COMPARING CARDINALITIES IN ZERMELO'S SYSTEM

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ABSTRACT. The aim of this note is to define and compare cardinalities in Zermelo's system of 1908 (without using the Axiom of Choice). The main tool to compare cardinalities in Set Theory (with or without the Axiom of Choice) is the Cantor-Bernstein Theorem. We shall present two different proofs of the Cantor-Bernstein Theorem and shall give some historical background. In particular, we shall see that this theorem was proved independently by Cantor, Dedekind, Bernstein, Korselt, Zermelo, and Peano, but not by Schröder — even though it is sometimes cited as Schröder-Bernstein Theorem.

Key words: Cantor-Bernstein theorem, cardinalities, Zermelo's system
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1. Axiomatisations of Set Theory

In order to define the notion of cardinality in a formal way, we first need some axioms. So, let us start by presenting a few axiomatic systems of Set Theory.

In 1908, Zermelo published in [34] his first axiomatic system of Set Theory, which we shall denote by ZC, consisting of the following seven axioms:

(1) Axiom der Bestimmtheit
which corresponds to the Axiom of Extensionality

(2) Axiom der Elementarmengen
which includes the Axiom of Empty Set as well as the Axiom of Pairing

(3) Axiom der Aussonderung
which corresponds to the Axiom Schema of Separation

(4) Axiom der Potenzmenge
which corresponds to the Axiom of Power Set

(5) Axiom der Vereinigung
which corresponds to the Axiom of Union

(6) Axiom der Auswahl
which corresponds to the Axiom of Choice

(7) Axiom des Unendlichen
which corresponds to the Axiom of Infinity

Zermelo's system ZC without the Axiom of Choice (AC), which is axiom (6) in the list, will be denoted by Z (*i.e.*, $\mathsf{Z} = \mathsf{ZC} - \mathsf{AC}$).

Some years later, in 1930, Zermelo presented in [33] his second axiomatic system, which he called ZF-system, in which he incorporated ideas of Fraenkel [11], Skolem [28], and von Neumann [20, 21, 22]. In fact, he added the Axiom Schema of Replacement and the Axiom of Foundation to his former system, cancelled the Axiom of Infinity (since he thought that it does not belong to the general theory of sets), and did not mention explicitly the Axiom of Choice (because of its different character and since he considered it as a general logical principle, rather than an axiom of Set Theory).

Zermelo's second axiomatic system ZF together with the Axiom of Choice is denoted by ZFC (*i.e.*, $\mathsf{ZFC} = \mathsf{ZF} + \mathsf{AC}$). The system ZFC is commonly considered as the standard axiomatic system of Set Theory as well as for Mathematics in general. (For Zermelo's published work in Set Theory, described and analysed in its historical context, we refer the reader to Kanamori [17] and Ebbinghaus [10].)

2. The Notion of Cardinality

The notion of cardinality is surely one of the most fundamental concepts in Set Theory. Cantor — who first discovered that there are infinite sets of different cardinalities — describes cardinalities as follows (*cf.* [5, §1] or [7, p. 282 f.]): *The general concept which with the aid of our active intelligence results from a set M, when we abstract from the nature of its various elements and from the order of their being given, we call the "power" or "cardinality" of M.* This double abstraction suggests his notation "$\overline{\overline{M}}$" for the cardinality of M.

Below we shall define cardinalities in a more formal way by following Zermelo [34, §2]: We say that two sets A and B are **equivalent**, denoted $A \sim B$, if there exists a bijection between A and B. Notice that "being equivalent" is indeed an equivalence relation (on the class of all sets).

Let us now define the **cardinality** of a set A, denoted by $|A|$. Firstly notice that every set A corresponds to a unique collection $\{B : B \sim A\}$, denoted by $\bar{\bar{A}}$. Since every set $B \in \bar{\bar{A}}$ is equivalent to A, we could define $|A| = \bar{\bar{A}}$. However, $\bar{\bar{A}}$ is in general not a set. In fact, $\bar{\bar{A}}$ is a set only in the case when $A = \emptyset$, otherwise, $\bar{\bar{A}}$ is a proper class. Thus, if we would like to define $|A|$ as a proper set, we would have to choose a unique member (or at least a well-defined subset) from each collection $\bar{\bar{A}}$; but this is not always possible and depends on the axiomatic system of Set Theory we work with:

- If we work in Z, then we cannot do better than just define
$$|A| = \bar{\bar{A}}$$
and accept that cardinalities are in general proper classes. This notion of cardinality was used for example by Frege (*cf.* [12, 13]) and Russell (*cf.* [24, p. 378] or [25, Section IX, p. 256]).
- If we work in ZF, then, using the cumulative hierarchy of sets V_α (for ordinals α), we can define $|A|$ as a set by stipultating
$$|A| = \big\{B \in \mathrm{V}_{\beta_0} : B \sim A\big\},$$
where β_0 is the least ordinal number for which there is a $B \in \mathrm{V}_{\beta_0}$ such that $B \sim A$.
- If we work in ZC or ZFC, then we can do even better by defining $|A|$ as follows:
$$|A| \text{ is the least ordinal number } \kappa \text{ such that } \kappa \sim A$$
Since ordinals numbers are well-ordered by $\in$, in this case, $|A|$ is a well-defined ordinal number.

Notice that no matter in which axiomatic system we are working, for any sets A and B we have $|A| = |B|$ *iff* $A \sim B$, *i.e.*, two sets have the same cardinality if and only if they are equivalent.

If $|A| = |B'|$ for some $B' \subseteq B$, then we say that the cardinality of A is less than or equal to the cardinality of B, denoted $|A| \leq |B|$. Notice that $|A| \leq |B|$ *iff* there is an injection (*i.e.*, a one-to-one function) from A into B. Furthermore, if $|A| \neq |B|$ but $|A| \leq |B|$, then the cardinality of A is said to be strictly smaller than the cardinality of B, denoted $|A| < |B|$. Notice that the relation "$\leq$" is reflexive and transitive. Moreover, the notation suggests that $|A| \leq |B|$ and $|B| \leq |A|$ implies $|A| = |B|$, which is indeed the case and will be shown below.

An alternative way to compare cardinalities we get by replacing the injection from A into B (in the definition of $|A| \leq |B|$) with a surjection from B onto A: Let us write $|A| \leq^* |B|$ to denote that there is a surjection from B onto A. Again, the notation suggests that $|A| \leq^* |B|$ and $|B| \leq^* |A|$ implies $|A| = |B|$, but this is not the case, except we have some form of the Axiom of Choice, *i.e.*, this implication is neither provable in Z nor in ZF (see for example Halbeisen and Shelah [14, Theorem 1 & 3]). On the other hand, it is not hard to verify that the implications

$$|A| \leq |B| \;\rightarrow\; |A| \leq^* |B| \;\rightarrow\; \big|\mathscr{P}(A)\big| \leq \big|\mathscr{P}(B)\big|$$

where $\mathscr{P}(A)$ denotes the *power set* of A, are provable without the aid of the Axiom of Choice, *i.e.*, these implications are provable in ZF as well as in Z.

Let us conclude this section by Cantor's Theorem which implies that there are arbitrarily large cardinalities — surely one of the most important results in Set Theory. In fact, Cantor's Theorem states that the cardinality of any set A is always strictly smaller than the cardinality of its power set. When Cantor first proved that there is no bijection between the set of natural numbers $\mathbb{N}$ and the set of real numbers $\mathbb{R}$, where $\mathbb{R}$ is in fact equivalent to $\mathscr{P}(\mathbb{N})$, he was using an argument with nested intervals (*cf.* [2, §2] or [7, p. 117]), but later, he improved that technique and showed the following more general result (*cf.* [4] or [7, III. 8]).

Theorem 2.1 (Cantor's Theorem)**.** *For all sets A, $|A| < |\mathscr{P}(A)|$.*

Proof. It is enough to show that there is an injection from A into $\mathscr{P}(A)$, but there is no surjection from A onto $\mathscr{P}(A)$.

Firstly, the function

$$\begin{array}{rccc} f: & A & \longrightarrow & \mathscr{P}(A) \\ & x & \longmapsto & \{x\} \end{array}$$

is obviously injective, and therefore we get $|A| \leq |\mathscr{P}(A)|$.

Secondly, let $g : A \rightarrow \mathscr{P}(A)$ be an arbitrary function. Consider the set

$$A' = \big\{x \in A : x \notin g(x)\big\}\,.$$

As a subset of A, the set A' is an element of $\mathscr{P}(A)$. If there would be an $x_0 \in A$ such that $g(x_0) = A'$, then $x_0 \in A' \leftrightarrow x_0 \notin g(x_0)$, but since $g(x_0) = A'$, $x_0 \notin g(x_0) \leftrightarrow x_0 \notin A'$. Thus, $x_0 \in A' \leftrightarrow x_0 \notin A'$, which is obviously a contradiction and shows that g is not surjective. ⊣

3. The Cantor-Bernstein Theorem

It is now time to discuss the main tool to compare cardinals — the so-called Cantor-Bernstein Theorem.

Theorem 3.1 (Cantor-Bernstein Theorem)**.** *Let A and B be any sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Below we shall give two different proofs of the CANTOR-BERNSTEIN THEOREM, where both proofs can be carried out in Z (*i.e.*, the proofs do not use the AXIOM OF CHOICE, the AXIOM OF FOUNDATION, or the AXIOM SCHEMA OF REPLACEMENT), but first let us say a few words about the history of this important result.

The CANTOR-BERNSTEIN THEOREM, unfortunately also known as SCHRÖDER-BERNSTEIN THEOREM, was first stated and proved by Cantor (*cf.* [3, VIII.4] or [7, p. 413], and [5, §2, Satz B] or [7, p. 285]). In order to prove this theorem, Cantor used the TRICHOTOMY OF CARDINALS, which states that for any sets A and B we have either $|A| < |B|$, or $|A| = |B|$, or $|A| > |B|$. In 1895, Cantor [5, §2] asserted the TRICHOTOMY OF CARDINALS without proof, and in a letter of 28 July 1899 (*cf.* [7, pp. 443–447]) he wrote to Dedekind that the TRICHOTOMY OF CARDINALS follows from the WELL-ORDERING PRINCIPLE. In fact, in 1915 Hartogs [15] showed that TRICHOTOMY OF CARDINALS is even equivalent to the WELL-ORDERING PRINCIPLE, and consequently to the AXIOM OF CHOICE (*cf.* Moore [19, p. 10], Cantor [7, p. 351, Anm. 2], and Zermelo [31, 32]). As a matter of fact we would like to mention that — according to Sierpinski [27, p. 99 f.] — Leśniewski showed that TRICHOTOMY OF CARDINALS is equivalent to the statement that for any two disjoint sets A and B, where at least one of these sets is infinite, we always have $|A \cup B| = |A|$ or $|A \cup B| = |B|$.

An alternative proof of the CANTOR-BERNSTEIN THEOREM, avoiding any form of the AXIOM OF CHOICE, was found by Bernstein, who was initially a student of Cantor's. Bernstein presented his proof around Easter 1897 in one of Cantor's seminars in Halle, and the result was published in 1898 in Borel [1, p. 103–106]. About the same time, Schröder gave a similar proof in [26] (submitted May 1896), but unfortunately, Schröder's proof was flawed by an irreparable error. While other mathematicians regarded his proof as correct, Korselt wrote to Schröder about the error in 1902. In his reply, Schröder admitted his mistake which he had already found some time ago but did not have the opportunity to make public. A few weeks later, Korselt submitted the paper [18] — which appeared almost a decade later — with a proof of the CANTOR-BERNSTEIN THEOREM which is quite different to the one given by Bernstein. A proof of the CANTOR-BERNSTEIN THEOREM, similar to Korselt's proof, was found in 1906 independently by Peano [23] (*cf.* [34, footnote p. 272 f.]) and Zermelo [34, §2] — who called it *Äquivalenzsatz*.

However, they could not know that they had just rediscovered the proof that had already been obtained twice by Dedekind in 1887 and 1897, since Dedekind's proof — in a slightly different terminology given below — was not published until 1932 (see [9, LXII & Erl. p. 448] and [7, p. 449]).

Below we shall present two proofs of CANTOR-BERNSTEIN THEOREM, first we give Dedekind's proof and then we sketch out Bernstein's proof.

3.1. **Dedekind's Proof.** The crucial point in Dedekind's proof is the following Lemma — whose proof uses what Dedeking called *Kettentheorie* (*i.e.*, concept of chains) which he developed in [8].

Lemma 3.2. *Let A_0, A_1, A be sets such that $A_0 \subseteq A_1 \subseteq A$. If $|A| = |A_0|$, then $|A| = |A_1|$.*

Proof. If $A_1 = A$ or $A_1 = A_0$, then the statement is trivial. So, let us assume that $A_0 \subsetneq A_1 \subsetneq A$ and let $C = A \setminus A_1$, *i.e.*, $A \setminus C = A_1$. Further, let $f : A \to A_0$ be a bijection and define $g : \mathscr{P}(A) \to \mathscr{P}(A_0)$ by stipulating $g(D) := f[D]$.

In Z we can construct a function $z_0 : \omega \to \mathscr{P}(A)$ such that $z_0(0) = C$ and for all $n \in \omega$ we have $z_0(n+1) = f\big[z_0(n)\big]$. Now, let

$$\bar{C} = \bigcup \{z_0(n) : n \in \omega\}$$

and define the function $\tilde{f} : A \to A$ by stipulating

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \bar{C}, \\ x & \text{otherwise.} \end{cases}$$

By definition of $\tilde{f}$ and since f is a bijection which maps C into A_0, $\tilde{f}[\bar{C}] = \bar{C} \setminus C$. Moreover, the function $\tilde{f}$ is injective. To see this, let $x, y \in A$ be distinct and consider the following three cases:

(1) If $x, y \in \bar{C}$, then $\tilde{f}(x) = f(x)$ and $\tilde{f}(y) = f(y)$, and since f is injective we get $\tilde{f}(x) \neq \tilde{f}(y)$.
(2) If $x, y \in A \setminus \bar{C}$ are distinct, then $\tilde{f}(x) = x$ and $\tilde{f}(y) = y$, and hence, $\tilde{f}(x) \neq \tilde{f}(y)$.
(3) If $x \in \bar{C}$ and $y \in A \setminus \bar{C}$, then $\tilde{f}(x) = f(x) \in \bar{C}$ and $\tilde{f}(y) = y \notin \bar{C}$, and therefore, $\tilde{f}(x) \neq \tilde{f}(y)$.

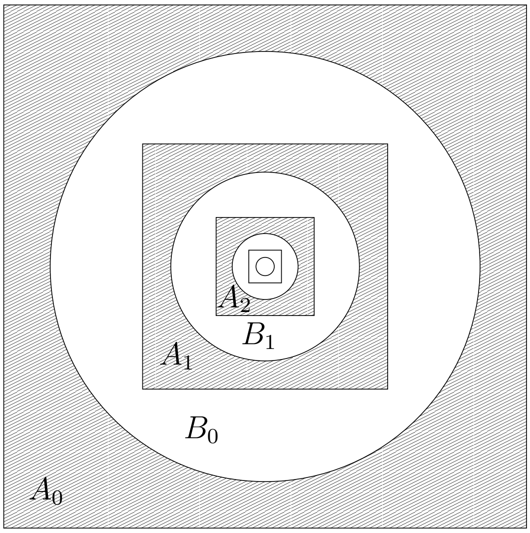
We already know that $\tilde{f}[\bar{C}] = \bar{C} \setminus C$ and by definition we have $\tilde{f}[A \setminus \bar{C}] = A \setminus \bar{C}$. Hence,

$$\tilde{f}[A] = (A \setminus \bar{C}) \,\dot{\cup}\, (\bar{C} \setminus C) = A \setminus C = A_1$$

which shows that $|A| = |A_1|$. ⊣

Proof of Cantor-Bernstein Theorem. Let $f : A \hookrightarrow B$ be an injection from A into B, and $g : B \hookrightarrow A$ be an injection from B into A. Further, let $A_0 := (g \circ f)[A]$ and $A_1 := g[B]$. Then $|A_0| = |A|$ and $A_0 \subseteq A_1 \subseteq A$, hence, by LEMMA 3.2, $|A| = |A_1|$, and since $|A_1| = |B|$ we have $|A| = |B|$. ⊣

3.2. **Bernstein's Proof.** Below we sketch out Bernstein's proof of the CANTOR-BERNSTEIN THEOREM as it was published by Borel in [1, p. 104 ff.] (see also Hausdorff [16, Kap. III, Satz I]): Let A and B be two arbitrary sets and let $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$ two injections. Further, let $A_0 := A$, $B_0 := g[B]$, and for $n \in \omega$ let $A_{n+1} := (g\circ f)[A_n]$ and $B_{n+1} := (g\circ f)[B_n]$; finally let $D := \bigcap_{n\in\omega} A_n$.
We get the following picture:



It is not hard to verify that the sets A_n and B_n have the following properties:

(a) $A_0 = D \cup (A_0 \setminus B_0) \cup (B_0 \setminus A_1) \cup (A_1 \setminus B_1) \cup (B_1 \setminus A_2) \cup \dots$
(b) $B_0 = D \cup (B_0 \setminus A_1) \cup (A_1 \setminus B_1) \cup (B_1 \setminus A_2) \cup (A_2 \setminus B_2) \cup \dots$
(c) For all $n \in \omega$, $|A_n \setminus B_n| = |A_{n+1} \setminus B_{n+1}|$.

Since the sets $(A_n \setminus B_n)$, $(B_n \setminus A_{n+1})$, and D, are pairwise disjoint, by (c), and by regrouping the representation of B_0 in (b), we get

$$|B_0| = |D \cup (A_0 \setminus B_0) \cup (B_0 \setminus A_1) \cup (A_1 \setminus B_1) \cup \dots| = |A_0|\,,$$

i.e., $|A| = |B|$, which completes the proof.

References

[1] Émile Borel, ***Leçons sur la Théorie des Fonctions***, Gauthier-Villars et Fils, Paris, 1898.

[2] Georg Cantor, *Über eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen*, ***Journal für die reine und angewandte Mathematik (Crelle)***, vol. 77 (1874), 258–262.

[3] ———, *Mitteilungen zur Lehre vom Transfiniten. I.&II.*, ***Zeitschrift für Philosophie und philosophische Kritik***, vol. 91/92 (1887/1888), 81–125 / 240–265.

[4] ———, *Über eine elementare Frage der Mannigfaltigkeitslehre*, ***Jahresbericht der Deutschen Mathematiker-Vereinigung***, vol. 1 (1891), 75–78.

[5] ———, *Beiträge zur Begründung der transfiniten Mengenlehre. I.&II.*, ***Mathematische Annalen***, vol. 46/49 (1895/1897), 481–512 / 207–246 (see [6] for a translation into English).

[6] ———, ***Contributions to the Founding of the Theory of Transfinite Numbers,*** (translation into English of [5]), [translated, and provided with an introduction and notes, by Philip E. B. Jourdain], Open Court Publishing Company, Chicago and London, 1915 [reprint: Dover Publications, New York, 1952].

[7] ———, ***Gesammelte Abhandlungen mathematischen und philosophischen Inhalts***, *Mit Erläuternden Anmerkungen sowie mit Ergänzungen aus dem Briefwechsel Cantor-Dedekind*, edited by E. Zermelo, Julius Springer, Berlin, 1932.

[8] Richard Dedekind, ***Was sind und was sollen die Zahlen***, Friedrich Vieweg & Sohn, Braunschweig, 1888 (see also [9, pp. 335–390]).

[9] ———, ***Gesammelte mathematische Werke III***, edited by R. Fricke, E. Noether, and Ö. Ore, Friedrich Vieweg & Sohn, Braunschweig, 1932.

[10] Heinz-Dieter Ebbinghaus (in cooperation with Volker Peckhaus), ***Ernst Zermelo, An Approach to His Life and Work***, Springer-Verlag, Berlin · Heidelberg, 2007.

[11] Adolf Fraenkel, *Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre*, ***Mathematische Annalen***, vol. 86 (1922), 230–237.

[12] Gottlob Frege, ***Die Grundlagen der Arithmetik, eine logisch-mathematische Untersuchung über den Begriff der Zahl***, Verlag von Wilhelm Koebner, Breslau, 1884. [English translation (with German original): *The foundations of arithmetic. A logo-mathematical enquiry into the concept of number*, translated by J. L. Austin, Basil Blackwell, Oxford, 1950].

[13] ———, *The concept of number,* (translation into English of parts of [12]), *in* ***Philosophy of Mathematics. Selected Readings*** (Paul Benacerraf and Hilary Putnam, eds.), Cambridge University Press, Cambridge, 2nd ed., 1983, pp. 130–159.

[14] Lorenz Halbeisen and Saharon Shelah, *Consequences of arithmetic for set theory*, ***The Journal of Symbolic Logic***, vol. 59 (1994), 30–40.

[15] Friedrich Hartogs, *Über das Problem der Wohlordnung*, ***Mathematische Annalen***, vol. 76 (1915), 438–443.

[16] Felix Hausdorff, ***Grundzüge der Mengenlehre***, de Gruyter, Leipzig, 1914 [reprint: Chelsea, New York, 1965].

[17] Akihiro Kanamori, *Zermelo and set theory*, ***The Bulletin of Symbolic Logic***, vol. 10 (2004), 487–553.

[18] Alwin Korselt, *Über einen Beweis des Äquvialenzsatzes*, ***Mathematische Annalen***, vol. 70 (1911), 294–296.

[19] Gregory H. Moore, ***Zermelo's Axiom of Choice. Its origins, development, and influence***, [Studies in the History of Mathematics and Physical Sciences 8], Springer-Verlag, New York, 1982.

[20] John von Neumann, *Eine Axiomatisierung der Mengenlehre*, ***Journal für die Reine und Angewandte Mathematik***, vol. 154 (1925), 219–240 (see [29] for a translation into English).

[21] ———, *Die Axiomatisierung der Mengenlehre*, ***Mathematische Zeitschrift***, vol. 27 (1928), 669–752.

[22] ———, *Über eine Widerspruchfreiheitsfrage in der axiomatischen Mengenlehre*, ***Journal für die Reine und Angewandte Mathematik***, vol. 160 (1929), 227–241.

[23] Giuseppe Peano, *Super theorema de Cantor-Bernstein*, ***Rendiconti del Circolo Matematico di Palermo***, vol. 21 (1906), 360–366.

[24] Bertrand Russell, *On finite and infinite cardinal numbers,* (Section III of Whitehead [30]), ***American Journal of Mathematics***, vol. 24 (1902), 378–383.

[25] ———, *Mathematical logic as based on the theory of types*, ***American Journal of Mathematics***, vol. 30 (1908), 222–262 (also published in [29]).

[26] Ernst Schröder, *Über zwei Definitionen der Endlichkeit und G. Cantor'sche Sätze*, ***Nova Acta, Abhandlungen der Kaiserlich Leopoldinisch-Carolinisch Deutschen Akademie der Naturforscher***, vol. 71 (1898), 301–362.

[27] Wacław Sierpiński, ***Leçons sur les nombres transfinis***, Gauthier-Villars et Fils, Paris, 1928.

[28] Thoralf Skolem, *Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre*, Matematikerkongressen i Helsingfors den 4–7 Juli 1922, Den femte skandinaviska matematikerkongressen, Akademiska Bokhandeln, Helsingfors, 1923, pp. 217–232 (see [29] for a translation into English).

[29] Jean van Heijenoort, ***From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931***, [Source Books in the History of Science], Harvard University Press, Cambridge, Massachusetts, 1967.

[30] Alfred North Whitehead, *On cardinal numbers*, ***American Journal of Mathematics***, vol. 24 (1902), 367–394.

[31] Ernst Zermelo, *Beweis, dass jede Menge wohlgeordnet werden kann*, ***Mathematische Annalen***, vol. 59 (1904), 514–516 (see [29] for a translation into English).

[32] ———, *Neuer Beweis für die Möglichkeit einer Wohlordnung*, ***Mathematische Annalen***, vol. 65 (1908), 107–128 (see [29] for a translation into English).

[33] ———, *Über Grenzzahlen und Mengenbereiche. Neue Untersuchungen über die Grundlagen der Mengelehre*, ***Fundamenta Mathematicae***, vol. 16 (1930), 29–47.

[34] ———, *Untersuchungen über die Grundlagen der Mengenlehre. I.*, ***Mathematische Annalen***, vol. 65 (1908), 261–281 (see [29] for a translation into English).

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