The general counterfeit coin problem

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Dedicated to the 60^{th} *birthday of Prof. Hans Läuchli*

Abstract Given c nickels among which there may be a counterfeit coin, which can only be told apart by its weight being different from the others, and moreover b balances. What is the minimal number of weighings to decide whether there is a counterfeit nickel, if so which one it is and whether it is heavier or lighter than a genuine nickel. We give an answer to this question for sequential and nonsequential strategies and we will consider the problem of more than one counterfeit coins.

1 Introduction

There is a well known problem of which one version reads like this: a man has twelve nickels among which there may be a counterfeit coin, which can only be told apart by its weight being different from the others. How can one tell in not more than three weighings whether there is a counterfeit nickel, if so which one it is and whether it is heavier or lighter than a genuine nickel. The balance we are allowed to use only gives the information whether two masses have the same weight or if not which one is heavier or lighter.
We generalize this problem in three directions:

(P1) What is the minimal number of weighings to decide c coins?

(P2) What is the minimal number of weighings to decide c coins when we are allowed to use $b \geq 1$ balances? This means that we may distribute the set of our coins on b balances to get b informations in one weighing. Note that this procedure differs from doing b weighings on one balance one after another.

(P3) What are the optimal strategies when more than one coin is counterfeit?

We will distinguish between *sequential* and *nonsequential* strategies:

- A strategy to decide a certain number of coins is called sequential if each weighing may depend on the results of the preceding steps.
- A strategy is called nonsequential if it satisfies the additional restriction that it states in advance exactly which coin is to put on which scale at each weighing, the choice being uninfluenced by the results of the previous weighings.

One of the results will be that for problem *P1* and *P2* sequential strategies are not shorter than nonsequential ones. This is not true for problem *P3*. Moreover we will give a complete answer to problem *P1* and *P2* and consider a special case of problem *P3*.

2 Best possible sequential solution

In this section we deal with problem *(P1)* and *(P2)* and make a rough estimate for the maximal number of coins which can be decided in w weighings on b balances by a sequential strategy. Although this estimate involves only simple combinatorial techniques it will turn out that the estimate is sharp even if we replace sequential by nonsequential.

Theorem 1 *Assume that there is at most one counterfeit coin. Then the maximal number c of coins which can be decided in w weighings on b balances by a sequential solution satisfies*

$$c \leq \frac{(2b+1)^w - 1}{2} - b\,.$$

Proof
Let c be a number for which a given sequential strategy allows to solve the problem with b balances for c coins. Consider the matrix (a_{ij}) with indices $i = 1, 2, \ldots, w$ and $j = 1, -1, 2, -2, \ldots, c, -c$ where the elements of the matrix have the following meaning:

(i) If, under the condition that coin j is heavier, at weighing i

- balance x is right hand down let $a_{ij} = x$
- balance x is left hand down let $a_{ij} = -x$
- all balances are balanced let $a_{ij} = 0$

(ii) If, under the condition that coin j is lighter, at weighing i

- balance x is right hand down let $a_{i-j} = x$
- balance x is left hand down let $a_{i-j} = -x$
- all balances are balanced let $a_{i-j} = 0$

Note that although the strategy is sequential the matrix (a_{ij}) is well defined. Since the given strategy is successful, in the matrix (a_{ij}) no column vector v_j with components $(v_j)_i := a_{ij}$ equals the zero vector and no two column vectors are equal. The fact that $a_{ij} \in \{-b, -b+1, -b+2, \ldots, b-2, b-1, b\}$ implies immediately

$$2c \leq (2b+1)^w - 1\,.$$

Note that the first row a_{1j} of the matrix may be considered as follows: In the first weighing coin j is placed

- on the right hand side of balance x if $a_{1j} = x > 0$
- on the left hand side of balance x if $a_{1j} = -x < 0$
- on no balance if $a_{1j} = 0$

So the first row a_{1j} of the matrix has obviously the property

$$|\{j > 0 \,:\, a_{1j} = p\}| = |\{j > 0 \,:\, a_{1j} = -p\}|$$

for all $p = 1, 2, \ldots, b$ since on every balance the same number of coins is placed on the left hand side as on the right hand side. Thus the number

$$|\{j \,:\, a_{1j} = p\}| = |\{j \,:\, a_{1j} = -p\}|$$

is even for every $p = 1, 2, \ldots, b$. Since for any $p = \pm 1, \pm 2, \ldots, \pm b$ the maximal number of column vectors having first component p is $(2b+1)^{w-1}$, i.e. odd, it follows

$$2c \leq (2b+1)^w - 1 - 2b\,.$$

$\square$

3 Mathematical formalism for nonsequential strategies

A nonsequential strategy for c coins with b balances and w weighings may be represented by a matrix (a_{ij}), $i = 1, 2, \ldots, w$, $j = 1, 2, \ldots, c$, with elements $a_{ij} \in \{-b, -b+1, \ldots, b\}$ when we give the elements a_{ij} the following meaning: If

- coin j is at weighing i on the right hand side of balance x let $a_{ij} = x$
- coin j is at weighing i on the left hand side of balance x let $a_{ij} = -x$
- coin j is at weighing i not in a scale of a balance let $a_{ij} = 0$

A column vector v_k with $(v_k)_i = a_{ik}$ may be interpreted as the results of the weighings for the given strategy provided that coin k is heavier than the other nickels – just give the elements $(v_k)_i$ the following meaning:

- if $(v_k)_i = x > 0$ the right hand side of balance x is down in weighing i
- if $(v_k)_i = -x < 0$ the left hand side of balance x is down in weighing i
- if $(v_k)_i = 0$ all balances are balanced in weighing i

Thus the matrix (a_{ij}) has the following fundamental properties:

(I) In each row the number of elements p equals the number of elements $-p$ for $p \in \{1, 2, \ldots, b\}$.

(II) If v_k and v_j are the column vectors $(v_k)_i = a_{ik}$ and $(v_j)_i = a_{ij}$ respectively, then $v_k = \pm v_j$ implies $j = k$.

(III) No column vector is the zero vector.

On the other hand any $w \times c$ matrix (a_{ij}) with elements $a_{ij} \in \{-b, -b+1, \ldots, b\}$ and properties (I) to (III) represents a nonsequential strategy to decide c nickels with b balances in w weighings.

4 Best possible nonsequential solution

Theorem 2 *Assume that there is at most one counterfeit coin. If the number $c > 2$ of coins satisfies*

$$c \leq \frac{(2b+1)^w - 1}{2} - b$$

then there exists a nonsequential strategy to decide these c coins with w weighings on b balances.

Remark 1 From section 2 it follows that this solution is best possible.

To prove Theorem 2, we need the following two Lemmas.

Lemma 1 *If the number $c > 2$ of coins satisfies*

$$c \leq \frac{(2b+1)^2 - 1}{2} - b = b(2b+1) =: c_{2,b}$$

then there exists a nonsequential strategy to decide these c coins with 2 weighings on b balances.

In the proof we shall use the following notations:
(1) If $M_1 = (a_{ij})$ is an $(n \times m_1)$-matrix and $M_2 = (b_{kl})$ is an $(n \times m_2)$-matrix, then the composition $M_1 \oplus M_2 = (d_{pq})$ denotes the $(n \times (m_1 + m_2))$-matrix with elements

$$d_{pq} = \begin{cases} a_{pq} & \text{for } q \leq m_1, \\ b_{p,(q-m_1)} & \text{otherwise.} \end{cases}$$

(2) By $a_1, a_2, \ldots, [a_k], \ldots$ we mean $a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots$.

Proof
By induction on c and b: For the cases $2 < c \leq 19$ and $1 \leq b \leq 3$ we give a complete list of matrices representing the nonsequential strategies.

$$\begin{array}{rrcl}
b=1: & M_{2,3} & := & \begin{pmatrix} +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix} \\
b=2: & M_{2,4} & := & \begin{pmatrix} +1 & -1 & +2 & -2 \\ +1 & +2 & -1 & -2 \end{pmatrix} \\
 & M_{2,5} & := & \begin{pmatrix} +1 & -1 & +2 & -2 & 0 \\ +1 & +2 & 0 & -2 & -1 \end{pmatrix} \\
 & M_{2,6} & := & \begin{pmatrix} +1 & -1 & +2 & -2 & 0 & 0 \\ +1 & 0 & 0 & -2 & -1 & +2 \end{pmatrix} \\
 & M_{2,7} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & -2 & 0 \\ +1 & +2 & 0 & +2 & -1 & -2 & -2 \end{pmatrix} \\
 & M_{2,8} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 \\ +1 & -1 & +2 & -2 & +1 & -1 & +2 & -2 \end{pmatrix} \\
 & M_{2,9} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 & 0 \\ +1 & -1 & +2 & -2 & +1 & -1 & +2 & 0 & -2 \end{pmatrix} \\
 & M_{2,10} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 & 0 & 0 \\ -1 & -2 & 0 & -2 & 0 & +1 & +1 & +2 & -1 & +2 \end{pmatrix} \\
b=3: & M_{2,11} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 & +3 & -3 & 0 \\ +1 & -1 & +2 & -2 & +1 & -1 & +2 & -2 & +3 & 0 & -3 \end{pmatrix} \\
 & M_{2,12} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 & +3 & +3 & -3 & -3 \\ +2 & -2 & +3 & -3 & +1 & -1 & +3 & -3 & +1 & -1 & +2 & -2 \end{pmatrix} \\
 & M_{2,13} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 & +3 & +3 & -3 & -3 & 0 \\ +2 & -2 & +3 & -3 & +1 & -1 & +3 & -3 & +1 & 0 & +2 & -2 & -1 \end{pmatrix} \\
 & M_{2,14} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 & +3 & +3 & -3 & -3 & 0 & 0 \\ +2 & 0 & +3 & -3 & +1 & -1 & +3 & -3 & +1 & 0 & +2 & -2 & -1 & -2 \end{pmatrix} \\
 & M_{2,15} & := & \begin{pmatrix} +1 & +1 & -1 & -1 & +2 & +2 & -2 & -2 & +3 & +3 & -3 & -3 & 0 & 0 & 0 \\ +2 & 0 & +3 & -3 & +1 & -1 & +3 & 0 & +1 & 0 & +2 & -2 & -1 & -2 & -3 \end{pmatrix} \\
 & M_{2,16} & := & \begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 & +2 & +2 & +2 & -2 & -2 & -2 & +3 & +3 & -3 & -3 \\ +1 & -1 & -2 & -2 & +3 & -3 & +1 & -1 & +2 & +2 & +3 & -3 & +1 & -1 & +2 & -2 \end{pmatrix} \\
 & M_{2,17} & := & \begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 & +2 & +2 & +2 & -2 & -2 & -2 & +3 & +3 & -3 & -3 & 0 \\ +1 & -1 & -2 & -2 & +3 & -3 & +1 & -1 & +2 & +2 & +3 & -3 & +1 & 0 & +2 & -2 & -1 \end{pmatrix} \\
 & M_{2,18} & := & \begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 & +2 & +2 & +2 & -2 & -2 & -2 & +3 & +3 & -3 & -3 & 0 & 0 \\ +1 & -1 & 0 & -2 & +3 & -3 & +1 & -1 & +2 & +2 & +3 & -3 & +1 & 0 & +2 & -2 & -1 & -2 \end{pmatrix} \\
 & M_{2,19} & := & \begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 & +2 & +2 & +2 & -2 & -2 & -2 & +3 & +3 & -3 & -3 & 0 & 0 & 0 \\ +1 & -1 & 0 & -2 & +3 & -3 & +1 & -1 & +2 & +2 & +3 & 0 & +1 & 0 & +2 & -2 & -1 & -2 & -3 \end{pmatrix}
\end{array}$$

Note that if $c = c_{2,1} = 3$ or $c = c_{2,2} = 10$ then neither $M_{2,3}$ nor $M_{2,10}$ has a constant column. Moreover, if $c = c_{2,2} - 1 = 9$ then neither $\left(\begin{smallmatrix} \pm 1 \\ 0 \end{smallmatrix}\right)$ nor $\left(\begin{smallmatrix} 0 \\ \pm 1 \end{smallmatrix}\right)$ occur in $M_{2,9}$.
Now we consider the case $c = c_{2,b}$ and $c = c_{2,b} - 1$ for $b \geq 3$. For $1 \leq x < y \leq b$ we define

$$N_{x,y} := \Bigg(\overbrace{\begin{matrix} -x & \ldots & -x \\ +1 & \ldots & +b \end{matrix}}^{b} \overbrace{\begin{matrix} +x & \ldots & \ldots & \ldots & +x \\ 0 & -1 & \ldots[-x] & \ldots & -b \end{matrix}}^{b} \overbrace{\begin{matrix} -(x+1) & \ldots & -(x+1) \\ +1 & \ldots & +b \end{matrix}}^{b} \overbrace{\begin{matrix} +(x+1) & \ldots & \ldots & \ldots & +(x+1) \\ 0 & -1 & \ldots[-(x+1)] & \ldots & -b \end{matrix}}^{b} \begin{matrix} \ldots \\ \ldots \end{matrix}$$
$$\begin{matrix} \ldots \\ \ldots \end{matrix} \overbrace{\begin{matrix} -y & \ldots & -y \\ +1 & \ldots & +b \end{matrix}}^{b} \overbrace{\begin{matrix} +y & \ldots & \ldots & \ldots & +y \\ 0 & -1 & \ldots[-y] & \ldots & -b \end{matrix}}^{b} \Bigg)$$

Now if b is odd the following matrix shows the nonsequential strategy for $c = c_{2,b}$:

$$M_{2,c_2} := N_{1,b-1} \oplus \Bigg(\overbrace{\begin{matrix} -b & \ldots & \ldots & \ldots & \ldots & \ldots & -b \\ +b & -1 & +1 & -2 & +2 & \ldots -\frac{b-1}{2} & +\frac{b-1}{2} \end{matrix}}^{b} \overbrace{\begin{matrix} +b & \ldots & \ldots & \ldots & \ldots & +b \\ -\frac{b+1}{2} & +\frac{b+1}{2} & \ldots & -(b-1) & +(b-1) & 0 \end{matrix}}^{b} \overbrace{\begin{matrix} 0 & \ldots & 0 \\ -1 & \ldots & -b \end{matrix}}^{b} \Bigg)$$

If b is even the following matrix shows the nonsequential strategy for $c = c_{2,b}$:

$$M_{2,c_2} := N_{1,b} \oplus \Big(\overbrace{\begin{smallmatrix} 0 & \cdots & 0 \\ -1 & \cdots & -b \end{smallmatrix}}^{b} \Big)$$

Now if b is odd the following matrix shows the nonsequential strategy for $c = c_{2,b} - 1$:

$$M_{2,c_2-1} := \Big(\overbrace{\begin{smallmatrix} -1 \ldots -1 \\ +1 \ldots +b \end{smallmatrix}}^{b} \overbrace{\begin{smallmatrix} +1 \ldots +1 \\ +1 \ldots +b \end{smallmatrix}}^{b} \overbrace{\begin{smallmatrix} -2 \ldots -2 \\ -1 \ldots -b \end{smallmatrix}}^{b} \overbrace{\begin{smallmatrix} +2 & \ldots & \ldots & \ldots +2 \\ 0 & -1 & [-2] & \ldots -b \end{smallmatrix}}^{b} \Big) \oplus N_{3,b-1} \oplus$$

$$\oplus \left(\overbrace{\begin{matrix} -b & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & -b \\ +b & -1 & +1 & -2 & +2 & \ldots & -\frac{b-1}{2} & +\frac{b-1}{2} \end{matrix}}^{b} \overbrace{\begin{matrix} +b & \ldots & \ldots & \ldots & \ldots & +b \\ -\frac{b+1}{2} & +\frac{b+1}{2} & \ldots & -(b-1) & +(b-1) & 0 \end{matrix}}^{b} \overbrace{\begin{matrix} 0 & \ldots & \ldots & 0 \\ [-1] & -2 & \ldots & -b \end{matrix}}^{b-1} \right)$$

If b is even the following matrix shows the nonsequential strategy for $c = c_{2,b} - 1$:

$$M_{2,c_2-1} \quad := \quad \Big(\overbrace{\begin{smallmatrix} -1 \ldots -1 \\ +1 \ldots +b \end{smallmatrix}}^{b} \overbrace{\begin{smallmatrix} +1 \ldots +1 \\ +1 \ldots +b \end{smallmatrix}}^{b} \overbrace{\begin{smallmatrix} -2 \ldots -2 \\ -1 \ldots -b \end{smallmatrix}}^{b} \overbrace{\begin{smallmatrix} +2 & \ldots & \ldots & \ldots +2 \\ 0 & -1 & [-2] & \ldots -b \end{smallmatrix}}^{b} \Big) \ \oplus \ N_{3,b} \ \oplus \ \Big(\overbrace{\begin{smallmatrix} 0 & \ldots & \ldots & 0 \\ [-1] & -2 & \ldots & -b \end{smallmatrix}}^{b-1} \Big)$$

Note that if $c = c_2$ then M_{2,c_2} has no constant column. Moreover, if $c = c_2 - 1$ then neither $\left(\begin{smallmatrix} \pm 1 \\ 0 \end{smallmatrix}\right)$ nor $\left(\begin{smallmatrix} 0 \\ \pm 1 \end{smallmatrix}\right)$ occur in M_{2,c_2-1}.

Let $c_{2,b'-(n+1)} < c \leq c_{2,b'-n}$ (for $0 \leq n < b'$) and $c_2 := c_{2,b'-n}$. For the further proceeding we can assume that we have more than three balances $b = (b'-n) \geq 4$ and that the number of coins c is less than $c_2 - 1$. To get a matrix representing a nonsequential strategy to decide these c coins, we start with the matrix M_{2,c_2} constructed as above and use the procedures P1 and P2 (explained below) to reduce the matrix M_{2,c_2} to a $(2 \times c)$-matrix with the desired property.

Procedure P1:

Let $M_{2,k}$ be a $(2 \times k)$-matrix with the following two properties

- The columns $\left(\begin{smallmatrix} +x \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ -x \end{smallmatrix}\right)$ occur in $M_{2,k}$.
- The column $\left(\begin{smallmatrix} +x \\ -x \end{smallmatrix}\right)$ does not occur in $M_{2,k}$.

To get the matrix $\mathrm{P1}(M_{2,k})$ we cancel in $M_{2,k}$ the column $\left(\begin{smallmatrix} 0 \\ -x \end{smallmatrix}\right)$ and replace the column $\left(\begin{smallmatrix} +x \\ 0 \end{smallmatrix}\right)$ by the column $\left(\begin{smallmatrix} +x \\ -x \end{smallmatrix}\right)$. Note that $\mathrm{P1}(M_{2,k})$ is a $(2 \times (k-1))$-matrix

Procedure P2:

Let $M_{2,k}$ be a $(2 \times k)$-matrix with the following property

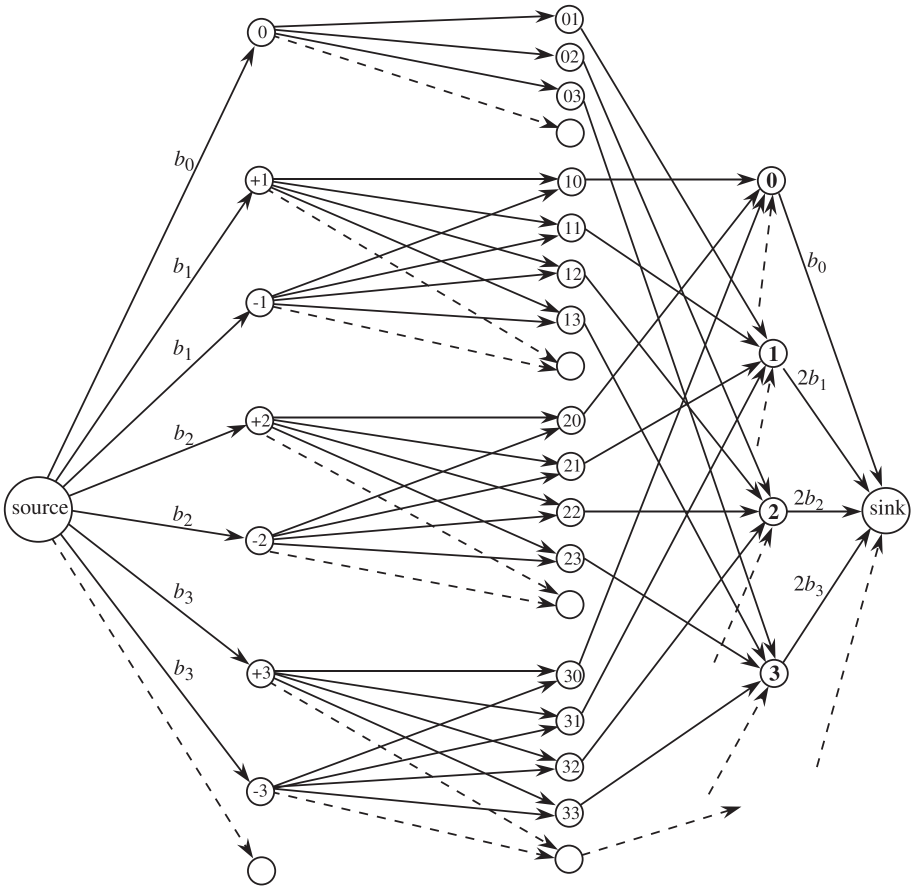
- The columns $\left(\begin{smallmatrix} +x \\ -y \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -x \\ +y \end{smallmatrix}\right)$ occur in $M_{2,k}$ where $x \neq y$ and $1 \leq x, y \leq b$.

To get the matrix $\mathrm{P2}(M_{2,k})$ we cancel in $M_{2,k}$ the two columns $\left(\begin{smallmatrix} +x \\ -y \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -x \\ +y \end{smallmatrix}\right)$. Note that $\mathrm{P2}(M_{2,k})$ is a $(2 \times (k-2))$-matrix

It is easy to see, that if $M_{2,k}$ represent a nonsequential strategy then $\mathrm{P1}(M_{2,k})$ and $\mathrm{P2}(M_{2,k})$ both represent a nonsequential strategy too. Hence, if we start with M_{2,c_2} and use the P1 and P2 in a suitable succession, we finally get a $(2 \times c)$-matrix representing a nonsequential strategy to decide the c coins with with 2 weighings on b balances. $\square$

Remark 2 Note that we constructed the matrices M_{2,c_2} and M_{2,c_2-1} such that M_{2,c_2} has no constant column and neither $\left(\begin{smallmatrix} \pm 1 \\ 0 \end{smallmatrix}\right)$ nor $\left(\begin{smallmatrix} 0 \\ \pm 1 \end{smallmatrix}\right)$ occur in M_{2,c_2-1}.

Now we present an interesting analogy of the problem in a completely different language which gives an additional insight to the considered structure. Let us look at the following network:



The set of successors of the source is $\{-b, -b+1, \ldots, b\}$;
the set of successors of 0 is $\{01, 02, \ldots, 0b\}$;
the set of successors of $\pm x$ is $\{x0, x1, \ldots, xb\}$ $(x > 0)$;
the set of successors of xy is $\{\boldsymbol{y}\}$;
the set of successors of $\boldsymbol{x}$ is $\{\text{sink}\}$.

With b_x, v_x as defined below the capacities are:

$$
\begin{aligned}
c(\text{source}, \pm x) &= b_x \\
c(\pm x, xy) &= \begin{cases} v_0 & \text{if } x = 1 \text{ and } y = 0, \\ v_1 & \text{if } x = 0 \text{ and } y = 1, \\ 3 & \text{otherwise.} \end{cases} \\
c(xy, \boldsymbol{y}) &= \begin{cases} v_= & \text{if } x = \boldsymbol{y}, \\ 1 & \text{if } x = 0 \text{ or } \boldsymbol{y} = 0, \\ 2 & \text{otherwise.} \end{cases} \\
c(\boldsymbol{x}, \text{sink}) &= \begin{cases} b_0 & \text{if } \boldsymbol{x} = 0, \\ 2b_x & \text{otherwise.} \end{cases}
\end{aligned}
$$

Let $c_2 := b(2b+1)$, the maximal number of coins decidable with two weighings.

1. If $c = c_2$, put $b_0 = b_1 = \ldots = b_b := b$, $v_= := 1$, $v_0 := 3$, $v_1 := 1$.
2. If $c = c_2 - 1$, put $b_1 = b_2 \ldots = b_b := b$, $b_0 := b - 1$, $v_= := 2$, $v_0 := 0$, $v_1 := 0$.
3. If $c < c_2 - 1$ and $c = 2m + k$ with $1 \leq k \leq b$ and $m \leq b^2$, choose $b_i \leq b$ such that $\sum_{i=1}^{b} b_i = m$ and $b_0 := k$, further $v_= := 2$, $v_0 := 3$, $v_1 := 1$.

As a consequence of the theory developed by Ford and Fulkerson in [2], we find that in the network described above the maximal flow value is less or equal than $b_0 + \sum_{i=1}^{b} 2b_i = c$. On the other hand Lemma 1 guarantees that a flow of the size c exists.
Given a flow Φ through the network, let $f(a,b)$ denote the flow (with respect to Φ) through the arc (a,b).
Now construct a maximal flow Φ_{max} through the network, such that there exists a $(2 \times c)$-matrix $M_{2,c}$ with the following properties:

(i) $\binom{+x}{+y}$ and $\binom{+x}{-y}$ are columns of $M_{2,c}$, if and only if $f(+x, xy) = 2$,

(ii) The same as (i), but replace $+x$ by $-x$.

(iii) If $f(+x, xy) = f(-x, xy) = 1$, then either $\binom{+x}{+y}$ and $\binom{-x}{+y}$ or $\binom{+x}{-y}$ and $\binom{-x}{-y}$ are columns of $M_{2,c}$.

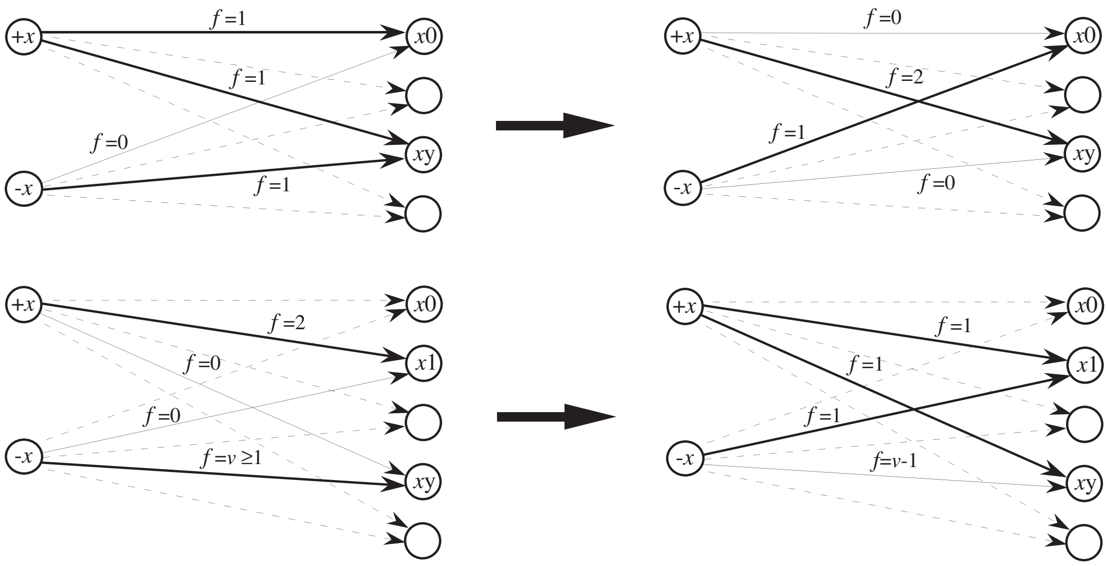
(iv) If $f(+x, xy) > 0$ then $\binom{+x}{+y}$ or $\binom{+x}{-y}$ is a column of $M_{2,c}$.

(v) The same as (iv), but replace $+x$ by $-x$.

(vi) For each $1 \leq x \leq b$, in the second row of $M_{2,c}$, the number of $+x$ equals the number of $-x$.

To construct such a flow, first consider an arbitrary maximal flow Φ through the network. With respect to Φ, there exists a $(2 \times c)$-matrix with the properties (i)-(v).

The following picture shows, how parts of Φ can be modified such that we get a maximal flow Φ', for which there exists a $(2 \times c)$-matrix $M_{2,c}$, (with respect to Φ') satisfying also the property (vi).



Note that we can find a maximal flow such that the corresponding matrix M_{2,c_2} has no constant column and in M_{2,c_2-1} neither $\left(\begin{smallmatrix}\pm1\\0\end{smallmatrix}\right)$ nor $\left(\begin{smallmatrix}0\\\pm1\end{smallmatrix}\right)$ occur.

Lemma 2 *If we have one balance and c coins such that $4 \leq c \leq 12$, then we can decide the c coins in three weighings.*

Proof
The following $(3 \times c)$-matrices $M_{3,c}$ represent nonsequential strategies for c coins with one balance and three weighings:

$$
\begin{aligned}
M_{3,4} &:= \begin{pmatrix} 0 & 0 & +1 & -1 \\ 0 & -1 & +1 & 0 \\ -1 & 0 & 0 & +1 \end{pmatrix} \\
M_{3,5} &:= \begin{pmatrix} 0 & +1 & +1 & -1 & -1 \\ +1 & -1 & 0 & +1 & -1 \\ 0 & -1 & 0 & 0 & +1 \end{pmatrix} \\
M_{3,6} &:= \begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & -1 & 0 & +1 & -1 \\ -1 & 0 & +1 & -1 & 0 & +1 \end{pmatrix} \\
M_{3,7} &:= \begin{pmatrix} 0 & +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & 0 & -1 & 0 & 0 & -1 & +1 \\ +1 & -1 & 0 & 0 & -1 & 0 & +1 \end{pmatrix} \\
M_{3,8} &:= \begin{pmatrix} 0 & 0 & +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & 0 & -1 & 0 & 0 & -1 & +1 \\ +1 & 0 & -1 & 0 & 0 & -1 & 0 & +1 \end{pmatrix} \\
M_{3,9} &:= \begin{pmatrix} 0 & 0 & 0 & +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & -1 & 0 & +1 & -1 & 0 & +1 & -1 \\ -1 & 0 & +1 & -1 & 0 & +1 & -1 & 0 & +1 \end{pmatrix} \\
M_{3,10} &:= \begin{pmatrix} 0 & 0 & 0 & 0 & +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & -1 & +1 & 0 & -1 & +1 & 0 & -1 & 0 \\ -1 & 0 & +1 & +1 & -1 & 0 & +1 & -1 & 0 & 0 \end{pmatrix} \\
M_{3,11} &:= \begin{pmatrix} 0 & 0 & 0 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ 0 & -1 & +1 & 0 & 0 & +1 & -1 & +1 & -1 & +1 & -1 \\ +1 & +1 & 0 & +1 & -1 & 0 & -1 & -1 & +1 & 0 & -1 \end{pmatrix} \\
M_{3,12} &:= \begin{pmatrix} 0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ 0 & +1 & -1 & -1 & 0 & +1 & -1 & 0 & 0 & +1 & -1 & +1 \\ -1 & 0 & +1 & -1 & -1 & 0 & +1 & 0 & -1 & 0 & +1 & +1 \end{pmatrix}
\end{aligned}
$$

□

Remark 3 Note that neither $\begin{pmatrix}\pm1\\0\\0\end{pmatrix}$ nor $\pm\begin{pmatrix}0\\1\\1\end{pmatrix}$ occur in $M_{3,11}$.

Notations:
Recall that if $M_1 = (a_{ij})$ is an $(n \times m_1)$-matrix and $M_2 = (b_{kl})$ is an $(n \times m_2)$-matrix, then the composition $M_1 \oplus M_2 = (d_{pq})$ denotes the $(n \times (m_1 + m_2))$-matrix with elements

$$d_{pq} = \begin{cases} a_{pq} & \text{for } q \leq m_1, \\ b_{p,(q-m_1)} & \text{otherwise.} \end{cases}$$

If M_i $(i = 1, 2, \ldots, r)$ are $(n \times m_i)$-matrices, then $\bigoplus_{i=1}^{r} M_i$ is the composition of the matrices M_i, hence an $(n \times \sum_{i=1}^{r} m_i)$-matrix.

If $M = a_{ij}$ is an $(n \times m)$-matrix, then let M^x denote the $((n+1) \times m)$-matrix (b_{ij}), with elements

$$b_{ij} = \begin{cases} x & \text{for } i = 1, \\ a_{i-1,j} & \text{otherwise.} \end{cases}$$

Let $\begin{pmatrix} x \\ \underline{y} \end{pmatrix}$ be the $((n+1) \times 1)$-matrix (d_{i1}) such that $d_{i1} = \begin{cases} x & \text{for } i = 1, \\ y & \text{otherwise.} \end{cases}$

If $M_{2,c}$ is a $(2 \times c)$-matrix and $n \geq 1$, then $\underline{M}_{2,c}$ is the $((n+1) \times c)$-matrix such that $\begin{pmatrix} x \\ \underline{y} \end{pmatrix}$ is a column of $\underline{M}_{2,c}$ if and only if $\begin{pmatrix} x \\ y \end{pmatrix}$ is a column of $M_{2,c}$.

Now we are prepared to prove Theorem 2:

Proof
Let c ($c > 2$) be the number of coins to be decided with b balances in w weighings.

If $b = 1$ and $w = 3$, then by Theorem 1, c has to be less or equal than 12 and for $4 \leq c \leq 12$ Lemma 2 gives us $(3 \times c)$-matrices representing nonsequential solutions.

If $b \geq 1$ and $w = 2$, then again by Theorem 1, c has to be less or equal than $b(2b+1)$ and for $3 \leq c \leq b(2b+1)$ Lemma 1 gives us $(2 \times c)$-matrices $M_{2,c}$ which represent nonsequential solutions.

Now consider the case $b \geq 1$ and $w = n+1$ for $n \geq 2$ (if $b = 1$ then $n > 2$). Assuming the existence of $(n \times c)$-matrices $M_{n,c}$ representing nonsequential solutions to decide c coins for $3 \leq c \leq \frac{(2b+1)^n - 1}{2} - b =: c_n$. We construct $((n+1) \times c)$-matrices $M_{n+1,c}$ for $3 \leq c \leq c_{n+1}$ as follows:

If $(2b+1)c_n + 3 \leq c \leq c_{n+1}$ and $c = (2b+1)c_n + d$, then let $M_{n+1,c} := \bigoplus_{x=-b}^{b} M^x_{n,c_n} \oplus \underline{M}_{2,d}$. By Remark 2, M_{n,c_n} has no constant column, hence $M_{n+1,c_{n+1}}$ has no constant column. Moreover if $b \geq 2$ neither $\begin{pmatrix} \pm 1 \\ \underline{0} \end{pmatrix}$ nor $\pm \begin{pmatrix} 0 \\ \underline{1} \end{pmatrix}$ occur in $M_{n+1,c_{n+1}-1}$.

If $c = (2b+1)c_n + 2$, let $N_1 := \begin{pmatrix} +1 & -1 \\ 0 & 0 \end{pmatrix}$, $N_2 := \begin{pmatrix} 0 & 0 \\ -1 & +1 \end{pmatrix}$ and

$$M_{n+1,c} := \bigoplus_{\substack{-b \leq x \leq b \\ |x| \neq 1}} M^x_{n,c_n} \oplus \underline{N}_1^{+1} \oplus \underline{N}_2^{-1} \oplus M^{+1}_{n,c_n-1} \oplus M^{-1}_{n,c_n-1}.$$

If $b = 1$, Remark 3 implies by induction that neither $\begin{pmatrix} \pm 1 \\ \underline{0} \end{pmatrix}$ nor $\pm \begin{pmatrix} 0 \\ \underline{1} \end{pmatrix}$ occur in $M_{n+1,c_{n+1}-1}$.

If $c = (2b+1)c_n + 1$, let $N_3 := \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$, $N_4 := \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $N_5 := \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and

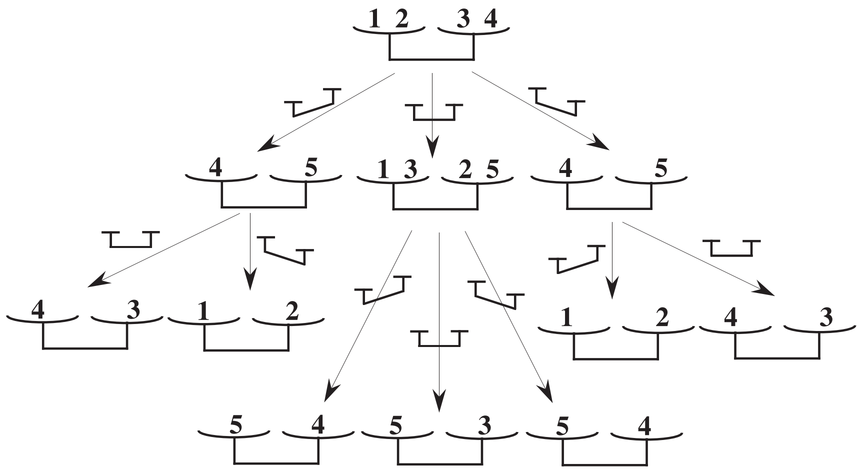
$$M_{n+1,c} := \bigoplus_{\substack{-b \leq x \leq b \\ |x| > 1}} M^x_{n,c_n} \oplus \underline{N}_4^{+1} \oplus \underline{N}_5^{-1} \oplus M^{+1}_{n,c_n-1} \oplus M^{-1}_{n,c_n-1} \oplus \underline{N}_3^{0} \oplus M^{0}_{n,c_n-1}.$$

If $3 \leq c \leq (2b+1)c_n$, then c is of the form $c = d_0 + \sum_{i=1}^{b} 2d_i$, where $3 \leq d_i \leq c_n$ or $d_i = 0$ for each $0 \leq i \leq b$. Then let $M_{n+1,c} := \bigoplus_{i=0}^{b} M^i_{n,d_i} \oplus \bigoplus_{i=1}^{b} M^{-i}_{n,d_i}$.

□

5 Two counterfeit coins

As a nontrivial case we consider five coins containing precisely two counterfeit nickels of the same kind. A similar reasoning as in section 2 shows that a sequential solution needs at least three weighings and that not more than five coins may be decided by three weighings. And in fact there exists a sequential solution that decides five coins in three weighings:



In contrast to the case of one counterfeit coin one can show that there does not exist a *nonsequential* solution with the same number of weighings for five coins. The minimal number of weighings for a nonsequential strategy to decide five coins is four:

$$\begin{pmatrix} +1 & +1 & -1 & -1 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \end{pmatrix}$$

6 Examples and unsolved problems

Example A rough estimate yields that the beach of Sicily consists of about $3 \cdot 10^{22}$ grains of sand. Assume that one of them has another weight than all the others. Given a (very large) balance, 48 weighings are enough to find the bad grain.

Problem 1 What is in general the best sequential or nonsequential strategy when more than one coin is counterfeit of possibly different kind? In particular, need sequential solutions always less weighings than nonsequential ones?

Problem 2 Assume that all possible cases of any coin to be heavier or lighter have the same probability. Let us call a sequential strategy $S1$ *better* than $S2$, if $S1$ needs less weighings than $S2$ in the average. What is the best strategy in this sense.

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