Silver Measurability and its relation to other regularity properties

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Abstract

For $a \subseteq b \subseteq \omega$ with $b \backslash a$ infinite, the set $D = \{x \in [\omega]^\omega : a \subseteq x \subseteq b\}$ is called a *doughnut*. Doughnuts are equivalent to conditions of Silver forcing, and so, a set $S \subseteq [\omega]^\omega$ is called *Silver measurable*, or *completely doughnut*, if for every doughnut D there is a doughnut $D' \subseteq D$ which is contained in or disjoint from S. In this paper, we investigate the Silver measurability of $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets of reals and compare it to other regularity properties like the Baire and the Ramsey property and Miller and Sacks measurability.

0. *Introduction*

Most forcings that are used in *Set Theory of the Reals* belong to a class called **arboreal forcing notions**. A forcing notion $\mathbb{P}$ is called **arboreal** if its conditions are trees on either $2 = \{0, 1\}$ or ω ordered by inclusion and for each $T \in \mathbb{P}$, the set $[T]$ of all branches through T is homeomorphic to either ${}^\omega 2$ or ${}^\omega\omega$.

Each arboreal forcing notion is canonically related to a notion of measurability and a notion of smallness:

If $\mathbb{P}$ is an arboreal forcing notion, we define

$$\mathfrak{A}_{\mathbb{P}} := \{A : \forall T \in \mathbb{P}\,(\exists S \leq T([S] \subseteq A \text{ or } [S] \cap A = \varnothing))\}, \text{ and}$$

$$\mathfrak{I}_{\mathbb{P}} := \{A : \forall T \in P\,(\exists S \leq T([S] \cap A = \varnothing))\}.$$

We call the elements of $\mathfrak{A}_{\mathbb{P}}$ **$\mathbb{P}$-measurable sets** and the elements of $\mathfrak{I}_{\mathbb{P}}$ **$\mathbb{P}$-null sets**.[1] Standard examples of arboreal forcing notions are **Cohen forcing** $\mathbb{C}$ (the set of basic open sets in the standard topology of either ${}^\omega 2$ or ${}^\omega\omega$), **Sacks forcing** $\mathbb{S}$ (the set of perfect trees), **Miller forcing** $\mathbb{M}$ (the set of superperfect trees), **Silver forcing** $\mathbb{V}$ (the set of uniform perfect trees), **Mathias forcing** $\mathbb{R}$ (the set of basic Ellentuck neighbourhoods).[2] The corresponding notions of measurability and smallness have been investigated in many contexts, and some of them are known under different names: the sets in $\mathfrak{I}_{\mathbb{S}}$ are also called **Marczewski null**, the sets in $\mathfrak{A}_{\mathbb{R}}$ are also said to be **completely Ramsey**, and the sets in $\mathfrak{A}_{\mathbb{V}}$ are said to be **completely doughnut** (*cf.* Section 1·2).[3]

Being $\mathbb{P}$-measurable is considered a regularity property of a set, and the extent of these regularity properties has been investigated: usually, all $\mathbf{\Sigma}^1_1$ sets are $\mathbb{P}$-measurable;[4] there are $\mathbf{\Delta}^1_2$ sets that are not $\mathbb{P}$-measurable in the constructible universe $\mathbf{L}$, and very often the statements "Every $\mathbf{\Delta}^1_2$ set is $\mathbb{P}$-measurable" and "Every $\mathbf{\Sigma}^1_2$ set is $\mathbb{P}$-measurable" can be characterized in terms of transcendence over $\mathbf{L}$ as exemplified in Fact 0·1.

In the following, we shall write $\Gamma(\mathscr{B})$ ($\Gamma(\mathscr{D})$, $\Gamma(\mathscr{L})$, $\Gamma(\mathscr{M})$, $\Gamma(\mathscr{R})$, $\Gamma(\mathscr{S})$) for "Every Γ set has the Baire property (is completely doughnut, is Lebesgue measurable, is Miller measurable, is completely Ramsey, is Sacks measurable)".

FACT 0·1.

(i) *(Solovay)* $\mathbf{\Sigma}^1_2(\mathscr{L})$ *is equivalent to "for all* $r \in {}^\omega\omega$ *there is a measure* 1 *set of random reals over* $\mathbf{L}[r]$*",*

(ii) *(Solovay)* $\mathbf{\Sigma}^1_2(\mathscr{B})$ *is equivalent to "for all* $r \in {}^\omega\omega$ *there is a comeagre set of Cohen reals over* $\mathbf{L}[r]$*",*

(iii) [**JudShe89**, Theorem 3.1 (i)/(ii)] $\mathbf{\Delta}^1_2(\mathscr{L})$ *is equivalent to "for all* $r \in {}^\omega\omega$ *there is a random real over* $\mathbf{L}[r]$*",*

(iv) [**JudShe89**, Theorem 3.1 (iii)/(iv)] $\mathbf{\Delta}^1_2(\mathscr{B})$ *is equivalent to "for all* $r \in {}^\omega\omega$ *there is a Cohen real over* $\mathbf{L}[r]$*",*

(v) [**JudShe89**, Theorem 2.10] $\mathbf{\Delta}^1_2(\mathscr{R})$ *and* $\mathbf{\Sigma}^1_2(\mathscr{R})$ *are equivalent,*

[1] This general approach to regularity properties connected to forcing notions was considered in [**Bre95**], and continued in [**Löw98**], [**BreLöw99**], [**Bre00**] and [**Löw03**]. Even more general are the notions of **Marczewski field** and **Marczewski ideal** from [**Bal+01/02**]. In these publications, $\mathfrak{I}_{\mathbb{P}}$ was denoted by (p^0), p^0 or $s^0(\mathbb{P})$. As the letter $\mathfrak{I}$ insinuates, in most applications, $\mathfrak{I}_{\mathbb{P}}$ turns out to be an ideal, or even a σ-ideal.

[2] *Cf.* Section 1·1 for more detailed definitions.

[3] Note that the measurability property connected to Cohen forcing is the Baire property (a set A has the **Baire property** if there is an open set P such that $A \Delta P$ is meagre) which is not the same as membership in $\mathfrak{A}_{\mathbb{C}}$.

[4] There is a uniform approach via game proofs of analytic measurability for these regularity properties in [**Löw98**].

(vi) [**JudShe89**, Theorem 3.5 (iv)] $\mathbf{\Sigma}^1_2(\mathscr{R})$ *does not imply* $\mathbf{\Delta}^1_2(\mathscr{B})$,

(vii) [**BreLöw99**, Theorem 6.1] $\mathbf{\Sigma}^1_2(\mathscr{M})$ *and* $\mathbf{\Delta}^1_2(\mathscr{M})$ *are equivalent, and equivalent to "for all* $r \in {}^\omega\omega$ $({}^\omega\omega \cap \mathbf{L}[r]$ *is not dominating)",*

(viii) [**BreLöw99**, Theorem 7.1] $\mathbf{\Sigma}^1_2(\mathscr{S})$ *and* $\mathbf{\Delta}^1_2(\mathscr{S})$ *are equivalent, and equivalent to "for all* $r \in {}^\omega\omega$ $({}^\omega\omega \cap \mathbf{L}[r] \neq {}^\omega\omega)$*".*

Abstractly, one could describe Fact 0·1 (i)/(ii) as "Measurability of $\mathbf{\Sigma}^1_2$ sets corresponds to the existence of a large set of generics over $\mathbf{L}[r]$," whilst one could describe Fact 0·1 (iii)/(iv) as "Measurability of $\mathbf{\Delta}^1_2$ sets corresponds to the existence of generics over $\mathbf{L}[r]$." We follow [**BreLöw99**] and call theorems of type (i)/(ii) **"Solovay-type characterizations"**. We shall call theorems of type (iii)/(iv) **"Judah-Shelah-type characterizations"**.

In this paper, we shall investigate Silver measurability, continuing research from the paper [**Hal03**], in order to give complete diagrams of the implications between the properties $\mathscr{B}$, $\mathscr{D}$, $\mathscr{R}$ and $\mathscr{D}$, $\mathscr{M}$, $\mathscr{S}$ for $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets. It will also be shown that $\mathbf{\Delta}^1_2(\mathscr{D})$ implies that there are splitting reals over each $\mathbf{L}[r]$, and that $\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that there are unbounded reals over each $\mathbf{L}[r]$.

We shall introduce some notation, prove crucial abstract results and list earlier results on Silver measurability in Section 1. In Section 2 and Section 3 we prove our results about $\mathbf{\Delta}^1_2(\mathscr{D})$ and $\mathbf{\Sigma}^1_2(\mathscr{D})$, respectively. Finally, in Section 4, we shall summarize our results and list some open questions.

1. *Definitions and the abstract setting*

Throughout this paper we shall use standard set theoretic terminology which the reader can find, *e.g.*, in textbooks like [**BarJud95**].

1·1. *Trees*

As usual, $X^{<\omega}$ is the set of all finite sequences of elements of X, and a **tree on** X is a subset of $X^{<\omega}$ closed under initial segments. If $x \in {}^\omega X$ is a function from ω to X and $n \in \omega$ is a natural number, we denote the finite sequence $\langle x(0), x(1), ..., x(n-1)\rangle$ by $x{\restriction}n$ and call it **the restriction of** x **to** n. If $s \in X^{<\omega}$ and $t \in X^{<\omega}$ or $x \in {}^\omega X$, we can define the **concatenation of** s **and** t **(of** s **and** x**)**, denoted by $s^\frown t$ ($s^\frown x$) in the obvious way.

A tree on $2 = \{0,1\}$ is called **uniform** if for all $s, t \in T$ of the same length and all $i \in \{0,1\}$, we have

$$s^\frown i \in T \iff t^\frown i \in T.$$

If T is a tree, then a function $x \in {}^\omega X$ is called a **branch** through T, if for all $n \in \omega$, we have that $x{\restriction}n \in T$. The set of all branches through T is denoted by $[T]$. A tree T on 2 is called **perfect**, if for every $s \in T$ there is a $t \in T$ with $s \subseteq t$ such that both $t^\frown 0$ and $t^\frown 1$ belong to T; such a sequence t is called a **splitting node** of T.

A perfect tree T is canonically (order) isomorphic to the full binary tree $2^{<\omega}$, and the order isomorphism induces a homeomorphism $\Theta_T : [T] \to {}^\omega 2$. Note that if $B \subseteq [T]$ is a Borel set with a Borel code in $\mathbf{L}[r]$, then $\Theta_T[B]$ is a Borel set with a Borel code in $\mathbf{L}[r, T]$ since the homeomorphism can be read off in a recursive way from the tree T. This will be used later.

Similarly, if T is a tree on ω, we can call $s \in T$ an ω**-splitting node** if s has infinitely many immediate successors. A tree T is called **superperfect** if for each $s \in T$ there is an ω-splitting node $t \supseteq s$ with $t \in T$.

We can now use the special kinds of trees just defined to define the forcing notions mentioned in the introduction:

Silver forcing $\mathbb{V}$ is the set of all uniform perfect trees ordered by inclusion,[5] **Sacks forcing** $\mathbb{S}$ is the set of all perfect trees ordered by inclusion, and **Miller forcing** $\mathbb{M}$ is the set of all superperfect trees ordered by inclusion.

1·2. *Doughnuts*

Investigating arrow partition properties, Carlos DiPrisco and James Henle introduced in [**DiPHen00**] the so-called doughnut property: Let $[\omega]^\omega := \{x \subseteq \omega : |x| = \omega\}$. Then, for $a \subseteq b \subseteq \omega$ with $b \setminus a \in [\omega]^\omega$, the set $D = \{x \in [\omega]^\omega : a \subseteq x \subseteq b\}$ is called a **doughnut**, or more precisely, the (a, b)-doughnut, denoted by $[a, b]^\omega$.

Doughnuts are equivalent to uniform perfect trees in the following sense (*cf.* [**Hal03**]):

Fact 1·1. *Each uniform perfect tree $T \subseteq 2^{<\omega}$ corresponds in a unique way to a doughnut, and vice versa.*

Di Prisco and Henle said that a set A has the **doughnut property** if it either contains or is disjoint from a doughnut, and that it is **completely doughnut** if for every doughnut D there is a doughnut $D^* \subseteq D$ such that either $D^* \subseteq A$ or $D^* \cap A = \varnothing$.

By virtue of Fact 1·1, being completely doughnut is equivalent to being Silver measurable in the sense of the introduction.[6]

Silver measurability or the doughnut property was investigated by the first author in [**Bre95**], for analytic sets in terms of games by the third author in [**Löw98**], and for $\boldsymbol{\Sigma}^1_2$ sets by the second author in [**Hal03**]. In particular, all Borel and all analytic sets are completely doughnut.

By work of the second author on Cohen reals and doughnuts in [**Hal03**], we know that Cohen forcing adds a doughnut of Cohen reals:

Lemma 1·2. *Suppose that A is a $\Sigma^1_2(r)$ set for some real number r and c is a Cohen real over $\mathbf{L}[r]$. Then there is a uniform perfect tree $T \in \mathbf{L}[r, c]$ such that either $[T] \subseteq A$ or $[T] \cap A = \varnothing$.*

Proof. See (the proof of) [**Hal03**, Lemma 2.1]. □

Corollary 1·3. $\boldsymbol{\Delta}^1_2(\mathscr{B})$ *implies* $\boldsymbol{\Sigma}^1_2(\mathscr{D})$.

Proof. Immediate from Lemma 1·2 and Fact 0·1 (iv). □

1·3. *Weak Measurability*

The notion of $\mathbb{P}$-measurability is a Π_2 notion. By dropping the first universal quantifier one arrives at a weaker Σ_1 notion that is called weak $\mathbb{P}$-measurability: A set A is said to be **weakly $\mathbb{P}$-measurable** if there is a $T \in \mathbb{P}$ such that either $[T] \subseteq A$ or $[T] \cap A = \varnothing$. In

[5] Uniform perfect trees have been used in recursion theory, and are called **Lachlan 1-trees** there. *Cf.* [**Lac71**].

[6] The Ramsey property, originally defined in terms of the Baire property in the Ellentuck topology or in terms of partitions (*cf.* [**Kec95**, 19.D]), can be equivalently defined in terms of doughnuts.

general, the notion of weak measurability is not a statement about the regularity of a set: a set can contain a $\mathbb{P}$-condition T and be completely irregular outside of T. Compare this to the doughnut property from Section 1·2: as Silver measurability is equivalent to being completely doughnut, weak Silver measurability is equivalent to the doughnut property.

Although weak measurability of a single set does not imply its regularity, classwise statements of weak measurability suffice to prove full measurability as the following general lemma from [**BreLöw99**] shows:

LEMMA 1·4 (Brendle-Löwe (1999)). *Let $\mathbf{\Gamma}$ be a boldface pointclass closed under intersections with closed sets (in this paper, $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ are the only examples). Then the following are equivalent:*

(i) *Every set in $\mathbf{\Gamma}$ is Silver measurable, and*

(ii) *every set in $\mathbf{\Gamma}$ is weakly Silver measurable.*

Lemma 1·4 was proved in an abstract setting in [**BreLöw99**, Lemma 2.1].

1·4. *Borel codes and the Borel decomposition of $\mathbf{\Sigma}^1_2$ sets*

We fix some coding of all Borel sets (*e.g.*, the one used in [**Jec03**, p. 504*sqq.*]), and use standard notation: if c is a Borel code, we denote the decoded set by A_c, or A_c^M if we want to stress that it is decoded in the model M.

Shoenfield's analysis of $\mathbf{\Sigma}^1_2$ sets [**Kan94**, p. 171–175] gives us for each $\mathbf{\Sigma}^1_2$ set X a decomposition into ω_1 Borel sets

$$X = \bigcup_{\alpha<\omega_1} X_\alpha$$

that is absolute in the following sense:

If X is $\Sigma^1_2(r)$, and $\alpha < \omega_1^{\mathbf{L}[r]}$, then X_α has a Borel code $c_\alpha \in \mathbf{L}[r]$, and

$$X \cap \mathbf{L}[r] = \bigcup_{\alpha<\omega_1^{\mathbf{L}[r]}} A_{c_\alpha}^{\mathbf{L}[r]}.$$

Consequently, if $\omega_1^{\mathbf{L}[r]} = \omega_1^{\mathbf{V}}$, the entire Borel decomposition is represented by Borel codes in $\mathbf{L}[r]$. Moreover, if for some $x \in {}^\omega\omega$, $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{V}}$ and X is $\Sigma^1_2(r)$, then we find a Borel decomposition of X with all Borel codes in $\mathbf{L}[x, r]$.

1·5. *Quasigenericity*

Let $\mathfrak{I}$ be an ideal (on ${}^\omega 2$ or ${}^\omega\omega$), and M be a model of (a rich enough fragment of) set theory. We write $\mathrm{N}(\mathfrak{I}, M)$ for the set of all Borel sets B such that

- $B \in \mathfrak{I}$, and
- there is a Borel code for the set B in M.

It is well-known that there are characterizations of the generics of random and Cohen forcing via the ideals $\mathfrak{N}$ of Lebesgue null and $\mathfrak{M}$ of meagre sets, respectively.[7]

FACT 1·5 (Solovay).

- *A real r is random over M if and only if $r \notin \bigcup \mathrm{N}(\mathfrak{N}, M)$, and*
- *a real c is Cohen over M if and only if $c \notin \bigcup \mathrm{N}(\mathfrak{M}, M)$.*

[7] *Cf.* [**Kan94**, Theorem 11.10].

For arbitrary arboreal forcings $\mathbb{P}$ on ω, the set ${}^\omega\omega\backslash\bigcup \mathrm{N}(\mathfrak{I}_{\mathbb{P}}, M)$ is not in general the set of generics. But we can define a notion of quasi-genericity in analogy to Fact 1·5:

Let $\mathfrak{I}$ be an ideal on ${}^\omega\omega$ and M be a model of set theory. We set

$$\mathrm{QG}(\mathfrak{I}, M) := {}^\omega\omega\backslash\bigcup \mathrm{N}(\mathfrak{I}, M),$$

and call the elements of $\mathrm{QG}(\mathfrak{I}, M)$ **$\mathfrak{I}$-M-quasigeneric**. Analogously, we define the set $\mathrm{QG}(\mathfrak{I}, M)$ for ideals $\mathfrak{I}$ on ${}^\omega 2$.

1·6. *Some ideals and Silver Homogeneity*

The equivalence relation E_0, defined by $xE_0y \iff \forall^\infty n(x(n) = y(n))$, is well-known from Descriptive Set Theory.[8] We call a Borel set $A \subseteq {}^\omega 2$ an **E_0-selector** if for any distinct $x, y \in A$ there are infinitely many $n \in \omega$ such that $x(n) \neq y(n)$. This makes sure that A selects at most one element from each equivalence class of E_0 (see [**Zap04**, Section 2.3.10]). Denote the set of E_0-selectors with Sel_{E_0}.

Now, let $\mathfrak{I}_{E_0}$ be the σ-ideal of sets σ-generated by Borel E_0-selectors.

We define two further ideals closer to the notion of Silver measurability: We call a Borel set $A \subseteq {}^\omega 2$ **G-independent** if for any distinct $x, y \in A$ there are at least two $n \in \omega$ such that $x(n) \neq y(n)$; we call a G-independent set **parity preserving** if for each $x, y \in A$ the number of n such that $x(n) \neq y(n)$ is even (including ω). We call them "parity preserving" because of the following fact:

If A is parity preserving, $z \in {}^\omega 2$ and $x, y \in A$ such that $x\Delta z$ and $y\Delta z$ are finite,[9] then

$$x\Delta z \text{ is odd if and only if } y\Delta z \text{ is odd.}$$

The sets of G-independent sets and parity preserving G-independent sets are denoted by Ind_G and $\mathrm{Ind}_G^{\mathrm{pp}}$, respectively. The ideals σ-generated by Ind_G and $\mathrm{Ind}_G^{\mathrm{pp}}$ are denoted by $\mathfrak{I}_G$ and $\mathfrak{I}_G^{\mathrm{pp}}$.

By a result of Zapletal's, the ideal $\mathfrak{I}_G$ is the ideal σ-generated by Borel sets in $\mathfrak{I}_{\mathbb{V}}$, whence the notions of $\mathfrak{I}_G$-quasigenericity and $\mathfrak{I}_{\mathbb{V}}$-quasigenericity coincide [**Zap04**, Lemma 2.3.37]. We have

$$\mathfrak{I}_{E_0} \subseteq \mathfrak{I}_G^{\mathrm{pp}} \subseteq \mathfrak{I}_G,$$

and so every $\mathfrak{I}_G$-quasigeneric is $\mathfrak{I}_G^{\mathrm{pp}}$-quasigeneric, and every $\mathfrak{I}_G^{\mathrm{pp}}$-quasigeneric is $\mathfrak{I}_{E_0}$-quasigeneric.

An ideal $\mathfrak{I}$ on ${}^\omega 2$ is called **Silver homogeneous** if for each $T \in \mathbb{V}$, the canonical homeomorphism $\Theta_T : [T] \to {}^\omega 2$ preserves membership in $\mathfrak{I}$, *i.e.*, if $A \in \mathfrak{I}$, then $\Theta_T[A] \in \mathfrak{I}$.[10]

Observation 1·6. *The ideals $\mathfrak{I}_{\mathbb{V}}$, $\mathfrak{I}_G$, $\mathfrak{I}_G^{\mathrm{pp}}$ and $\mathfrak{I}_{E_0}$ are Silver homogeneous.*

Lemma 1·7 (First Homogeneity Lemma). *Let $\mathfrak{I}$ be a Silver homogeneous ideal on ${}^\omega 2$ and $T \in \mathbb{V}$. Suppose that there is an $\mathfrak{I}$-$\mathbf{L}[r,T]$-quasigeneric real x, then $\Theta_T^{-1}(x)$ is also $\mathfrak{I}$-$\mathbf{L}[r,T]$-quasigeneric.*

[8] It is the least non-smooth countable Borel equivalence relation and as such the object of the famous Generalized Glimm-Effros Dichotomy of Harrington, Kechris and Louveau [**HarKecLou90**]; *cf.* the survey paper [**Kec99**, p. 166-167].

[9] Here we interpret elements of ${}^\omega 2$ as sets of natural numbers and let $x\Delta y := \{n : x(n) \neq y(n)\}$.

[10] This is a slight generalization of Zapletal's notion of homogeneity [**Zap04**].

Proof. Let $x \in \mathrm{QG}(\mathfrak{I}, \mathbf{L}[r,T])$. We claim that $y := \Theta_T^{-1}(x)$ is also $\mathfrak{I}$-$\mathbf{L}[r,T]$-quasigeneric. This is a direct consequence of Silver homogeneity: take any Borel set $B \in \mathfrak{I}$ coded in $\mathbf{L}[r,T]$, then $B \cap [T]$ is still a Borel set from $\mathfrak{I}$ coded in $\mathbf{L}[r,T]$. We shift it from $[T]$ to ${}^\omega 2$ via Θ_T. By Silver homogeneity, it is still in $\mathfrak{I}$. But since Θ_T is recursively defined from T, $\Theta_T[B \cap [T]]$ is in $\mathrm{N}(\mathfrak{I}, \mathbf{L}[r,T])$. If $y \in B$, then $x \in \Theta_T[B \cap [T]]$, contradicting x's quasigenericity; thus, y cannot lie in B. $\square$

Note that Θ_T and Θ_T^{-1} preserve the property of being a uniform perfect tree: If S is a uniform perfect tree, then $\Theta_T^{-1}[S]$ is the set of branches through a uniform perfect subtree of T.

LEMMA 1·8 (Second Homogeneity Lemma). *Let $\mathfrak{I}$ be an ideal on ${}^\omega 2$, let $A = {}^\omega 2 \setminus \bigcup \mathrm{QG}(\mathfrak{I}, \mathbf{L}[r])$ and suppose that the following conditions are met:*

(i) *A is weakly Silver measurable,*

(ii) *$\mathfrak{I}$ is Silver homogeneous,*

(iii) *for each s there is an $\mathfrak{I}$-$\mathbf{L}[r,s]$-quasigeneric.*

Then there is a uniform perfect tree of $\mathfrak{I}$-$\mathbf{L}[r]$-quasigenerics.

Proof. Since A is weakly Silver measurable, there is either a uniform perfect tree whose branches are disjoint from A or one whose branches are all in A.

In the former case, all of the branches of that tree are quasigeneric by definition of A and we are done immediately.

In the latter case, all of the branches of T are non-quasigeneric. By the assumption, we can pick some $\mathfrak{I}$-$\mathbf{L}[r,T]$-quasigeneric real. Now the assumptions of the First Homogeneity Lemma 1·7 are satisfied, so we get a $\mathfrak{I}$-$\mathbf{L}[r,T]$-quasigeneric inside $[T]$. But since $\mathrm{QG}(\mathfrak{I}, \mathbf{L}[r,T]) \subseteq \mathrm{QG}(\mathfrak{I}, \mathbf{L}[r])$, this is absurd. $\square$

1·7. *Mansfield-Solovay statements and inaccessibility of ω_1 by reals*

For an arboreal forcing notion $\mathbb{P}$, we call the statement "For every $\Sigma^1_2(r)$ set A, either there is some $T \in \mathbb{P}$ such that $[T] \subseteq A$, or A does not contain any $\mathfrak{I}_\mathbb{P}$-$\mathbf{L}[r]$-quasigenerics" the **Mansfield-Solovay statement for $\mathbb{P}$ and r**. The reason for this name is the fact that the classical Mansfield-Solovay theorem [**Kan94**, Corollary 14.9] is equivalent to the Mansfield-Solovay statement for Sacks forcing $\mathbb{S}$: Since every real in ${}^\omega\omega \setminus \mathbf{L}[r]$ is $\mathfrak{I}_\mathbb{S}$-$\mathbf{L}[r]$-quasigeneric, the "or" condition is equivalent to "$A \subseteq \mathbf{L}[r]$".

Unpublished work of Zapletal shows that the Mansfield-Solovay statement for Silver forcing $\mathbb{V}$ cannot hold for r if $\omega_1^{\mathbf{L}[r]}$ is countable: if $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{V}}$, then there is a (lightface) Σ^1_2 set that contains $\mathfrak{I}_\mathbb{V}$-$\mathbf{L}$-quasigenerics but no uniform perfect tree. The Mansfield-Solovay statement for $\mathbb{V}$ is true, however, for those r with uncountable $\omega_1^{\mathbf{L}[r]}$ (Lemma 1·10).

These two facts require us to distinguish between the case "ω_1 is *inaccessible* by reals" and the case "ω_1 is *accessible* by reals" several times in the sequel. Therefore, let us state two observations about Silver measurability for these proofs by cases:

LEMMA 1·9. *If ω_1 is inaccessible by reals (i.e., for all reals r, $\omega_1^{\mathbf{L}[r]}$ is countable), then $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ holds.*

Proof. Clear by Corollary 1·3 and Fact 0·1 (iv): The existence of Cohen reals gives us $\boldsymbol{\Delta}^1_2(\mathscr{B})$, and that in turn yields $\boldsymbol{\Sigma}^1_2(\mathscr{D})$. $\square$

LEMMA 1·10. *The Mansfield-Solovay statement for $\mathbb{V}$ is true for all $r \in {}^\omega\omega$ such that $\omega_1^{\mathbf{L}[r]} = \omega_1$, i.e., every $\Sigma^1_2(r)$ set either contains the branches through a uniform perfect tree or does not contain any $\mathfrak{I}_\mathbb{V}$-$\mathbf{L}[r]$-quasigenerics.*

Proof. Let X be a $\Sigma^1_2(r)$ set and $X = \bigcup_{\alpha<\omega_1} X_\alpha$ be the Borel decomposition with Borel codes in $\mathbf{L}[r]$. By Silver measurability of Borel sets, each X_α either contains the branches through a uniform perfect tree or is in $\mathfrak{I}_{\mathbb{V}}$.

Case 1. If for some uniform perfect tree T and some $\alpha < \omega_1$, we have $[T] \subseteq X_\alpha$, then $[T] \subseteq X$.

Case 2. If all X_α are in $\mathfrak{I}_{\mathbb{V}}$, then none of them can contain any $\mathfrak{I}_{\mathbb{V}}$-$\mathbf{L}[r]$-quasigeneric (because they all have Borel codes in $\mathbf{L}[r]$), so $\mathrm{QG}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r]) \cap X = \varnothing$. □

2. $\mathbf{\Delta}^1_2$ *sets*

We connect the existence of quasigenerics to Silver measurability and deduce some consequences for the relationship of Silver measurability to other regularity properties.

Proposition 2·1. *If for all $r \in {}^\omega\omega$ there is an $\mathfrak{I}_{\mathbb{V}}$-$\mathbf{L}[r]$-quasigeneric real, then every $\mathbf{\Delta}^1_2$ set is Silver measurable.*

Proof. If ω_1 is inaccessible by reals, we are done by Lemma 1·9. So, let us assume that there is some x such that $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{V}}$.

By Lemma 1·4 we only have to show that for every $\mathbf{\Delta}^1_2$ set X there is a T in $\mathbb{V}$ such that either $[T] \subseteq X$ or $[T] \cap X = \varnothing$. Given a $\Delta^1_2(r)$ set X, let Y be its complement and $X = \bigcup_{\alpha<\omega_1} X_\alpha$ and $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$ be the Borel decompositions of X and Y. By our assumption and the absoluteness of the Borel decomposition, all of these sets have a Borel code in $\mathbf{L}[x, r]$.

Case 1. There is an α such that $X_\alpha \notin \mathfrak{I}_{\mathbb{V}}$. Since X_α is Borel and thus has the doughnut property, there is $T \in \mathbb{V}$ such that $[T] \subseteq X_\alpha \subseteq X$.

Case 2. There is an α such that $Y_\alpha \notin \mathfrak{I}_{\mathbb{V}}$. Since Y_α is Borel, this means that there is $T \in \mathbb{V}$ such that $[T] \subseteq Y_\alpha \subseteq Y$.

Case 3. For all α, both X_α and Y_α are Silver null.

Then $\bigcup_{\alpha<\omega_1} (X_\alpha \cup Y_\alpha) \subseteq \bigcup \mathrm{N}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[x, r])$, hence it cannot contain a quasigeneric. But

$$\bigcup_{\alpha<\omega_1} (X_\alpha \cup Y_\alpha) = X \cup Y = {}^\omega 2,$$

contradicting the existence of quasigenerics over $\mathbf{L}[x, r]$. □

It is easy to see that both in the Silver model and in the random model, we have $\mathfrak{I}_{\mathbb{V}}$-quasigenerics, so we get two consequences:

Corollary 2·2. *An ω_1-iteration with countable support of Silver forcing, starting from $\mathbf{L}$, yields a model in which every $\mathbf{\Delta}^1_2$ set is Silver measurable. Similarly for an ω_1-iteration of random forcing starting from $\mathbf{L}$.*

Corollary 2·3. *$\mathbf{\Delta}^1_2(\mathscr{L})$ implies $\mathbf{\Delta}^1_2(\mathscr{D})$.*

Proof. Note that the generators of $\mathfrak{I}_G$ are Lebesgue null sets, so every random real is $\mathfrak{I}_G$-quasigeneric (and thus $\mathfrak{I}_{\mathbb{V}}$-quasigeneric). The claim now follows by Proposition 2·1 and Fact 0·1 (iii). □

Proposition 2·4. *$\mathbf{\Delta}^1_2(\mathscr{D})$ implies that for all $r \in {}^\omega 2$ there is an $\mathfrak{I}_G^{\mathrm{pp}}$-$\mathbf{L}[r]$-quasigeneric (and thus, a fortiori, an $\mathfrak{I}_{E_0}$-$\mathbf{L}[r]$-quasigeneric).*

Proof. Assume towards a contradiction that there is an r such that $\mathrm{QG}(\mathfrak{I}_G^{\mathrm{pp}}, \mathbf{L}[r]) = \varnothing$. Now, for each $x \in {}^\omega 2$ define the set

$$C_x := \{c \in \mathbf{L}[r] : c \text{ is a Borel code } \& \ A_c \in \mathrm{Ind}_G^{\mathrm{pp}} \ \& \ \exists y \in A_c (y \Delta x \text{ is finite})\}.$$

We fix some $x \in {}^\omega 2$. By our assumption, x is not $\mathfrak{I}_G^{\mathrm{pp}}$-$\mathbf{L}[r]$-quasigeneric, so is in some set in $\mathfrak{I}_G^{\mathrm{pp}}$, hence in some parity preserving G-independent set, so C_x is a non-empty $\Sigma^1_2(r,x)$ set. Pick the $<_{\mathbf{L}[r]}$-least element of C_x and call it c_x. If $y_0, y_1 \in A_{c_x}$ and $y_0 \Delta x$ and $y_1 \Delta x$ are both finite, then they have the same parity since A_{c_x} is parity preserving. Let $n_x := 0$ if $y_0 \Delta x$ is even and $n_x := 1$ if it is odd.

Define $C_0 := \{x : n_x = 0\}$ and $C_1 := \{x : n_x = 1\}$. Since the canonical wellordering $<_{\mathbf{L}[r]}$ of $\mathbf{L}[r]$ is a $\Delta^1_2(r)$-wellordering, both of these sets are $\Sigma^1_2(r)$ sets, and hence $\boldsymbol{\Delta}^1_2$ sets (by our assumption, we have $C_0 \cup C_1 = {}^\omega 2$).

But neither C_0 nor C_1 contains a uniform perfect tree: If $z \in C_0$ and T is a uniform perfect tree with $z \in [T]$, then $[T]$ contains infinitely many elements $\{z_n : n \in \omega\}$ that differ in exactly one place from z (say, $z(k_n) \neq z_n(k_n)$).

Note that $c_z = c_{z_m}$. Pick some $y \in A_{c_z}$ such that $z \Delta y$ is finite and even. Choose k_m such that $k_m \notin z \Delta y$, then $z_m \Delta y$ and $z \Delta y$ have different parity, so $z_m \notin C_0$.

The same argument works for C_1. Consequently, neither C_0 nor C_1 contain a uniform perfect tree, and thus they cannot be Silver measurable. □

With a similar technique, we can show:

Proposition 2·5. *$\boldsymbol{\Delta}^1_2(\mathscr{D})$ implies that for all reals r there is a splitting real over $\mathbf{L}[r]$.*

Proof. For $x \in [\omega]^\omega$ let $\tau_x \in {}^\omega\omega$ be the increasing enumeration of x and let $\widehat{x} \in [\omega]^\omega$ be defined as follows:

$$k \in \widehat{x} \iff \exists n \in \omega \big(\tau_x(2n) < k \leq \tau_x(2n+1)\big).$$

Assume towards a contradiction that there is $r \in [\omega]^\omega$ such that there is no splitting real over $\mathbf{L}[r]$, which is equivalent to

$$\exists r \in [\omega]^\omega \, \forall x \in [\omega]^\omega \, \exists y \in [\omega]^\omega \cap \mathbf{L}[r] \, \big(y \cap x \text{ or } y \setminus x \text{ is finite}\big).$$

Now, for each $x \in [\omega]^\omega$ pick the $<_{\mathbf{L}[r]}$-least $y_x \in [\omega]^\omega \cap \mathbf{L}[r]$ such that $y_x \cap \widehat{x}$ or $y_x \setminus \widehat{x}$ is finite, and let $A \subseteq [\omega]^\omega$ be the set of all x for which the former case holds. It is easy to see that A is a $\Delta^1_2(r)$ set and that A does neither contain nor is disjoint from any uniform perfect tree, which completes the proof. □

3. $\boldsymbol{\Sigma}^1_2$ *sets*

We can use the Second Homogeneity Lemma 1·8 to derive a result about $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ and the existence of quasigenerics:

Lemma 3·1. *The following are equivalent:*

(i) *For all r, we have $\mathrm{QG}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r]) \neq \varnothing$ and $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ holds, and*

(ii) *for all r, the set $\mathrm{QG}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r])$ is co-Silver null (i.e., its complement is in $\mathfrak{I}_{\mathbb{V}}$).*

Proof. "$\Rightarrow$": Consider the $\boldsymbol{\Sigma}^1_2$ set $X = \bigcup \mathrm{N}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r])$. Our assumption $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ implies that X is weakly Silver measurable. Let T be an arbitrary uniform perfect tree. We have to show that there is a uniform perfect subtree $S \subseteq T$ that consists of quasigenerics.

We can apply the Second Homogeneity Lemma 1·8, and get a uniform perfect tree of quasigenerics. Now we can use the First Homogeneity Lemma 1·7 to copy that tree into T.

"$\Leftarrow$": If ω_1 is inaccessible by reals, then Lemma 1·9 yields the claim. So, let $x \in {}^\omega\omega$ be such that $\omega_1^{\mathbf{L}[x]} = \omega_1$.

Now we can view a given $\Sigma^1_2(r)$ set X as a $\Sigma^1_2(x,r)$ set and apply our weak version of the Mansfield-Solovay theorem for Silver forcing, Lemma 1·10. By Lemma 1·4, we only have to show that either X or its complement contains the branches through a uniform perfect tree.

If X does not contain the branches through a uniform perfect tree, then by Lemma 1·10, $\mathrm{QG}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[x,r]) \cap X = \varnothing$. But by our assumption, $\mathrm{QG}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[x,r])$ contains a uniform perfect tree. □

The next Proposition 3·2 is not exactly a characterization of $\mathbf{\Sigma}^1_2(\mathscr{D})$, but very close to one, since the ideals $\mathfrak{I}_G$ and $\mathfrak{I}_G^{\mathrm{pp}}$ are very similar, and thus the notions of $\mathfrak{I}_{\mathbb{V}}$-quasigenerics and $\mathfrak{I}_G^{\mathrm{pp}}$-quasigenerics are very close. (*Cf.* Question 6.)

Proposition 3·2.
(i) *If for each r the set of $\mathfrak{I}_{\mathbb{V}}$-$\mathbf{L}[r]$-quasigenerics is co-Silver null, then $\mathbf{\Sigma}^1_2(\mathscr{D})$ holds.*
(ii) *If $\mathbf{\Sigma}^1_2(\mathscr{D})$ holds, then for each r the set of $\mathfrak{I}_G^{\mathrm{pp}}$-$\mathbf{L}[r]$-quasigenerics is co-Silver null (and hence also the set of $\mathfrak{I}_{E_0}$-$\mathbf{L}[r]$-quasigenerics).*

Proof. "(i)": This is an immediate consequence of Lemma 3·1.

"(ii)": For the second implication, we apply the Homogeneity Lemmas again as in Lemma 3·1: Consider the $\mathbf{\Sigma}^1_2$ set $X = \bigcup \mathrm{N}(\mathfrak{I}_G^{\mathrm{pp}}, \mathbf{L}[r])$. $\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that X is weakly Silver measurable. This time, we use the Silver homogeneity of $\mathfrak{I}_G^{\mathrm{pp}}$ (Observation 1·6). After we fixed a uniform perfect tree T, we can use the quasigenerics given by Proposition 2·4, and then apply the Second Homogeneity Lemma 1·8. We again get a uniform perfect tree of quasigenerics which we paste into T by use of the First Homogeneity Lemma 1·7. □

We can also connect $\mathbf{\Sigma}^1_2(\mathscr{D})$ to splitting reals, and almost get a converse to Proposition 2·5.

Lemma 3·3. *If $s \in [\omega]^\omega$ splits the set A (i.e., for all $a \in A$, both $a \cap s$ and $a \backslash s$ are infinite), then there is a uniform perfect tree T such that $[T] \cap A = \varnothing$.*

Proof. Define

$$U_s := \left\{ t \in 2^{<\omega} : (n \notin s \ \&\ n \in \mathrm{dom}(t)) \to t(n) = 0 \right\}.$$

Since s is an infinite set, U_s is a uniform perfect tree. If now $a \in A$, then by the assumption there is an n such that $n \in a \backslash s$, so the real associated to a cannot belong to $[U_s]$. □

Proposition 3·4. *If for each r there is a splitting real over $\mathbf{L}[r]$, then every $\mathbf{\Sigma}^1_2$ set either contains the branches through a perfect tree or its complement contains the branches through a uniform perfect tree.*

Proof. By Mansfield-Solovay [**Kan94**, Corollary 14.9], every $\mathbf{\Sigma}^1_2$ set A either contains a perfect subset or is contained in $\mathbf{L}[r]$. But if it is contained in $\mathbf{L}[r]$, we can take the splitting real and construct a uniform perfect tree in the complement of A by Lemma 3·3. □

We shall see later that Proposition 3·4 cannot be improved to "If for each r there is a splitting real over $\mathbf{L}[r]$, then every $\mathbf{\Sigma}^1_2$ set is weakly Silver measurable" (Corollary 3·6 proves that $\mathbf{\Delta}^1_2(\mathscr{D})$ and $\mathbf{\Sigma}^1_2(\mathscr{D})$ are not equivalent).

PROPOSITION 3·5. *$\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that for each $r \in {}^\omega\omega$ there is an unbounded real over $\mathbf{L}[r]$.*

Proof. We shall construct a tree $P_f \subseteq 2^{<\omega}$ which belongs to $\mathfrak{I}_{E_0}$ for every strictly increasing function $f \in {}^\omega\omega$; and for every uniform perfect tree T we shall construct a function $g_T \in {}^\omega\omega$, such that $f > g_T$ implies $[P_f] \cap [T] \neq \varnothing$. These assignments $f \mapsto P_f$ and $T \mapsto g_T$ form a Galois-Tukey connection and thus give us the claim by Proposition 3·2 (ii).

For $T \in \mathbb{V}$, g_T is just the increasing enumeration of the split levels of $[T]$.

For $f \in {}^\omega\omega$, let $k_0 = 0$ and $k_{n+1} = f(k_n + 1)$. We construct the tree P_f by induction. For $n = 0$, let $P^n_f = 2^{<\omega}$ be the full binary tree. Assume we have already constructed P^n_f for some $n \in \omega$. Let $P^n_f|_{k_{n+1}} = \{t \in P^n_f : |t| \leq k_{n+1}\}$. Further, for every $t \in 2^{<\omega}$ with $|t| = k_{n+1}$ let $\xi^t_n \in 2$ be defined as follows:

$$\xi^t_n = \begin{cases} 0 & \text{if } \ t(n) \equiv |\{m : n < m < k_{n+1} \text{ and } \ t(m) = 0\}| \mod 2, \\ 1 & \text{otherwise.} \end{cases}$$

Now, define

$$(P^n_f)^* := \{s \in P^n_f : \exists t, t' \in 2^{<\omega}\,(\,|t| = k_{n+1} \ \&\ s = t^\frown\xi^t_n{}^\frown t'\,)\}, \text{ and}$$

$$P^{n+1}_f = P^n_f|_{k_{n+1}} \cup (P^n_f)^*.$$

Finally, let $P_f = \bigcap_{n\in\omega} P^n_f$, then, by construction, $[P_f]$ is a closed set in $\mathfrak{I}_{E_0}$ with parameter f. To see that $[P_f] \in \mathfrak{I}_{E_0}$, assume towards a contradiction that there are two distinct $x, y \in [P_f]$ and an $m \in \omega$ such that $x(m) \neq y(m)$ and for all $m' > m$, $x(m') = y(m')$. Then, by construction, we get $x(k_{m+1}) \neq y(k_{m+1})$, and since $k_{m+1} > m$, this is a contradiction.

Further, if $f > g_T$, then $g_T(k_n) < k_{n+1}$, which implies that for any $n \in \omega$, there is a split level of T between k_n and k_{n+1}, and thus, by construction, we have $[P_f] \cap [T] \neq \varnothing$. □

As a consequence we get:

COROLLARY 3·6. *An ω_1-iteration with countable support of Silver forcing, starting from $\mathbf{L}$, yields a model $\mathbf{W}$ in which we have $\mathbf{\Delta}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Sigma}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{B})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{R})$.*

Proof. Firstly recall that Silver forcing does not add unbounded reals. Thus, since $\mathbf{\Delta}^1_2(\mathscr{R})$ implies that for all $r \in {}^\omega\omega$ there is a dominating real over $\mathbf{L}[r]$, we have $\mathbf{W} \models \neg\mathbf{\Delta}^1_2(\mathscr{R})$. Secondly, in Corollary 2·2 we have seen that an ω_1-iteration of Silver forcing with countable support, starting from $\mathbf{L}$, yields a model $\mathbf{W}$ in which every $\mathbf{\Delta}^1_2$ set is Silver measurable, and in Proposition 3·5 we have seen that $\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that for every real r, there are unbounded reals over $\mathbf{L}[r]$. Hence, since Silver forcing does not add unbounded reals, by Corollary 1·3 we have $\mathbf{W} \models \mathbf{\Delta}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Sigma}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{B})$. □

PROPOSITION 3·7. *Let $\mathbb{C}_{\omega_1}$ be the ω_1-product with finite support of Cohen forcing. Then*

$$\mathbf{V}^{\mathbb{C}_{\omega_1}} \models \text{“all projective sets are Silver measurable”}.$$

Proof. Let $A = \{y : \varphi(y)\}$, where φ is a $\mathbf{\Sigma}^1_n$-formula with some parameter r. Given $[a,b]^\omega \in \mathbf{V}^{\mathbb{C}_{\omega_1}}$, we want to find $[a',b']^\omega \subseteq [a,b]^\omega$ such that either $[a',b']^\omega \subseteq A$ or $[a',b']^\omega \cap A = \varnothing$. Without loss of generality, let us assume that $a, b, r \ldots$ belong to $\mathbf{V}$. Recall that $\mathbb{C}_{\omega_1}$ is homogeneous, and therefore, for every sentence σ of the forcing language with parameters in $\mathbf{V}$ we have either $[\![\sigma]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{1}$ or $[\![\sigma]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{0}$. Notice also that if $c \in \mathbf{V}^{\mathbb{C}_{\omega_1}}$ is Cohen-generic over $\mathbf{V}$, say, added by $\mathbb{C}$, and if we decompose $\mathbb{C}_{\omega_1} = \mathbb{C} * \dot{\mathbb{A}}$, then in $\mathbf{V}[c]$, $\dot{\mathbb{A}}[c]$ is isomorphic to $\mathbb{C}_{\omega_1}$.

Let us consider $\varphi(c)$: By homogeneity, in $\mathbf{V}[c]$ we have either $[\![\varphi(c)]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{1}$ or $[\![\varphi(c)]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{0}$. Hence, in $\mathbf{V}$, we have $p(\mathbf{1}) \vee p(\mathbf{0}) = \mathbf{1}$ and $p(\mathbf{1}) \wedge p(\mathbf{0}) = \mathbf{0}$, where $p(\mathbf{1}) = [\![\,[\![\varphi(\dot{c})]\!]_{\dot{\mathbb{C}}_{\omega_1}} = \dot{\mathbf{1}}]\!]_{\mathbb{C}}$ and $p(\mathbf{0}) = [\![\,[\![\varphi(\dot{c})]\!]_{\dot{\mathbb{C}}_{\omega_1}} = \dot{\mathbf{0}}]\!]_{\mathbb{C}}$. Now, in $\mathbf{V}[c]$ we find a doughnut $[a',b']^\omega \subseteq [a,b]^\omega$ such that for all $x \in [a',b']^\omega$, x is Cohen-generic over $\mathbf{V}$ (because Cohen forcing adds a uniform perfect tree of Cohen reals). By shrinking $[a',b']^\omega$ if necessary, we may assume that $[a',b']^\omega \subseteq p(\mathbf{1})$ or $[a',b']^\omega \subseteq p(\mathbf{0})$. Let us consider just the former case, since the latter case is similar.

We claim that $[a',b']^\omega \subseteq A$: If $x \in [a',b']^\omega \subseteq p(\mathbf{1})$, then x is Cohen-generic over $\mathbf{V}$ (no matter where x is). Thus, $\mathbf{V}[x] \models [\![\varphi(x)]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{1}$. But the extension leading to $\mathbf{V}^{\mathbb{C}_{\omega_1}}$ is a $\mathbb{C}_{\omega_1}$-extension, hence, $\mathbf{V}^{\mathbb{C}_{\omega_1}} \models \varphi(x)$. $\square$

4. *Conclusion*

THEOREM 4·1. *The only implications between the properties $\mathscr{D}$, $\mathscr{B}$ and $\mathscr{R}$ of $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets are given in the following transitive diagram:*

$$\begin{tikzcd}
\mathbf{\Sigma}^1_2(\mathscr{B}) \arrow[dr, Rightarrow] & & & & \\
& \mathbf{\Delta}^1_2(\mathscr{B}) \arrow[dr, Rightarrow] & & \mathbf{\Delta}^1_2(\mathscr{R}) \arrow[dl, Rightarrow] \arrow[r, Leftrightarrow] & \mathbf{\Sigma}^1_2(\mathscr{R}) \\
& & \mathbf{\Sigma}^1_2(\mathscr{D}) \arrow[d, Rightarrow] & & \\
& & \mathbf{\Delta}^1_2(\mathscr{D}) & &
\end{tikzcd}$$

Proof. For the implications between the Baire and Ramsey property of $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets see [**Jud88**] and [**JudShe89**].

$\mathbf{\Sigma}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{R})$: This follows from $\mathbf{\Sigma}^1_2(\mathscr{R}) \Leftrightarrow \mathbf{\Delta}^1_2(\mathscr{R})$ (*cf.* Fact 0·1 (v)) and $\mathbf{\Sigma}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Sigma}^1_2(\mathscr{R})$ (*cf.* [**Hal03**]).

$\mathbf{\Sigma}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{B})$: This follows from the obvious implication $\mathbf{\Sigma}^1_2(\mathscr{R}) \Rightarrow \mathbf{\Sigma}^1_2(\mathscr{D})$ and $\mathbf{\Sigma}^1_2(\mathscr{R}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{B})$ (*cf.* Fact 0·1 (vi)).

$\mathbf{\Delta}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Sigma}^1_2(\mathscr{D})$: This follows from Corollary 3·6. $\square$

PROPOSITION 4·2. *The only implications between the properties $\mathscr{D}$, $\mathscr{M}$ and $\mathscr{S}$ of $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets are given in the following diagram:*

$$\begin{tikzcd}[column sep=small]
 & & \mathbf{\Sigma}^1_2(\mathscr{D}) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathbf{\Sigma}^1_2(\mathscr{M}) \arrow[r, Leftrightarrow] & \mathbf{\Delta}^1_2(\mathscr{M}) \arrow[dr, Rightarrow] & & \mathbf{\Delta}^1_2(\mathscr{D}) \arrow[dl, Rightarrow] \\
 & & \mathbf{\Sigma}^1_2(\mathscr{S}) \arrow[r, Leftrightarrow] & \mathbf{\Delta}^1_2(\mathscr{S})
\end{tikzcd}$$

Proof. The equivalences for $\mathscr{M}$ and $\mathscr{S}$ are Fact 0·1 (vii) & (viii).

$\mathbf{\Sigma}^1_2(\mathscr{D}) \Rightarrow \mathbf{\Delta}^1_2(\mathscr{M})$: This follows from Proposition 3·5 and Fact 0·1 (vii).

$\mathbf{\Delta}^1_2(\mathscr{D}) \Rightarrow \mathbf{\Sigma}^1_2(\mathscr{S})$: This follows from Proposition 2·4 and Fact 0·1 (viii).

$\mathbf{\Delta}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Sigma}^1_2(\mathscr{D})$: This is Corollary 3·6.

$\mathbf{\Sigma}^1_2(\mathscr{S}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{D})$: Sacks forcing doesn't add splitting reals, so by Proposition 2·5, $\mathbf{\Delta}^1_2(\mathscr{D})$ can't be true in the Sacks model which is a model of $\mathbf{\Sigma}^1_2(\mathscr{S})$.

$\mathbf{\Sigma}^1_2(\mathscr{S}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{M})$: Similarly, Sacks forcing doesn't add unbounded reals, so by Fact 0·1 (vii), the Sacks model can't be a model of $\mathbf{\Delta}^1_2(\mathscr{M})$.

$\mathbf{\Delta}^1_2(\mathscr{M}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{D})$: Miller forcing doesn't add splitting reals, so the Miller model is a model of $\mathbf{\Delta}^1_2(\mathscr{M})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{D})$.

$\mathbf{\Delta}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{M})$: As mentioned in the proof of Corollary 3·6, Silver forcing doesn't add unbounded reals, so the Silver model is a model of $\mathbf{\Delta}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{M})$. □

Note that in the proof of Proposition 4·2, we get

$$\mathbf{\Sigma}^1_2(\mathscr{S})\,\&\,\mathbf{\Delta}^1_2(\mathscr{M})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Sigma}^1_2(\mathscr{D})$$

in the Miller model,

$$\mathbf{\Sigma}^1_2(\mathscr{S})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{M})\,\&\,\mathbf{\Delta}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Sigma}^1_2(\mathscr{D})$$

in the Silver model, and

$$\mathbf{\Sigma}^1_2(\mathscr{S})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{M})\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Sigma}^1_2(\mathscr{D})$$

in the Sacks model. This suggests the question about the natural dual to the situation in the Sacks model:

QUESTION 1. *Is*

$$\mathbf{\Sigma}^1_2(\mathscr{S})\,\&\,\mathbf{\Delta}^1_2(\mathscr{M})\,\&\,\mathbf{\Delta}^1_2(\mathscr{D})\,\&\,\neg\mathbf{\Sigma}^1_2(\mathscr{D})$$

consistent?

In [**Jud88**] it is proved that

$$\mathbf{\Sigma}^1_2(\mathscr{K}_\sigma) \iff \forall r \in {}^\omega\omega\,({}^\omega\omega \cap \mathbf{L}[r] \text{ is bounded}),$$

so $\mathbf{\Sigma}^1_2(\mathscr{K}_\sigma)$ implies $\mathbf{\Sigma}^1_2(\mathscr{M})$.[11] By "$\mathbf{\Delta}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{M})$" from Proposition 4·2, $\mathbf{\Delta}^1_2(\mathscr{D})$ doesn't imply $\mathbf{\Sigma}^1_2(\mathscr{K}_\sigma)$, but we don't know anything about the converse:

QUESTION 2. *Does $\mathbf{\Sigma}^1_2(\mathscr{K}_\sigma)$ imply $\mathbf{\Delta}^1_2(\mathscr{D})$? In particular, is the Laver model a model of $\mathbf{\Sigma}^1_2(\mathscr{K}_\sigma)\,\&\,\neg\mathbf{\Delta}^1_2(\mathscr{D})$?*

[11] By $\mathbf{\Sigma}^1_2(\mathscr{K}_\sigma)$ we denote the statement "Every $\mathbf{\Sigma}^1_2$ set is K_σ-regular".

We have succeeded in determining the strength of $\boldsymbol{\Delta}^1_2(\mathscr{D})$ and $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ in terms of other regularity properties. What is still missing are results of Solovay- and Judah-Shelah-type: Propositions 2·1, 2·4, and 3·2 yield almost equivalences since the ideals $\mathfrak{I}_{\mathbb{V}}$ and $\mathfrak{I}^{\mathrm{pp}}_G$ are very close to each other.

QUESTION 3. *Is there an ideal $\mathfrak{I}$ such that the following hold:*

(i) $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ *is equivalent to "for all r, the set of $\mathfrak{I}$-$\mathbf{L}[r]$-quasigenerics is co-Silver null", and*

(ii) $\boldsymbol{\Delta}^1_2(\mathscr{D})$ *is equivalent to "for all r, there is an $\mathfrak{I}$-$\mathbf{L}[r]$-quasigeneric real".*

In the following diagram, we focus on existence statements of special real numbers. We shall abbreviate "there is an $\mathfrak{I}_{\mathbb{V}}$-quasigeneric ($\mathfrak{I}^{\mathrm{pp}}_G$-quasigeneric, $\mathfrak{I}_{E_0}$-quasigeneric, splitting real)" by $[\mathsf{v}]$ ($[\mathsf{p}]$, $[\mathsf{e}]$, $[\mathsf{s}]$, respectively), and get the following diagram from Propositions 2·1, 2·4 and 2·5:

$$\begin{tikzcd}
& {[\mathsf{v}]} \arrow[d, Rightarrow] & \\
& \boldsymbol{\Delta}^1_2(\mathscr{D}) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
{[\mathsf{p}]} \arrow[d, Rightarrow] & & {[\mathsf{s}]} \\
{[\mathsf{e}]} & &
\end{tikzcd}$$

Can we get the reverse directions anywhere in this diagram?

QUESTION 4. *Does $\boldsymbol{\Delta}^1_2(\mathscr{D}) \Rightarrow [\mathsf{v}]$ hold (the converse to Proposition 2·1) ?*

Note that if $\boldsymbol{\Delta}^1_2(\mathscr{D}) \Rightarrow [\mathsf{v}]$, then we can also characterize $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ in terms of quasigenerics by Lemma 3·1: In that case, the converse to Proposition 3·2 (i) holds as well.

QUESTION 5. *Does the existence of a splitting real over each $\mathbf{L}[r]$ imply $\boldsymbol{\Delta}^1_2(\mathscr{D})$ (the converse to Proposition 2·5) ?*

QUESTION 6. *Can we reverse any of the arrows between $\boldsymbol{\Delta}^1_2(\mathscr{D})$ and the existence statements of the different quasigenerics, i.e., can we prove any of the implications*

$$[\mathsf{e}] \Rightarrow [\mathsf{p}] \Rightarrow \boldsymbol{\Delta}^1_2(\mathscr{D})\ ?^{12}$$

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[12] James Hirschorn (Vienna) has a simple argument that the three notions of quasigenericity are not equivalent. For instance, for a countable model M, he gives an example of a real which is $\mathfrak{I}_{E_0}$-M-quasigeneric, but not $\mathfrak{I}^{\mathrm{pp}}_G$-$M$-quasigeneric. Note that this does not refute $[\mathsf{e}] \Rightarrow [\mathsf{p}]$.

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