varepsilon_4$ (recall that K_0 is the cardinality of the support of x^0); and further choose $N > 8K_0M$ so that $(pN)^{1/3} < N/8$.

We next choose $\gamma_n \downarrow 0$ with $\sum_{n\in\omega\setminus\{0\}} \gamma_n < 1$. For each $n \in \omega$ choose $\bar\gamma_{n+1} > 0$ so that if $g = \frac{1_E}{\sqrt{m_i}}$ is a term of some $f \in \mathcal{F}$ with the property that $|g(z)| \geq \gamma_n$ for some $\|z\|_Y \leq 1$ with $|\operatorname{supp}(z)| \leq K_n$ then $|g(y)| < \gamma_{n+1}$ whenever $\|y\|_Y \leq 1$ and $\|y\|_\infty < \bar\gamma_{n+1}$. By passing to a subsequence of the rows again and relabeling and not changing the first row of x_i^0's we may assume that $x^{n*}|_p = x|_p$ for all positive n (this actually introduces a slight error which we shall ignore in that it is insignificant to what follows) and

$$x^{n*} = x|_p + x|_{[p,p_n]} + x^{n*}|_{(p_n,K_n]} \tag{4}$$

where $\|x^{n*}|_{[p_n,K_n]}\|_\infty < \bar\gamma_n$, the $\|\cdot\|_\infty$ being calculated relative to the (u_j)-coordinates, where $p < p_1 < K_1 < p_2 < K_2 < p_3 \cdots$. Now $\|x_{i_0}^0 + \frac{1}{N}\sum_{n=1}^N x_{i_n}^n\| > 2\lambda_4$, provided $i_0 < i_1 < \cdots < i_N$ are large enough. We fix these elements and use (4) to write each $x_{i_n}^n = x_{i_n}^n(1) + x_{i_n}^n(2) + x_{i_n}^n(3)$, where the three terms are disjointly supported and each has, respectively, the same distribution as the three terms in (4), thus, $x_{i_n}^n(2)^* = x|_{[p,p_n]}^*$. Choose disjointly supported $(g_k)_{k\in m} \subseteq \mathcal{F}$ with

$$\left(\sum_{k\in m} g_k\Big(x_{i_0}^0 + \frac{1}{N}\sum_{n=1}^N x_{i_n}^n\Big)^3\right)^{1/3} > 2\lambda_4\ . \tag{5}$$

It follows that

$$\left(\sum_{k\in m} g_k\big(x_{i_0}^0\big)^3\right)^{1/3} > 1 - 2\delta_4\ .$$

Write $g_k = \sum_{j\in n_k} \frac{1_{E_{i_j^k}}}{\sqrt{m_{i_j^k}}}$ as in the definition of $\mathcal{F}$. We shall call $\frac{1_{E_{i_j^k}}}{\sqrt{m_{i_j^k}}}$ a **term** of g_k. By reordering the g_k's we may assume for some $\bar{m} \le m$ that if $k \le \bar{m}$, then some term $\frac{1_E}{\sqrt{m_j}}$ of g_k satisfies $|\frac{1_E}{\sqrt{m_j}}(x^0_{i_0})| \ge \frac{\varepsilon_4}{K_0}$. In particular, this forces $\bar{m} \le K_0$ and $j < M$ and so $n_k < M$ for $k \le \bar{m}$. If $k \in m \setminus \bar{m}$, then for each term $\frac{1_E}{\sqrt{m_j}}$ of g_k we have $|\frac{1_E}{\sqrt{m_j}}(x^0_{i_k})| \ge \frac{\varepsilon_4}{K_0}$ and so, since at most K_0 such terms could be non-zero on $x^0_{i_0}$,

$$(6) \qquad \Big(\sum_{k\in m\setminus\bar{m}} g_k(x^0_{i_0})^3\Big)^{1/3} \le \sum_{k\in m\setminus\bar{m}} |g_k(x^0_{i_0})| < \frac{\varepsilon_4}{K_0}\cdot K_0 = \varepsilon_4 \ .$$

From $(*)$, (5) and our choice of λ_4,

$$\Big(\sum_{k\in m}\Big(g_k(x^0_{i_0}) - g_k\Big(\frac{1}{N}\sum_{n=1}^N x^n_{i_n}\Big)\Big)^3\Big)^{1/3} < \varepsilon_4 \ ,$$

and so, from (6) and the triangle inequality in ℓ_3,

$$(7) \qquad \Big(\sum_{k\in m\setminus\bar{m}} g_k\Big(\frac{1}{N}\sum_{n=1}^N x^n_{i_n}\Big)^3\Big)^{1/3} < 2\varepsilon_4 \ .$$

Thus, by (7) and (5),

$$(8) \qquad \Big(\sum_{k\in\bar{m}} g_k\Big(\frac{1}{N}\sum_{n=1}^N x^n_{i_n}\Big)^3\Big)^{1/3} > 1 - 2\delta_4 - 2\varepsilon_4 > \lambda_5 \, .$$

Now $\bar{m} \le K_0$ and each $n_k \le M$. So we have amongst $(g_k)_{k\in\bar{m}}$ at most K_0M terms of the form $\frac{1_E}{\sqrt{m}}$. We shall show that

$$(9) \quad \Big(\sum_{k\in\bar{m}} g_k\Big(\frac{1}{N}\sum_{n=1}^N x^n_{i_n}(1)\Big)^3\Big)^{1/3} + \Big(\sum_{k\in\bar{m}} g_k\Big(\frac{1}{N}\sum_{n=1}^N x^n_{i_n}(2)\Big)^3\Big)^{1/3} + \Big(\sum_{k\in\bar{m}} g_k\Big(\frac{1}{N}\sum_{n=1}^N x^n_{i_n}(3)\Big)^3\Big)^{1/3} < \lambda_5 \, ,$$

which will contradict (8). The second term is easiest to estimate, it is

$$\leq \frac{1}{N}\sum_{n=1}^{N}\left\|x^n_{i_n}(2)\right\| < \frac{1}{N}\sum_{n=1}^{N}\varepsilon_4 = \varepsilon_4 \ .$$

We next estimate the third term in (9). If for a term $\frac{1_E}{\sqrt{m_j}}$ of some g_k, $k \in \bar{m}$ we have $|\frac{1_E}{\sqrt{m_j}}(x^n_{i_n}(3))| \geq \gamma_n$, then $|\frac{1_E}{\sqrt{m_j}}(x^l_{i_l}(3))| \leq \gamma_l$ for $l \neq n$. Thus,

$$\Big|\frac{1_E}{\sqrt{m_j}}\Big(\frac{1}{N}\sum_{n=1}^{N}x^n_{i_n}(3)\Big)\Big| \leq \frac{1}{N}\Big(1+\sum_{j=1}^{N}\gamma_j\Big)$$

and therefore the third term in (9) is

$$\leq \frac{1}{N}(K_0M)\Big(1+\sum_{j=1}^{N}\gamma_j\Big) < \frac{2K_0M}{N} \ .$$

Finally, $\frac{1}{N}\sum_{n=1}^{N}x^n_{i_n}(1)$ consists of the vector $\frac{1}{N}x|_p$ repeated N times on disjoint blocks. Hence, its norm is less than or equal to twice the norm of the vector in Y which consists of $\frac{1}{N}$ repeated pN times. Since $\sum_{n\in\omega\setminus\{0\}}\frac{1}{\sqrt{m_n}} < 1$, this is at most $2(pN)^{1/3}/N < 1/8$. Thus, the left hand side of (9) is

$$\leq \frac{1}{8}+\varepsilon_4+\frac{2K_0M}{N} < \frac{1}{8}+\frac{1}{4}+\frac{1}{8} = \frac{1}{2} < \lambda_5$$

and we have a contradiction which completes the proof of Example 4.13. □

In summary, asymptotic models generalize spreading models. Certain positive theorems that one would like to have for spreading models are just not true. This was one motivation behind the development of asymptotic structures $\{X\}_n$ in [MMT95]. In that setting, the theorems are more complete, yet a sacrifice is made in that certain infinite dimensional structural properties are lost. Asymptotic models provide a somewhat fuller theory than spreading models, although some of the same deficiencies remain. They also provide a context in which some of the long outstanding problems in spreading models may prove tractable in this new setting (see Section 6.2 below for some of these problems). We believe that the stronger type of convergence one has in strong asymptotic models, as opposed to the convergence of arrays should enter into the solution of some of these problems.

5. Asymptotic Models Under Renormings

In this section we extend some of the results of [OS98$_2$] to the settings of asymptotic models. Information about the spreading models of a space X does not usually yield information about the subspace structure of X. For example, every $X \subseteq T$ (Tsirelson's space) has a spreading model 1-equivalent to the unit vector basis of ℓ_1, but T does not contain an isomorph of ℓ_1 [OS98$_1$]. But something can be said if one strengthens the hypothesis to include all equivalent norms as the following theorem of Th. Schlumprecht and the second named author illustrates.

Theorem 5.1. [OS98$_2$] *For every X there exists an equivalent norm $|\!|\!| \cdot |\!|\!|$ on X, so that we have: If $(X, |\!|\!| \cdot |\!|\!|)$ admits a spreading model $(e_n)_{n\in\omega}$ satisfying*

a) *$(e_n)_{n\in\omega}$ is 1-equivalent to the unit vector basis of c_0 (or even just $|\!|\!| e_0 + e_1 |\!|\!| = 1$, where $(e_n)_{n\in\omega}$ is generated by a weakly null sequence), then X contains an isomorph of c_0;*
b) *$(e_n)_{n\in\omega}$ is 1-equivalent to the unit vector basis of ℓ_1 (or even just $|\!|\!| e_0 \pm e_1 |\!|\!| = 2$), then X contains an isomorph of ℓ_1;*
c) *$(e_n)_{n\in\omega}$ is such that $|\!|\!| \sum_{i\in\omega} a_i e_i |\!|\!| = \sum_{i\in\omega} a_i$ for all $(a_i) \in c_{00}$ with $a_i \geq 0$ for $i \in \omega$ (or even just $|\!|\!| e_0 + e_1 |\!|\!| = 2$), then X is not reflexive.*

We shall develop an asymptotic model version of each part. Part of our construction will mirror that in [OS98$_2$], but we need some new tricks as well. We begin by recalling the construction of the equivalent norm $|\!|\!| \cdot |\!|\!|$ from [OS98$_2$].

For $c \in X$ and $x \in X$ define $\|x\|_c := \big\| c\|x\| + x \big\| + \big\| c\|x\| - x \big\|$, where $\| \cdot \|$ is the original norm on X. Then $\|x\|_c$ is an equivalent norm on X and in fact, for all $x \in X$, $2\|x\| \leq \|x\|_c \leq 2(1 + \|c\|)\|x\|$. Let C be a countable dense set in X and for $c \in C$ choose $p_c > 0$ so that $\sum_{c\in C} p_c(1+\|c\|)<\infty$. Define for $x\in X$,

$$\|\!|x|\!\|:=\sum_{c\in C} p_c\|x\|_c\,. \tag{10}$$

This is an equivalent norm on X. We call $\|\!|\cdot|\!\|$ the **asymptotic norm** generated by $\|\cdot\|$. We may assume $\|\!|x|\!\|\ge\|x\|$.

Theorem 5.2. *X contains an isomorph of c_0 if there exists a weakly null basic array $(x_i^n)_{n,i\in\omega}\subseteq X$ generating—in $(X,\|\!|\cdot|\!\|)$—an asymptotic model $(e_i)_{i\in\omega}$ which is 1-equivalent to the unit vector basis of c_0.*

Lemma 5.3. *Let $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$ be $\|\!|\cdot|\!\|$ normalized weakly null sequences in X with $\lim_{m\to\infty}\lim_{n\to\infty}\|\!|x_m+y_n|\!\|=1$. Then there exist integers $k(0)<k(1)<\cdots$ so that setting $a:=\lim_{m\to\infty}\|x_{k(m)}\|$ and $x'_m=x_{k(m)}/\|x_{k(m)}\|$, for all $y\in X$ we have*

$$\lim_{m\to\infty}\lim_{n\to\infty}\|y+x'_m+a^{-1}y_{k(n)}\|=\lim_{m\to\infty}\|y+x'_m\|\,. \tag{11}$$

Proof. By Ramsey's Theorem there exist $k(0)<k(1)<\cdots$ so that for all $y\in X$ and $\alpha,\beta\in\mathbb{R}$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\|y+\alpha x_{k(m)}+\beta y_{k(n)}\|\ \text{ exists.}$$

To simplify notation we write $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$ for $(x_{k(m)})_{m\in\omega}$ and $(y_{k(n)})_{n\in\omega}$ and thus $a:=\lim_{m\to\infty}\|x_m\|$. Now

$$\begin{aligned}1&=\lim_{m\to\infty}\lim_{n\to\infty}\|\!|x_m+y_n|\!\|=\lim_{m\to\infty}\lim_{n\to\infty}\sum_{c\in C}p_c\|x_m+y_n\|_c\\&=\lim_{m\to\infty}\sum_{c\in C}p_c\|x_m\|_c\,.\end{aligned}$$

Thus

$$1=\sum_{c\in C}p_c\Big(\lim_{m\to\infty}\lim_{n\to\infty}\|x_m+y_n\|_c\Big)=\sum_{c\in C}p_c\Big(\lim_{m\to\infty}\|x_m\|_c\Big)\,. \tag{12}$$

Since $y_n\to 0$ weakly, $\lim_{m\to\infty}\lim_{n\to\infty}\|x_m+y_n\|_c\ge\lim_{m\to\infty}\|x_m\|_c$ for all $c\in C$. From this and (12) we get

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty}\|x_m+y_n\|_c\\&\qquad=\lim_{m\to\infty}\|x_m\|_c\text{ for all }c\in X\text{ (since }C\text{ is dense).}\end{aligned} \tag{13}$$

Letting $c=0$, this yields

$$\lim_{m\to\infty}\lim_{n\to\infty}\|x_m+y_n\|=a\ .$$

Thus, for all $y\in X$,

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty}&\big[\|ay+x_m+y_n\|+\|-ay+x_m+y_n\|\big]\\ &=\lim_{m\to\infty}\big[\|ay+x_m\|+\|-ay+x_m\|\big]\ .\end{aligned}\tag{14}$$

Again, since $(y_n)_{n\in\omega}$ is weakly null,

$$\begin{gathered}\lim_{n\to\infty}\|ay+x_m+y_n\|\geq\|ay+x_m\|\ \text{and}\\ \lim_{n\to\infty}\|-ay+x_m+y_n\|\geq\|-ay+x_m\|\quad\text{for}\quad m\in\omega\ .\end{gathered}$$

Thus, by (14), for all $y\in X$ we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\|ay+x_m+y_n\|=\lim_{m\to\infty}\|ay+x_m\|$$

which completes the proof. $\square$

Note that it follows from Lemma 5.3 that if $\lim_{m\to\infty}\lim_{n\to\infty}\|x_m\pm y_n\|=1$, then for all $y\in X$ we can obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\|y\pm x'_m\pm a^{-1}y_{k(n)}\|=\lim_{m\to\infty}\|y\pm x'_m\|\tag{15}$$

for all choices of sign (keeping the sign of x'_m the same on both sides of (15)).

Proof of Theorem 5.2. By passing to a subarray of $(x_i^n)_{n,i\in\omega}$ we may assume that for each $n\in\omega$ we have $\lim_{i\to\infty}\|x_i^n\|=a_n$ (for some a_n). Let $\varepsilon_n\downarrow 0$ with $\sum_{n\in\omega}\varepsilon_n<\infty$. By passing to a subsequence of the rows we may assume that for all n, $a_n\to a>0$, $|\frac{1}{a_n}-\frac{1}{a}|<\frac{\varepsilon_n}{3}$ and $a_n>a/2$. In addition we may assume that for all $y\in X$, $\alpha,\beta\in\mathbb{R}$ and $i,j\in\omega$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\|y+\alpha x_m^i+\beta x_n^j\|$$

exists, and moreover, by Lemma 5.3 (actually (15)) we may assume that for $y\in X$ and $p,q\in\omega$ with $p<q$ we have

$$\lim_{i\to\infty}\lim_{j\to\infty}\left\|y\pm\frac{x_i^p}{a_p}\pm\frac{x_j^q}{a_p}\right\|=\lim_{i\to\infty}\left\|y\pm\frac{x_i^p}{a_p}\right\|\ .$$

Hence, from the triangle inequality using $|\frac{1}{a} - \frac{1}{a_n}| < \frac{\varepsilon_n}{3}$ we get

$$\lim_{i\to\infty}\lim_{j\to\infty}\left\|y \pm \frac{x_i^p}{a} \pm \frac{x_j^q}{a}\right\| < \lim_{i\to\infty}\left\|y + \frac{x_i^p}{a}\right\| + \varepsilon_i \ . \tag{16}$$

By passing to another subarray and setting $x_i = \frac{x_i^i}{a}$ for $i \in \omega$ we may assume that for all $m \in \omega$ and $y \in 2mB_{\langle x_i\rangle_{i\in m}}$,

$$\|y \pm x_m \pm x_{m+1}\| < \|y \pm x_m\| + 2\varepsilon_m \ . \tag{17}$$

This is accomplished using (16). If i is large enough and $i < j$, then $\|\frac{x_i^0}{a} \pm \frac{x_j^1}{a}\| < \|\frac{x_0^i}{a}\| + \varepsilon_0$. This fixes i and $x_0 = x_i^0/a$ (under relabeling) and then we increase j large enough so that for $j < k$ and $y \in 2B_{\langle x_0\rangle}$,

$$\left\|y \pm \frac{x_j^1}{a} \pm \frac{x_k^2}{a}\right\| < \left\|y + \frac{x_j^1}{a}\right\| + \varepsilon_1 \ .$$

This fixes j and $x_1 = x_j^1/a$ (under relabeling) and so on.

We claim that

$$\sup\left\{\left\|\sum_{i\in m} \pm x_i\right\| : \text{ all choices of } \pm\right\} < \infty \ ,$$

which will yield the theorem ($(x_i)_{i\in\omega}$ is then equivalent to the unit vector basis of c_0). Indeed, from (16) we get

$$\begin{aligned}\left\|\sum_{i\in m} \pm x_i\right\| &\le \left\|\sum_{i\in m-1} \pm x_i\right\| + 2\varepsilon_{m-2} \\ &\le \cdots \le \|x_0\| + \sum_{m\in\omega} 2\varepsilon_m < \infty \ . \quad \square\end{aligned}$$

Remark 5.4. In the proof of Theorem 5.2 we only used

$$\lim_{m\to\infty}\lim_{n\to\infty}\|x_m^p \pm x_n^q\| = 1$$

for all $p < q$. In other words $|||e_p \pm e_q||| = 1$ for $p \neq q$. In the case of spreading models (Theorem 5.1(a)) one only needs $|||e_p + e_q||| = 1$ for $p \neq q$. We do not know if this is sufficient to obtain c_0 inside X for asymptotic models.

The proof of Theorem 5.2 was the most similar to the spreading model analogue of the three results we present in this section. Our next proof is more difficult.

Theorem 5.5. *For every separable infinite dimensional Banach space X, there exists an equivalent norm $|||\cdot|||$ on X with the following property. If there exist $|||\cdot|||$-normalized basic sequences $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$ with* $\lim_{m\to\infty}\lim_{n\to\infty}|||x_m+y_n|||=2$*, then X is not reflexive.*

Corollary 5.6. *X is reflexive if and only if there exists an equivalent norm $|||\cdot|||$ on X such that if $(e_n)_{n\in\omega}$ is an asymptotic model of $(X,|||\cdot|||)$, then $|||e_0+e_1|||<2$.*

Proof of Theorem 5.5. We first construct the norm $|||\cdot|||$ on X. We begin by assuming that $X=\langle x_0\rangle\oplus_\infty Y$ where Y is a subspace of a Banach space with a bimonotone normalized basis (d_i) and we let $)\!\!)\cdot(\!\!($ be the inherited norm on Y. We assume the norm $\|\cdot\|$ on X is given as follows. If $x=ax_0+y\in X$ with $a\in\mathbb{R}$ and $y\in Y$, then $\|x\|=\max(|a|,)\!\!)y(\!\!(+\sum_{i\in\omega}|y(i)|2^{-i})$ if $y=\sum_{i\in\omega}y(i)d_i$. We have the following:

(18) Let $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$ be weakly null $\|\cdot\|$ normalized sequences in X.
Let $\alpha+\beta=1$, $\alpha,\beta>0$ and $\alpha\neq\frac{1}{2}$.
Then $\lim_{m\to\infty}\lim_{n\to\infty}\|x_0+\frac{1}{2}x_m+\frac{1}{2}y_n\|=1$ while

$$\lim_{m\to\infty}\left\|\alpha x_0+\tfrac{1}{2}x_m\right\|+\lim_{n\to\infty}\left\|\beta x_0+\tfrac{1}{2}y_n\right\|$$
$$=\max\left(\alpha,\tfrac{1}{2}\right)+\max\left(\beta,\tfrac{1}{2}\right)=\tfrac{1}{2}+\max(\alpha,\beta)>1\ .$$

(19) Let $y\in Y$, $y\neq0$ and let $(x_m)_{m\in\omega}$ be a $\|\cdot\|$-normalized weakly null sequence in X.
Then, presuming the limit exists,

$$\lim_{n\to\infty}\|y+x_n\|\geq1+\sum_{i\in\omega}2^{-i}|y(i)|>1\ .$$

Let $\|\!|\cdot|\!\|$ be the asymptotic norm on X generated by $\|\cdot\|$ (see (10) above), and let $|||\cdot|||$ be the equivalent asymptotic norm on X generated by $\|\!|\cdot|\!\|$. $\square$

Before proceeding we present a lemma. The lemma is valid in any $(X,\|\cdot\|)$, not just in our space above.

Lemma 5.7. *Let $|||\cdot|||$ be the equivalent asymptotic norm on $(X, \|\cdot\|)$ generated by $\|\cdot\|$ as in* (10). *Let $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$ be $|||\cdot|||$-normalized sequences in X.*

a) *If $\lim_{m\to\infty}\lim_{n\to\infty} |||x_m + y_n||| = 2$, then there exist integers $k(0) < k(1) < \cdots$ so that setting $x'_m = x_{k(m)}/\|x_{k(m)}\|$ and $y'_n = y_{k(n)}/\|y_{k(n)}\|$, then for all $y \in Y$ and $\beta_1, \beta_2 \geq 0$ (not both 0) we have*

$$\lim_{m\to\infty}\lim_{n\to\infty} \|y + \beta_1 x'_m + \beta_2 y'_n\| = \lim_{m\to\infty} \left\| \frac{\beta_1}{\beta_1+\beta_2} y + \beta_1 x'_m \right\| + \lim_{n\to\infty} \left\| \frac{\beta_2}{\beta_1+\beta_2} y + \beta_2 y'_n \right\| . \tag{20}$$

b) *If $\lim_{m\to\infty}\lim_{n\to\infty} |||x_m \pm y_n||| = 2$, then there exist integers $k(0) < k(1) < \cdots$ so that setting $x'_m = x_{k(m)}/\|x_{k(m)}\|$ and $y'_n = y_{k(n)}/\|y_{k(n)}\|$, then for all $y \in X$, $\beta_1, \beta_2 \in \mathbb{R}$ (not both 0) we have*

$$\begin{aligned} &\lim_{m\to\infty}\lim_{n\to\infty} \|y + \beta_1 x'_m + \beta_2 y'_n\| \\ &= \lim_{m\to\infty} \left\| \frac{|\beta_1|}{|\beta_1|+|\beta_2|} y + \beta_1 x'_m \right\| \\ &\quad + \lim_{n\to\infty} \left\| \frac{|\beta_2|}{|\beta_1|+|\beta_2|} y + \beta_2 y'_n \right\| . \end{aligned} \tag{21}$$

Proof. Again by Ramsey's Theorem we can find $k(0) < k(1) < \cdots$ so that relabeling $x_{k(m)} = x_m$ and $y_{k(n)} = y_n$, $\lim_{m\to\infty}\lim_{n\to\infty} \|y + \alpha x_m + \beta y_n\|$ exists for all $y \in X$ and $\alpha, \beta \in \mathbb{R}$. Let $a = \lim_{m\to\infty} \|x_m\|$, $b = \lim_{n\to\infty} \|y_n\|$ and let $x'_m = x_m/a$, $y'_n = y_n/b$. We will prove the conclusion of the lemma for these sequences which will yield the lemma.

a) We first suppose that $\beta_1 + \beta_2 = 1$. Set $\bar\beta_1 = \beta_1/a$, $\bar\beta_2 = \beta_2/b$. From our hypothesis,

$$\lim_{m\to\infty}\lim_{n\to\infty} |||\bar\beta_1 x_m + \bar\beta_2 y_n||| = \bar\beta_1 + \bar\beta_2 \ .$$

From the definition of $|||\cdot|||$ and the triangle inequality in each $\|\cdot\|_c$ we obtain for $c \in C$,

$$\lim_{m\to\infty}\lim_{n\to\infty} \|\bar\beta_1 x_m + \bar\beta_2 y_n\|_c = \lim_{m\to\infty} \|\bar\beta_1 x_m\|_c + \lim_{n\to\infty} \|\bar\beta_2 y_n\|_c \, . \tag{22}$$

By the density of C in X this holds for all $c \in X$.

Setting $c = 0$ in (22) yields

$$\lim_{m\to\infty}\lim_{n\to\infty} \|\beta_1 x'_m + \beta_2 y'_n\| = \beta_1 + \beta_2 = 1 \ . \tag{23}$$

From (22), using (23), for all $c \in X$,

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} &\Big[\big\|c + \beta_1 x'_m + \beta_2 y'_n\big\| + \big\|c - (\beta_1 x'_m + \beta_2 y'_n)\big\|\Big] \\ &= \lim_{m\to\infty} \Big[\big\|\beta_1 c + \beta_1 x'_m\big\| + \big\|\beta_1 c - \beta_1 x'_m\big\|\Big] \\ &\quad + \lim_{n\to\infty} \Big[\big\|\beta_2 c + \beta_2 y'_n\big\| + \big\|\beta_2 c - \beta_2 y'_n\big\|\Big] \ .\end{aligned} \tag{24}$$

From (24) and the triangle inequality we obtain (20) in the case $\beta_1 + \beta_2 = 1$. To get the general case from this we note that for $y \in X$, $\beta_1, \beta_2 \in \mathbb{R}$ (not both 0) we have

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} &\left\|\frac{y}{\beta_1+\beta_2} + \frac{\beta_1}{\beta_1+\beta_2} x'_m + \frac{\beta_2}{\beta_1+\beta_2} y'_n\right\| \\ &= \lim_{m\to\infty} \left\|\frac{\beta_1}{\beta_1+\beta_2}\Big(\frac{y}{\beta_1+\beta_2}\Big) + \frac{\beta_1}{\beta_1+\beta_2} x'_m\right\| \\ &\quad + \lim_{n\to\infty} \left\|\frac{\beta_2}{\beta_1+\beta_2}\Big(\frac{y}{\beta_1+\beta_2}\Big) + \frac{\beta_2}{\beta_1+\beta_2} y'_n\right\|\end{aligned}$$

and (20) follows by multiplying by $\beta_1 + \beta_2$.

b) We continue the argument from a). As in that case we may assume that $|\beta_1| + |\beta_2| = 1$. The case $\beta_1, \beta_2 \le 0$ is covered by a) using

$$\lim_{m\to\infty}\lim_{n\to\infty} \|y + \beta_1 x'_m + \beta_2 y'_n\| = \lim_{m\to\infty}\lim_{n\to\infty} \| - y - \beta_1 x'_m - \beta_2 y'_n\| \ .$$

Similarly, the only case left to consider is $\beta_1 > 0$ and $\beta_2 < 0$. We prefer to take $\beta_1, \beta_2 > 0$, $\beta_1 + \beta_2 = 1$ and work with "$\beta_1 x'_m - \beta_2 y'_n$". As in a), we obtain from the hypothesis for $c \in X$,

$$\lim_{m\to\infty}\lim_{n\to\infty} \|\beta_1 x'_m - \beta_2 y'_n\|_c = \lim_{m\to\infty} \|\beta_1 x'_m\|_c + \lim_{n\to\infty} \|\beta_2 y'_n\| \ .$$

Thus, for $y \in X$ we get

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty}&\Big[\big\|y+(\beta_1 x'_m-\beta_2 y'_n)\big\|+\big\|y-(\beta_1 x'_m-\beta_2 y'_n)\big\|\Big]\\ &=\lim_{m\to\infty}\Big[\big\|\beta_1 y+\beta_1 x'_m\big\|+\big\|\beta_1 y-\beta_1 x'_m\big\|\Big]\\ &\quad+\lim_{n\to\infty}\Big[\big\|\beta_2 y-\beta_2 y'_n\big\|+\big\|\beta_2 y+\beta_2 y'_n\big\|\Big]\ .\end{aligned}$$

Again from the triangle inequality we obtain (21) in this case. □

We return to the proof of Theorem 5.5. Suppose that $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$ are $|||\cdot|||$ normalized basic sequences in X with

$$\lim_{m\to\infty}\lim_{n\to\infty}|||x_m+y_n|||=2\ .$$

Assume towards a contradiction that X is reflexive. Then $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$ are both weakly null. We may assume that

$$\lim_{m\to\infty}\lim_{n\to\infty}\|y+\alpha x_m+\beta y_n\|$$

exists for all $y\in X$, $\alpha,\beta\in\mathbb{R}$ (and for all of the norms we have constructed). By Lemma 5.7 we may also assume that setting $x'_m = x_m/|||x_m|||$ and $y'_n = y_n/|||y_n|||$, for $y\in X$ and $\alpha,\beta\geq 0$ (not both 0) we have

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty}|||y+\alpha x'_m+\beta y'_n|||\\ &\qquad=\lim_{m\to\infty}\Big|\Big|\Big|\frac{\alpha}{\alpha+\beta}y+\alpha x'_m\Big|\Big|\Big|+\lim_{n\to\infty}\Big|\Big|\Big|\frac{\beta}{\alpha+\beta}y+\beta y'_n\Big|\Big|\Big|\ .\end{aligned}$$

Thus,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{c\in C}p_c\Big[\Big\|c\|y+\alpha x'_m+\beta y'_n\|+y+\alpha x'_m+\beta y'_n\Big\|$$
$$+\Big\|c\|y+\alpha x'_n+\beta y'_n\|-(y+\alpha x'_m+\beta y'_n)\Big\|\Big]$$
$$=\lim_{m\to\infty}\sum_{c\in C}p_c\Big[\Big\|c\|\frac{\alpha}{\alpha+\beta}y+\alpha x'_m\|+\frac{\alpha}{\alpha+\beta}y+\alpha x'_m\Big\|$$
$$+\Big\|c\|\frac{\alpha}{\alpha+\beta}y+\alpha x'_m\|-\Big(\frac{\alpha}{\alpha+\beta}y+\alpha x'_m\Big)\Big\|\Big]$$
$$+\lim_{n\to\infty}\sum_{c\in C}p_c\Big[\Big\|c\|\frac{\beta}{\alpha+\beta}y+\beta y'_n\|+\frac{\beta}{\alpha+\beta}y+\beta y'_n\Big\|$$
$$+\Big\|c\|\frac{\beta}{\alpha+\beta}y+\beta y'_n\|-\Big(\frac{\beta}{\alpha+\beta}y+\beta y'_n\Big)\Big\|\Big]\ .$$

From this and the triangle inequality we have for all $c\in X$, $y\in X$ and $\alpha,\beta\geq 0$ (not both 0) that

$$\lim_{m\to\infty}\lim_{n\to\infty}\big\|c\|y+\alpha x'_m+\beta y'_n\|+y+\alpha x'_m+\beta y'_n\big\| \tag{25}$$
$$=\lim_{m\to\infty}\Big\|c\|\frac{\alpha}{\alpha+\beta}y+\alpha x'_m\|+\frac{\alpha}{\alpha+\beta}y+\alpha x'_m\Big\|$$
$$+\lim_{n\to\infty}\Big\|c\|\frac{\beta}{\alpha+\beta}y+\beta y'_n\|+\frac{\beta}{\alpha+\beta}y+\beta y'_n\Big\|\ .$$

Setting $c=y=0$ in (25) yields

$$\lim_{m\to\infty}\lim_{n\to\infty}\|\alpha x'_m+\beta y'_n\|=\alpha a+\beta b\ , \tag{26}$$

where $a=\lim_m\|x'_m\|$ and $b=\lim_n\|y'_n\|$. Let $x''_m=x'_m/\|x'_m\|$ and $y''_n=y'_n/\|y'_n\|$. Then

$$\lim_{m\to\infty}\lim_{n\to\infty}\|\alpha x''_m+\beta y''_n\|=\alpha+\beta\ . \tag{27}$$

Letting $y=0$ and replacing c by $\frac{c}{\alpha+\beta}$ in (25), using (27), we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\|c+\alpha x''_m+\beta y''_n\| \tag{28}$$
$$=\lim_{m\to\infty}\Big\|\frac{\alpha}{\alpha+\beta}c+\alpha x''_m\Big\|+\lim_{n\to\infty}\Big\|\frac{\beta}{\alpha+\beta}c+\beta y''_n\Big\|\ .$$

We claim that $a = b$. Indeed, let us assume $a \neq b$. By (27) we get $\lim_{m\to\infty} \lim_{n\to\infty} \|\frac{1}{2}x''_m + \frac{1}{2}y''_n\| = 1$ and further we have $\lim_{m\to\infty}\lim_{n\to\infty} \|x_0 + \frac{1}{2}x''_m + \frac{1}{2}y''_n\| = 1$, see (18). But from (25), taking $y = x_0$ and $c = 0$, we get

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty} \left\|x_0 + \tfrac{1}{2}x''_m + \tfrac{1}{2}y''_n\right\| &= \lim_{m\to\infty}\lim_{n\to\infty} \left\|x_0 + \tfrac{1}{2a}x'_m + \tfrac{1}{2b}y'_n\right\| \\
&= \lim_{m\to\infty} \left\| \frac{\frac{1}{2a}}{\frac{1}{2a}+\frac{1}{2b}} x_0 + \tfrac{1}{2}x''_m \right\| + \lim_{n\to\infty} \left\| \frac{\frac{1}{2b}}{\frac{1}{2a}+\frac{1}{2b}} x_0 + \tfrac{1}{2}y''_n \right\| \\
&> 1 \quad (\text{for } a \neq b) \text{ using (18).}
\end{aligned}$$

From (25) we obtain for all $c, y \in X$ and $\alpha, \beta > 0$ (not both 0), by replacing α, β by α/a and β/a since $\beta/b = \beta/a$,

$$\begin{aligned}
(29)\qquad \lim_{m\to\infty}\lim_{n\to\infty} &\left\| c\|y + \alpha x''_m + \beta y''_n\| + y + \alpha x''_m + \beta y''_n \right\| \\
&= \lim_{m\to\infty} \left\| c\left\|\frac{\alpha}{\alpha+\beta}y + \alpha x''_m\right\| + \frac{\alpha}{\alpha+\beta}y + \alpha x''_m \right\| \\
&\quad = \lim_{n\to\infty} \left\| c\left\|\frac{\beta}{\alpha+\beta}y + \beta y''_n\right\| + \frac{\beta}{\alpha+\beta}y + \beta y''_n \right\| .
\end{aligned}$$

Next, we wish to show that $(x''_m)_{m\in\omega}$ and $(y''_n)_{n\in\omega}$ generate the same type over Y, i.e., if $y \in Y$, $\delta := \lim_{m\to\infty} \|y + x''_m\|$ and $\gamma := \lim_{n\to\infty} \|y + y''_m\|$, then $\delta = \gamma$. Clearly, $\delta = \gamma = 1$ if $y = 0$, so assume $y \neq 0$ and $\delta \neq \gamma$. Let $\alpha + \beta = 1$. Now from (28) we get

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty} \|y + \alpha x''_m + \beta y''_n\| &= \lim_{m\to\infty} \|\alpha y + \alpha x''_m\| + \lim_{n\to\infty} \|\beta y + \beta y''_n\| \\
&= \alpha\delta + \beta\gamma .
\end{aligned}$$

Thus, from (29) we get for $c \in X$, $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

$$\begin{aligned}
(30)\qquad \lim_{m\to\infty}\lim_{n\to\infty} &\|c(\alpha\delta + \beta\gamma) + y + \alpha x''_m + \beta y''_n\| \\
&= \lim_{m\to\infty} \|(\alpha\delta)c + \alpha y + \alpha x''_m\| \\
&\quad + \lim_{n\to\infty} \|(\beta\gamma)c + \beta y + \beta y''_n\| .
\end{aligned}$$

Let $\alpha = \beta = \frac{1}{2}$ and $c = \frac{-1}{\frac{\gamma}{2}+\frac{\delta}{2}} y = \frac{-2y}{\delta+\gamma}$. Using this in (30), from (27) we have

$$\begin{aligned} \lim_{m\to\infty} \lim_{n\to\infty} \Big\| \frac{1}{2} x''_m + \frac{1}{2} y''_n \Big\| &= 1 = \\ = \lim_{m\to\infty} \Big\| \Big(\frac{1}{2} - \frac{\delta}{\delta+\gamma}\Big) y + \frac{1}{2} x''_m \Big\| \\ + \lim_{n\to\infty} \Big\| \Big(\frac{1}{2} - \frac{\gamma}{\delta+\gamma}\Big) y + \frac{1}{2} y''_n \Big\| \end{aligned} \tag{31}$$

and since $\delta \neq \gamma$, both coefficients of $y \in Y$ on the right side of (31) are nonzero. Therefore, by (19), the right side exceeds 1, a contradiction.

It follows that $\lim_{m\to\infty} \lim_{n\to\infty} \|x''_m + x''_n\| = 2$ and moreover, $(x''_n)_{n\in\omega}$ can be substituted for $(y''_n)_{n\in\omega}$ in our above equations. So, we are in the same situation as the proof of Theorem 4.1 c) in [OS98$_2$] and it follows that for some subsequence $(x''_{n_i})_{i\in\omega}$,

$$\Big\| \sum_{i\in\omega} a_i x''_{n_i} \Big\| > \frac{1}{2} \ \text{ if } \ (a_i)_{i\in\omega} \subseteq [0,\infty)\ , \qquad \sum_{i\in\omega} a_i = 1\ .$$

Hence, $(x''_{n_i})_{i\in\omega}$ is not weakly null and X is not reflexive, which completes the proof of Theorem 5.5. $\square$

Theorem 5.8. *Let X have a basis $(b_i)_{i\in\omega}$. There exists an equivalent norm $|||\cdot|||$ on X so that if $(X, |||\cdot|||)$ admits $|||\cdot|||$ normalized block bases of $(b_i)_{i\in\omega}$, say $(x_m)_{m\in\omega}$ and $(y_n)_{n\in\omega}$, satisfying $\lim_{m\to\infty} \lim_{n\to\infty} |||x_m \pm y_n||| = 2$, then X contains an isomorph of ℓ_1.*

Corollary 5.9. *If X has a basis and does not contain an isomorph of ℓ_1, then X can be given an equivalent norm so that if $(e_n)_{n\in\omega}$ is any asymptotic model generated by a block basic array, then $(e_n)_{n\in\omega}$ is not 1-equivalent to the unit vector basis of ℓ_1.*

Proof of Theorem 5.8. The norm $|||\cdot|||$ is constructed as in the proof of Theorem 5.5 where we begin with $X = \langle b_0 \rangle \oplus_\infty [(b_i)]_{i\in\omega\setminus\{0\}}$ and $(b_i)_{i\in\omega}$ is bimonotone. Everything we did in the proof of Theorem 5.5 remains valid and in addition we have the use of (21). It follows that not only do $(x''_m)_{m\in\omega}$ and $(y''_n)_{n\in\omega}$ generate the same type over Y, but so do $(x''_m)_{m\in\omega}$ and $(-y''_n)_{n\in\omega}$ and thus, as in the case of

Theorem 5.5, the proof reduces to the situation in [OS98$_2$]. Hence, some subsequence of $(x''_m)_{m\in\omega}$ is an ℓ_1 basis. $\square$

The arguments easily generalize to the case where X is a subspace of a space with a basis $(b_m)_{m\in\omega}$, and $(e_n)_{n\in\omega}$ is generated by an array $(x^n_i)_{n,i\in\omega}$, where for all n, m:

$$\lim_{i\to\infty} b^*_m(x^n_i) = 0 \ .$$

6. Odds and Ends

In this section we first consider some stronger versions of convergence one might hope for but, as we shall see, one cannot always achieve. We also raise a number of open questions.

6.1. **Could We Get More?** There are very many possible strengthenings of asymptotic models that one could hope for. One such question is as follows:

Suppose we are given a normalized basic sequence $(y_i)_{i\in\omega}$ and $(\boldsymbol{a}^i)_{i\in\omega}$. *Does there exist a subsequence $(x_i)_{i\in\omega}$ of $(y_i)_{i\in\omega}$ with the following property: for all $n \in \omega$, $(b_i)_{i\in n} \in [-1,1]^n$ and $\varepsilon > 0$, there is an $N \in \omega$ so that if $N \le j_0 < \cdots < j_{n-1}$, $N \le k_0 < \cdots < k_{n-1}$ are integers and $Q \in \langle\omega\rangle^\omega$, then*

$$\left| \Big\| \sum_{i\in n} b_i x\big(Q(j_i), \boldsymbol{a}^i\big) \Big\| - \Big\| \sum_{i\in n} b_i x\big(Q(k_i), \boldsymbol{a}^i\big) \Big\| \right| < \varepsilon \ ?$$

Indeed, this is true if for each $i \in \omega$, $\boldsymbol{a}^i$ is finitely supported, for one can then take $(x_i)_{i\in\omega}$ to be a subsequence of $(y_i)_{i\in\omega}$ generating a spreading model $(\tilde{e}_i)_{i\in\omega}$. The limit will exists in the above sense (it will be just $\| \sum_{i\in n} b_i \tilde{f}_i \|$ where $(\tilde{f}_i)_{i\in\omega}$ is the normalized block basis of $(\tilde{e}_i)_{i\in\omega}$ determined by the $\boldsymbol{a}^i$'s).

In general, however, this is false, even if $(y_i)_{i\in\omega}$ is weakly null and $\boldsymbol{a}^i = \boldsymbol{a}$ for all $i \in \omega$ and some $\boldsymbol{a}$. Indeed (cf. [LT77, p. 123]) one can embed $\ell_p \oplus \ell_2$ ($p \ne 2$) into a space Y with a normalized symmetric basis $(y_i)_{i\in\omega}$ in such a way that the unit vector basis of $\ell_p \oplus \ell_2$ is equivalent to a normalized block basis of the form $(y(P(i), \boldsymbol{a}))_{i\in\omega}$ where $|P(i)| \to \infty$ and $\boldsymbol{a} = (1, 1, 1, \ldots)$. Thus, for appropriate $Q_1, Q_2 \in \langle\omega\rangle^\omega$ with $|Q_1(i)|, |Q_2(i)| \to \infty$, every subsequence $(x_i)_{i\in\omega}$

of $(y_i)_{i\in\omega}$ contains block bases $\big(x(Q_1(i),\boldsymbol{a})\big)_{i\in\omega}$ and $\big(x(Q_2(i),\boldsymbol{a})\big)_{i\in\omega}$ which are equivalent to the unit vector basis of ℓ_p and ℓ_2 respectively.

On the other hand, there are of course variations of our construction of asymptotic models in Theorem 4.3 that do succeed. For example, given a basic array $(x_i^n)_{n,i\in\omega}$, one might stabilize

$$\Big\| \sum_{i\in n} b_i x^{k(i,P)} \boldsymbol{a}_{P(i)}^{k(i,P)} \Big\|$$

where the row now depends upon i and $P \in \langle\omega\rangle^\omega$. In this more general setting, one has that $(e_i)_{i\in n} \in \{X_n\}$ iff there exists a block basic array $(x_i^n)_{n,i\in\omega}$ and $k(i,P)$, $\boldsymbol{a}_{P(i)}^j$'s, so that the above expression converges (as in Theorem 4.3) to $\|\sum_{i\in n} b_i e_i\|$.

Indeed, suppose for example that the tree $T_2 = \{x_{(m_0,m_1)} : 0 \le m_0 < m_1\}$ converges to (e_1,e_2) as in (4.7.3). Let $x_i^0 = x_{(i)}$, $x_i^1 = x_{(0,i)}$ for $i > 0$, $x_i^2 = x_{(1,i)}$ for $i > 1$, and so on. (Notice that there is no need to define the first part of each row.) Set $k(0,P) := 0$ and $k(1,P) := j+1$ if $\min P(0) = j$, and let $\boldsymbol{a}_{P(i)}^j := (1,0,0,\dots)$.

One could also relax the conditions defining a basic array (x_i^n) by deleting the requirement that the rows be K-basic. This would yield many more "asymptotic models." For example every normalized basic sequence (x_i) in X would be an "asymptotic model" of X; take $(x_i^n) = (x_i)$ for all n. Proposition 4.5 would also hold in this relaxed setting.

6.2. **Open Problems.**

Problem 6.1. X is **asymptotic** ℓ_p (respectively, **asymptotic** c_0) if there exists K so that for all $(e_i)_{i\in n} \in \{X\}_n$, $(e_i)_{i\in n}$ is K-equivalent to the unit vector basis of ℓ_p^n (respectively, ℓ_∞^n) (see [MMT95]). Assume that there exists K and $1 \le p \le \infty$ so that if $(e_i)_{i\in\omega}$ is an asymptotic model of X, then $(e_i)_{i\in\omega}$ is K-equivalent to the unit vector basis of ℓ_p (c_0, if $p=\infty$). *Does X contain an asymptotic ℓ_p (or c_0) subspace?* The analogous problem for spreading models is also open.

Problem 6.2. Suppose X has a basis and that there is a unique, in the isometric sense, asymptotic model for all normalized block basic arrays. In this case, even if one replaces asymptotic model by spreading model, it follows from Krivine's Theorem [Kr76] that this unique asymptotic model is 1-equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \le p < \infty$. *Must X contain an isomorphic copy of this space?* The analogous problem for spreading models is known to be true for the case of c_0 and ℓ_1 (see [OS98$_2$]). Also the asymptotic structure version of the question is true: if $|\{X\}_2| = 1$, then X contains an isomorphic copy of c_0 or ℓ_p (see [MMT95]).

Problem 6.3. *Can one stabilize the asymptotic models of a space X?* Precisely, does there exist a basic sequence $(x_i)_{i\in\omega}$ in X so that for all block bases $(y_i)_{i\in\omega}$ of $(x_i)_{i\in\omega}$, if $(e_i)_{i\in\omega}$ is an asymptotic model of some normalized block basic array of $(x_i)_{i\in\omega}$, then $(e_i)_{i\in\omega}$ is equivalent to an asymptotic model of a normalized block basic array of $(y_i)_{i\in\omega}$? We do not even know if there is some basic sequence $(x_i)_{i\in\omega}$ and an asymptotic model $(e_i)_{i\in\omega}$ of $(x_i)_{i\in\omega}$ such that every block basis $(y_i)_{i\in\omega}$ of $[(x_i)_{i\in\omega}]$ admits an asymptotic model equivalent to $(e_i)_{i\in\omega}$. The analogous questions for spreading models are open. It is known that one can stabilize the asymptotic structures $\{X\}_n$ for all $n \in \omega$ by passing to a block basis (see [MMT95]).

Problem 6.4. Assume that in X, every asymptotic model $(e_i)_{i\in\omega}$ of any normalized basic block sequence is 1-unconditional (this is $\|\sum \pm a_i e_i\| = \|\sum a_i e_i\|$). *Does X contain an unconditional basic sequence? Does X contain an asymptotically unconditional subspace?* (i.e., a basic sequence $(x_i)_{i\in\omega}$ so that for some $K < \infty$ and for all $n \in \omega$, every block basis $(y_i)_{i\in n}$ of $(x_i)_{i\in\omega\setminus n}$ is K-unconditional).

Problem 6.5. *For any space X, does there exist a finite chain of asymptotic models $X = X_0, X_1, \ldots, X_n$, so that X_{i+1} is an asymptotic model of X_i (for $i \in n$) and X_n is isomorphic to c_0 or ℓ_p for some $1 \le p < \infty$?* The analogous problem for spreading models is also open.

Problem 6.6. For $1 < p < \infty$, ℓ_p is arbitrarily distortable [OS94]: Given $K > 1$ there exists an equivalent norm $\|\cdot\|$ on ℓ_p so that for all $X \subseteq \ell_p$, $(X, \|\cdot\|)$ is not K-isomorphic to ℓ_p. Is this true for asymptotic models as well? *Given $K > 1$ (or for even some $K > 1$)*

does there exist an equivalent norm $\|\cdot\|$ on ℓ_p so that if $(e_i)_{i\in\omega}$ is an asymptotic model of $(\ell_p, \|\cdot\|)$, then $(e_i)_{i\in\omega}$ is not K-equivalent to the unit vector basis of ℓ_p ? The analogue for spreading models is also open.

Problem 6.7. *If X has the property that every normalized bimonotone basic sequence is an asymptotic model of X, does X contain an isomorphic copy of c_0 ?*

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