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Thoralf **Räsch**, Johan **van Benthem** (*eds.*)
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Infinite Games

A Playful Approach to Silver and Mathias Forcings

Lorenz Halbeisen*

Universität Bern
Intitut für Informatik und angewandte Mathematik
Neubrückstrasse 10
`halbeis@iam.unibe.ch`

ABSTRACT. Forcing is a method to extend models of Set Theory in order to get independence or at least consistency results. For generalized Silver and Mathias forcings it is shown how infinite games between two players, say Death and the Maiden, and in particular the absence of a winning strategy for the Maiden, can be used to predict combinatorial properties of the extended model. For example it is shown that Mathias forcing restricted to certain game families adds dominating reals, has pure decision, and does not add Cohen reals, and that Silver forcing restricted to some weaker game families does not add unbounded reals, adds splitting reals, and is minimal.

Outline

The aim of this paper is to show how infinite games can be used in the investigation of forcing extensions of models of Set Theory. In particular two types of forcing notions are considered, namely Mathias and Silver forcings, and it is shown how infinite two-player games, especially the absence of a winning strategy for one of the players, can be used to predict combinatorial properties of the corresponding extended models.

Our system of set-theoretic axioms includes the axioms of Zermelo and Fraenkel as well as the Axiom of Choice. This system is usually denoted ZFC. All our set-theoretic notations and definitions are standard and can be found in textbooks such as [Jec03], [Kun83] or [Bar_1Jud95]. A brief

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introduction to the forcing technique can be found for example in [Jec86] and [She98, Chapter I, § 1]. However, to make this paper self-contained, we also provide a short introduction to forcing here.

The paper is organized as follows: In the first section, a brief introduction to forcing is given and some combinatorial properties of forcing extensions are discussed. Then in Section 2, two types of forcing notions are introduced which are investigated in the last two sections. In Section 3, families defined by the absence of a winning strategy for player I are introduced. These families play the key rôle in the investigation of Mathias and Silver forcings in Section 4 and Section 5 respectively.

1 The Notion of Forcing

In modern set theory, one usually gets consistency results by a forcing construction. Forcing was invented by Paul Cohen in the early 1960s to show that the Axiom of Choice AC as well as the Continuum Hypothesis CH are not provable in ZF (which is Zermelo-Fraenkel Set Theory without AC). In fact he showed that $\neg\mathsf{AC}$ is relatively consistent with ZF and that $\neg\mathsf{CH}$ is relatively consistent with ZFC. Forcing is a technique to extend models of set theory in such a way that certain statements become true in the extension, no matter if they were true or false in the ground model. In other words, forcing adds new sets to some ground model and by choosing the right forcing notion, which is essentially a partial ordering, we can make sure that the new sets have some desired properties. So, the main ingredients of a forcing construction are a model of ZFC, usually denoted by $\mathbf{V}$, and a partial ordering $\mathbb{P} = (P, \leq)$.

1.1 Partial orderings, generic filters, and names

Let $\mathbf{V}$ be a model of Set Theory and let $\mathbb{P} = (P, \leq)$ be a partial ordering defined in this model. The elements of P are usually called **conditions**. Two conditions p_1 and p_2 of P are called **incompatible**, denoted $p_1 \perp p_2$, if there is no $q \in P$ such that $p_1 \leq q \geq p_2$. A set $D \subseteq P$ is called **dense** if for every condition $p \in P$ there is a $q \in D$ such that $p \leq q$. A set $D \subseteq P$ is called **open** (or upwards closed) if $p \in D$ and $q \geq p$ implies $q \in D$. A set $F \subseteq P$ is called a **filter** if it is directed (*i.e.*, for all $p_1, p_2 \in F$ there is a $q \in F$ such that $p_1 \leq q \geq p_2$) and downwards closed (*i.e.*, if $p \in F$ and $q \leq p$ implies $q \in F$). A filter G is called a **generic filter** if for each dense open set $D \subseteq P$ which belongs to $\mathbf{V}$ we have $G \cap D \neq \varnothing$. If $G \subseteq P$ is a generic filter, then we say that G is **$\mathbb{P}$-generic over $\mathbf{V}$**. Notice that if the partial ordering $\mathbb{P}$ has no trivial branch in the sense that for every condition $p \in P$ there are $q_1, q_2 \in P$ such that $q_1 \geq p \leq q_2$ and $q_1 \perp q_2$, then a generic filter cannot belong to $\mathbf{V}$ (otherwise, the set $P \setminus G$ would be a dense open subset of P belonging to $\mathbf{V}$).

Theorem 1 (The Generic Model Theorem). Let $\mathbf{V}$ be a model of ZFC, called the **ground model**, and let $\mathbb{P} = (P, \leq)$ be a partial ordering defined in $\mathbf{V}$. If G is $\mathbb{P}$-generic over $\mathbf{V}$, then there is a model $\mathbf{V}[G]$ of ZFC, called the **generic extension** of $\mathbf{V}$, such that $\mathbf{V} \subseteq \mathbf{V}[G]$ and $G \in \mathbf{V}[G]$, and every model of ZFC containing $\mathbf{V}$ and G contains also $\mathbf{V}[G]$.

In the sequel, let $\mathbf{V}$ be a model of ZFC, let $\mathbb{P}$ be a partial ordering defined in $\mathbf{V}$, and let G be $\mathbb{P}$-generic over $\mathbf{V}$. In general, $G \notin \mathbf{V}$ and therefore people living in $\mathbf{V}$ do not have knowledge of all the sets in $\mathbf{V}[G]$. On the other hand, people living in $\mathbf{V}$ have so-called "names" for each member of $\mathbf{V}[G]$, but before we introduce the notion of names and their interpretation in $\mathbf{V}[G]$, let us recall the definition of the rank of a set: The **rank** of a set $x \in \mathbf{V}$, denoted $\mathrm{rk}(x)$, is $\bigcup \{ \mathrm{rk}(y) + 1 : y \in x \}$ where $\mathrm{rk}(\varnothing) = 0$. Notice that since the union of a set of ordinals is an ordinal, $\mathrm{rk}(x)$ is an ordinal if defined, and by the Axiom of Foundation $\mathrm{rk}(x)$ is defined for every set $x \in \mathbf{V}$.

Now we can define by induction on α what is a $\mathbb{P}$-name $\underset{\sim}{\tau}$ of rank less than or equal to α as well as its interpretation $\underset{\sim}{\tau}[G]$ in $\mathbf{V}[G]$: $\underset{\sim}{\tau}$ is a $\mathbb{P}$-name with $\mathrm{rk}(\underset{\sim}{\tau}) \leq \alpha$ if it has the form

$$\underset{\sim}{\tau} = \{ (p_\iota, \underset{\sim}{\tau_\iota}) : \iota \in I \}$$

where I is some set and for each $\iota \in I$ we have $p_\iota \in P$ and $\mathrm{rk}(\underset{\sim}{\tau_\iota}) < \alpha$. The interpretation $\underset{\sim}{\tau}[G]$ of $\underset{\sim}{\tau}$ in $\mathbf{V}[G]$ is

$$\{ \underset{\sim}{\tau_\iota}[G] : (p_\iota, \underset{\sim}{\tau_\iota}) \in \underset{\sim}{\tau} \text{ and } p_\iota \in G \}.$$

Since $G \in \mathbf{V}[G]$ and $V \subseteq \mathbf{V}[G]$, there is a $\mathbb{P}$-name for G as well as for each set in $\mathbf{V}$. For a set $x \in \mathbf{V}$, $\dot{x}$ is a $\mathbb{P}$-name for x defined by induction on $\mathrm{rk}(x)$ as follows:

$$\dot{x} = \{ (p, \dot{y}) : p \in P \text{ and } y \in x \}.$$

Further, the $\mathbb{P}$-name for G, denoted by $\underset{\sim}{G}$, is defined as follows:

$$\underset{\sim}{G} = \{ (p, \dot{p}) : p \in P \}.$$

Notice that for every $\mathbb{P}$-generic filter G we have $\dot{x}[G] = x$ and $\underset{\sim}{G}[G] = G$. In the sequel we identify the names for sets in the ground model with the corresponding sets and omit the dots.

Let $\underset{\sim}{\tau_1}, \ldots, \underset{\sim}{\tau_n}$ be $\mathbb{P}$-names and $\varphi(x_1, \ldots, x_n)$ be a first-order formula of the language of Set Theory. For a condition $p \in P$ we write

$$p \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{\tau_1}, \ldots, \underset{\sim}{\tau_n})\text{”}$$

and say p **forces** $\varphi(\underset{\sim}{\tau}_1, \ldots, \underset{\sim}{\tau}_n)$, if for every $\mathbb{P}$-generic filter G containing p we have $\varphi\big(\underset{\sim}{\tau}_1[G], \ldots, \underset{\sim}{\tau}_n[G]\big)$ is true in $\mathbf{V}[G]$, in symbols:

$$\mathbf{V}[G] \models \varphi\big(\underset{\sim}{\tau}_1[G], \ldots, \underset{\sim}{\tau}_n[G]\big).$$

Notice that for any conditions $p, q \in \mathbb{P}$ and any first-order sentence of the forcing language φ, if $p \Vdash_{\mathbb{P}} \varphi$ and $q \geq p$, then also $q \Vdash_{\mathbb{P}} \varphi$.

Theorem 2 (The Forcing Theorem). If $\varphi(\underset{\sim}{\tau}_1, \ldots, \underset{\sim}{\tau}_n)$ is a first-order sentence of the forcing language, then for every $\mathbb{P}$-generic filter G we have

$$\mathbf{V}[G] \models \varphi\big(\underset{\sim}{\tau}_1[G], \ldots, \underset{\sim}{\tau}_n[G]\big) \iff \exists p \in G\big(p \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{\tau}_1, \ldots, \underset{\sim}{\tau}_n)\text{”}\big).$$

1.2 Dominating, unbounded, and splitting reals

In the following we characterize a few real numbers which might appear in a generic extension, but first we have to introduce some notations.

The set $\{0, 1, 2, \ldots\}$ of **natural numbers** is denoted by ω and we usually consider a natural number n as the set of all numbers smaller than n, so, $n = \{k \in \omega : k < n\}$ and $n + 1 = n \cup \{n\}$, which is also denoted n^+. The **cardinality** of a set x is denoted by $|x|$. In particular, for every natural number n we have $|n| = n$.

The set of all infinite subsets of ω is denoted by $[\omega]^\omega$, the set of all functions from ω to ω is denoted by ${}^\omega\omega$, and the set of all functions from ω to $\{0, 1\} = 2$ is denoted by ${}^\omega 2$. Each of the sets $[\omega]^\omega$, ${}^\omega\omega$, and ${}^\omega 2$ can be identified with the set of real numbers and in the sequel we usually call their members just “reals”.

For two functions $f, g \in {}^\omega\omega$ we say that g is **dominated** by f, denoted $g <^* f$, if there is an $n \in \omega$ such that for all $k \geq n$ we have $g(k) < f(k)$. For two sets $x, y \in [\omega]^\omega$ we say that x **splits** y if both sets $y \cap x$ and $y \setminus x$ are infinite.

Now let $\mathbf{V}$ be any model of ZFC and let $\mathbf{V}[G]$ be a generic extension (with respect to some forcing notion $\mathbb{P}$). A function $f \in {}^\omega\omega$ in $\mathbf{V}[G]$ is called a **dominating real** if each function $g \in {}^\omega\omega \cap \mathbf{V}$ is dominated by f, and f is called an **unbounded real** if it is not dominated by any function $g \in {}^\omega\omega \cap \mathbf{V}$. Further, a set $x \in [\omega]^\omega$ in $\mathbf{V}[G]$ is called a **splitting real** if it splits each set $y \in [\omega]^\omega \cap \mathbf{V}$.

Proposition 3. If $\mathbf{V}[G]$ contains a dominating real, then it also contains a splitting real.

Proof. We can just follow the proof of [vDo84, Theorem 3.1 (a)]: Since $\mathbf{V}[G]$ is a model of ZFC we have that if a function $f \in {}^\omega\omega$ belongs $\mathbf{V}[G]$, then also the set

$$\sigma_f = \bigcup \Big\{ \big[f^{2n}(0), f^{2n+1}(0)\big) : n \in \omega \Big\}$$

belongs to $\mathbf{V}[G]$, where $[a,b) = \{k \in \omega : a \leq k < b\}$ and $f^{n+1}(0) = f\big(f^n(0)\big)$ with $f^0(0) := 0$. Now let $f \in {}^\omega\omega$ be a dominating real. Without loss of generality we may assume that f is strictly increasing and that $f(0) > 0$. Fix any $x \in [\omega]^\omega \cap \mathbf{V}$ and let g_x be the (unique) strictly increasing bijection $\omega \to x$. Since f is dominating we have $g_x <^* f$, which implies that there is an $n \in \omega$ such that for all $k \geq n$ we have $g_x(k) < f(k)$. Because $k \leq g_x(k)$ for all $k \in \omega$, we get that if $k \geq n$ then

$$f^n(0) \leq g_x\big(f^n(0)\big) < f\big(f^n(0)\big) = f^{n+1}(0)\,.$$

Hence $g_x\big(f^n(0)\big) \in \sigma_f$ if n is even and $g_x\big(f^n(0)\big) \notin \sigma_f$ if n is odd, which shows that σ_f is splitting. q.e.d.

A forcing notion is called ${}^\omega\omega$**-bounding** if there are no unbounded reals in the generic extension, or in other words, if every function is dominated by some function in the ground model. Obviously, a forcing notion which adds a dominating real also adds unbounded reals and therefore cannot be ${}^\omega\omega$-bounding, and by Proposition 3, such a forcing notion also adds splitting reals. On the other hand, none of these implications is reversible. For example a forcing notion which is ${}^\omega\omega$-bounding but adds splitting reals is Silver forcing (investigated in Section 5), and Cohen forcing, discussed below, is an example of a forcing notion which adds unbounded and splitting reals but does not add dominating reals.

1.3 Cohen forcing

The Cohen partial ordering is certainly one of the simplest non-trivial forcing notions. Cohen forcing is denoted by $\mathbb{C} = (C, \leq)$ and defined as follows: The set of conditions C consists of all functions from some $n \in \omega$ to $\{0,1\}$, and for two conditions $p, q \in C$ we define

$$p \leq q \iff q|_{\mathrm{dom}(p)} \equiv p\,,$$

or in other words, $p \leq q$ if q extends p. If G is $\mathbb{C}$-generic over $\mathbf{V}$, then G generates a function $c \in {}^\omega 2$. To see this, notice that for every $n \in \omega$, the set $D_n = \big\{p \in C : n \in \mathrm{dom}(p)\big\}$ is dense open, and therefore, for any $n \in \omega$ there is a $p \in G$ such that $n \in \mathrm{dom}(p)$. Further, since any two members of G are compatible, all conditions $p \in G$ which are defined on n must agree at this point. Thus,

$$c(n) = \begin{cases} 0 & \text{if } \exists p \in G\,\big(p(n) = 0\big), \\ 1 & \text{if } \exists p \in G\,\big(p(n) = 1\big), \end{cases}$$

is a well-defined function from ω to $\{0,1\}$. The real $c \in {}^{\omega}2$ is called a **Cohen real** (over $\mathbf{V}$). Thus, a $\mathbb{C}$-generic filter generates a Cohen real, and vice versa, the $\mathbb{C}$-generic filter can be reconstructed from the corresponding Cohen real.

The following proposition gives some basic properties of Cohen forcing and Cohen reals respectively. Even though the proofs are straightforward, they involve some standard techniques which will be also used later in the investigation of Mathias and Silver forcings.

Proposition 4. Cohen forcing does not add dominating reals, but every Cohen real is unbounded and splitting.

Proof. First we show that a Cohen real is always unbounded: Let $\underset{\sim}{c}$ be a name for a Cohen real, then for any condition $p \in C$ we have

$$p \Vdash_{\mathbb{C}} \text{“}\underset{\sim}{c}|_{\mathrm{dom}(p)} \equiv p\text{”}\,.$$

Let $g \in {}^{\omega}2 \cap \mathbf{V}$ and $n \in \omega$, then there exists a $k \geq n$ and a condition $q \geq p$ such that $k \in \mathrm{dom}(q)$ and $q(n) > g(n)$, which shows that for every $n \in \omega$ the set of conditions $q \in C$ such that

$$q \Vdash_{\mathbb{C}} \text{“}\exists k \geq n \left(g(k) < \underset{\sim}{c}(k)\right)\text{”}$$

is dense open in C, which implies that no condition forces that c is dominated by g, and since g was arbitrary, c is not dominated by any real in the ground model.

In a similar way one can show that a Cohen real is always splitting: Let c be a Cohen real and let $\sigma_c := \{k \in \omega : c(k) = 1\}$, then for any infinite set $x \in [\omega]^{\omega} \cap \mathbf{V}$ and any $n \in \omega$, the set of conditions $p \in C$ such that

$$p \Vdash_{\mathbb{C}} \text{“}|x \cap \underset{\sim}{\sigma_c}| > n \text{ and } |x \setminus \underset{\sim}{\sigma_c}| > n\text{”}$$

is dense open, and therefore, σ_c splits every real in the ground model.

Now let $f \in {}^{\omega}\omega$ be a function in $\mathbf{V}[c]$ and let $\underset{\sim}{f}$ be a $\mathbb{C}$-name for f. In order to show that f is not dominating we have to find a function $g \in {}^{\omega}\omega \cap \mathbf{V}$ such that for every $n \in \omega$ there is a $k \geq n$ such that $g(k) \not< f(k)$. We can just follow the proof of [Bar$_1$Jud95, Lemma 3.1.2 (2)]: Let $\{p_k : k \in \omega\}$ be a countable dense subset of C. For every $k \in \omega$ define

$$g(k) = \min\left\{n : \exists q \geq p_k \left(q \Vdash_{\mathbb{C}} \underset{\sim}{f}(k) = n\right)\right\}.$$

For every condition $p \in C$ and every $n \in \omega$ there is a $k \geq n$ such that $p_k \geq p$, and we find a $q \geq p_k$ such that $q \Vdash_{\mathbb{C}} \underset{\sim}{f}(k) = g(k)$. Consequently, for every $n \in \omega$, the set

$$D_n = \left\{q \in C : q \Vdash_{\mathbb{C}} \text{“}\underset{\sim}{f}(k) = g(k) \text{ for some } k \geq n\text{”}\right\}$$

is dense open in C. Hence, by the Forcing Theorem 2 and since the Cohen real c meets every dense open subset of C we have

$$\mathbf{V}[c] \models \forall n \in \omega\, \exists k \geq n \left(g(k) \not< f(k)\right)$$

which shows that g is not dominated by f. q.e.d.

2 Mathias and Silver Forcings

2.1 Free families

Before we introduce the forcing notions of Mathias and Silver, we consider certain families of subsets of ω.

In the following the "ground set" will be ω and consequently for $x \subseteq \omega$ we define $x^c := \omega \setminus x$. Now a family $\mathscr{F} \subseteq [\omega]^\omega$ is called a **filter** if it is closed under intersections and supersets, or in other words, if for any $x, y \in [\omega]^\omega$ we have

1. if $x \in \mathscr{F}$ and $y \in \mathscr{F}$, then $x \cap y \in \mathscr{F}$,
2. if $x \in \mathscr{F}$ and $x \subseteq y$, then $y \in \mathscr{F}$.

The **Fréchet filter** is the filter consisting of all co-finite subsets of ω, *i.e.*, all $x \in [\omega]^\omega$ such that x^c is finite, and a filter $\mathscr{F} \subseteq [\omega]^\omega$ is called a **free filter** if it contains the Fréchet filter. For a filter $\mathscr{F} \subseteq [\omega]^\omega$, $\mathscr{F}^+$ denotes the collection of all subsets $x \subseteq \omega$ such $x^c \notin \mathscr{F}$. It is useful to notice that for a free filter $\mathscr{F}$, $\mathscr{F}^+ = \{x \subseteq \omega : \forall z \in \mathscr{F}\, (|x \cap z| = \omega\}$ (*cf.* [Laf96, p. 52]). Further, a family $\mathscr{E}$ of subsets of ω is called a **free family** if there is a free filter $\mathscr{F} \subseteq [\omega]^\omega$ such that $\mathscr{E} = \mathscr{F}^+$. In particular, $[\omega]^\omega$ and all ultrafilters are free families.

Notice that a free family does not contain any finite sets and is closed under supersets. A filter $\mathscr{F} \subseteq [\omega]^\omega$ is called an **ultrafilter** if for all $x \subseteq [\omega]^\omega$ either x or x^c belongs to $\mathscr{F}$. It is easy to see that for a filter $\mathscr{F} \subseteq [\omega]^\omega$, $\mathscr{F} = \mathscr{F}^+$ if and only if $\mathscr{F}$ is an ultrafilter. Hence, a free family $\mathscr{E}$ is closed under intersections if and only if $\mathscr{E}$ is an ultrafilter. However, all free families have the following slightly weaker property.

Lemma 5. If $\mathscr{E}$ is a free family, $x \in \mathscr{E}$, $y \subseteq x$, and $y \notin \mathscr{E}$, then $x \setminus y$ belongs to $\mathscr{E}$.

Proof. Let $\mathscr{E} = \mathscr{F}^+$ where $\mathscr{F}$ is some free filter, let $x \in \mathscr{E}$, and let $y \subseteq x$ be such that $y \notin \mathscr{E}$. By definition, $x^c \notin \mathscr{F}$ and $y^c \in \mathscr{F}$. Since $x \in \mathscr{E}$, $x \cap z$ is infinite for all $z \in \mathscr{F}$. In particular, since $y^c \in \mathscr{F}$ and $\mathscr{F}$ is a free filter, $x \cap y^c$ as well as $x \cap (y^c \cap z) = (x \cap y^c) \cap z$ is infinite for all $z \in \mathscr{F}$, which implies that $x \cap y^c = x \setminus y$ belongs to $\mathscr{E}$. q.e.d.

2.2 Mathias forcing restricted to free families

In the sequel let $\mathscr{E}$ be an arbitrary free family. **Mathias forcing** restricted to $\mathscr{E}$, denoted $\mathbb{M}_{\mathscr{E}} = (M_{\mathscr{E}}, \leq)$, is defined as follows:

$$M_{\mathscr{E}} = \big\{(s,x) : s \subseteq \omega \text{ is finite}, x \in \mathscr{E}, \max(s) < \min(x)\big\}$$

and

$$(s,x) \leq (t,y) \iff s \subseteq t,\ y \subseteq x,\ t \setminus s \subseteq x\,.$$

The finite set s of a Mathias condition (s,x) is called the **stem** of the condition. Similar to Cohen forcing we can identify every $\mathbb{M}_{\mathscr{E}}$-generic filter with a real number, called **Mathias real**, which is in fact just the union of the stems of the conditions which belong to the generic filter.

In [Mat$_0$77], Mathias introduced and investigated rigorously his forcing notion in the case when $\mathscr{E}$ is a so-called happy family (defined and discussed in Section 6). Special cases of happy families are when $\mathscr{E} = [\omega]^\omega$ (in which case $\mathbb{M}_{\mathscr{E}}$ is known as unrestricted Mathias forcing) and when $\mathscr{E}$ is a Ramsey ultrafilter. Mathias showed that if $\mathscr{E}$ is a happy family, then $\mathbb{M}_{\mathscr{E}}$ has many interesting combinatorial properties. In the next section, so-called Ramsey families, defined in terms of infinite games, will be introduced and in Section 4 it will be shown that Mathias forcing restricted to such families has essentially the same combinatorial features as for example unrestricted Mathias forcing.

2.3 Silver forcing restricted to free families

In the sequel let again $\mathscr{E}$ be an arbitrary free family. For a set $x \subseteq \omega$, let ${}^x2$ denote the set of all functions $f : x \to \{0,1\}$. **Silver forcing** restricted to $\mathscr{E}$, denoted $\mathbb{S}_{\mathscr{E}} = (S_{\mathscr{E}}, \leq)$, is defined as follows:

$$S_{\mathscr{E}} = \bigcup \big\{\, {}^x2 : x^{\mathrm{c}} \in \mathscr{E} \big\}$$

and for $p, q \in S_{\mathscr{E}}$ we stipulate

$$p \leq q \iff q|_{\mathrm{dom}(p)} \equiv p\,.$$

Again we can identify every $\mathbb{S}_{\mathscr{E}}$-generic filter with a real number, called **Silver real**, which is in fact just the union of the functions which belong to the generic filter.

The original (or unrestricted) Silver forcing we get when $\mathscr{E} = [\omega]^\omega$ (*cf.* [Mat$_0$79, p. 112] or [Jec86, Part I, 3.10]). For $\mathscr{E}$ a P-point (defined below), restricted Silver forcing, also known as **Grigorieff forcing**, was introduced and investigated in depth by Grigorieff in [Gri71] (see also [Jec86, Part I, 3.22]). Unrestricted Silver forcing has essentially the same combinatorial properties as Grigorieff forcing. Moreover, there are families between

$[\omega]^\omega$ and P-points (introduced in the next section) such that Silver forcing restricted to such families has still the same combinatorial features as unrestricted Silver forcing or as Grigorieff forcing.

3 Infinite Games

Let $\mathscr{E}$ be an arbitrary free family. Associated with $\mathscr{E}$ we define two quite similar games between two players, say DEATH and the MAIDEN.

$$\mathcal{G}_{\mathscr{E}}: \quad \begin{tikzcd}[column sep=small] \text{MAIDEN} & x_0 \arrow[dr] & & x_1 \arrow[dr] & & x_2 \arrow[dr] & & \cdots \\ \text{DEATH} & & a_0 \arrow[ur] & & a_1 \arrow[ur] & & a_2 \arrow[ur, dash] & \end{tikzcd}$$

The rules for the game $\mathcal{G}_{\mathscr{E}}$ are as follows: For each $i \in \omega$, $x_i \in \mathscr{E}$ and $a_i \in x_i$, and further we require that $x_{i+1} \subseteq x_i$ and $a_i < a_{i+1}$. Finally, DEATH wins the game $\mathcal{G}_{\mathscr{E}}$ if and only if the sequence $\{a_i : i \in \omega\}$ belongs to the family $\mathscr{E}$.

A free family $\mathscr{E}$ is called a **Ramsey family** if the MAIDEN has no winning strategy for the game $\mathcal{G}_{\mathscr{E}}$. In other words, if $\mathscr{E}$ is a Ramsey family then DEATH can always defeat any given strategy of the MAIDEN, no matter how sophisticated her strategy is. (The only possibility for the MAIDEN to win against DEATH is to play randomly, *i.e.*, not according to any strategy.) Notice that this does not imply that DEATH has a winning strategy. Obviously, $[\omega]^\omega$ is a Ramsey family. On the other hand, there are also ultrafilters which are Ramsey families, namely the so-called Ramsey ultrafilters (and vice versa, every Ramsey family which is an ultrafilter is a Ramsey ultrafilter). According to [Bar_1Jud95], this was shown by Galvin and Shelah (*cf.* [Bar_1Jud95, Theorem 4.5.3]). So, Ramsey families are a kind of generalized Ramsey ultrafilters. These families were first introduced and studied by Laflamme in [Laf96] (where the filters associated to a Ramsey family are called +-Ramsey filters). As we will see in Section 6, every Ramsey family is a happy family (in the sense of [$\text{Mat}_0$77]), but not vice versa.

In the game $\mathcal{G}^*_{\mathscr{E}}$, DEATH has slightly more freedom, since he can play now finite sequences instead of just singletons.

$$\mathcal{G}^*_{\mathscr{E}}: \quad \begin{tikzcd}[column sep=small] \text{MAIDEN} & x_0 \arrow[dr] & & x_1 \arrow[dr] & & x_2 \arrow[dr] & & \cdots \\ \text{DEATH} & & s_0 \arrow[ur] & & s_1 \arrow[ur] & & s_2 \arrow[ur, dash] & \end{tikzcd}$$

Again, the sets x_i played by the MAIDEN must belong to the free family $\mathscr{E}$ and each finite set s_i played by DEATH must be a subset of the corresponding x_i. Further, for each $i \in \omega$ we require that $x_{i+1} \subseteq x_i$ and $\max(s_i) < \min(s_{i+1})$. Finally, DEATH wins the game $\mathcal{G}^*_{\mathscr{E}}$ if and only if $\bigcup\{s_i : i \in \omega\}$ belongs to the family $\mathscr{E}$.

A free family $\mathscr{E}$ is called a ***P*-family** if the MAIDEN has no winning strategy for the game $\mathcal{G}^*_{\mathscr{E}}$. Obviously, $[\omega]^\omega$ is a P-family. On the other hand, there are also P-families which are ultrafilters, namely the so-called P-points (*cf.* [Bar$_1$Jud95, Theorem 4.4.4]). So, P-families can be considered as a generalization of P-points, which are a weaker form of Ramsey ultrafilters. P-families were first introduced and studied by Laflamme in [Laf96] (where the filters associated to a P-family are called $P+$-filters).

4 Properties of Mathias Forcing Notions

Throughout this section let $\mathscr{E}$ be an arbitrary but fixed Ramsey family. It will be shown that the forcing notion $\mathbb{M}_{\mathscr{E}}$ adds dominating reals, does not add Cohen reals, and that every infinite subset of a Mathias real is also a Mathias real.

Since every Ramsey family is happy (*cf.* Fact 19), the main results of this section follow from Mathias' investigations [Mat$_0$77, Section 4]. However, the technique used here provides a new and uniform approach to Mathias forcing notions and may be applied also in more general contexts.

4.1 $\mathbb{M}_{\mathscr{E}}$ adds dominating reals

Theorem 6. The forcing notion $\mathbb{M}_{\mathscr{E}}$ adds dominating reals.

Proof. We show that a Mathias real is always dominating: Let m be $\mathbb{M}_{\mathscr{E}}$-generic over $\mathbf{V}$, let $p = (s, x)$ be an arbitrary $\mathbb{M}_{\mathscr{E}}$-condition, and let $g \in {}^\omega\omega \cap \mathbf{V}$ be any function in the ground model. It is enough to show that there exists a condition $q \geq p$ such that $q \Vdash_{\mathbb{M}_{\mathscr{E}}}$ "$\underset{\sim}{m}$ dominates g". In order to construct the condition q we run the game $\mathcal{G}_{\mathscr{E}}$ where the MAIDEN plays according to the following strategy: The MAIDEN's first move is $x_0 := x \setminus \big(g(n_0)^+\big)$, where $n_0 = |s|$, and for $i \in \omega$ she plays $x_{i+1} := x_i \setminus \max\big\{g(n_0 + i)^+, a_i^+\big\}$. Since this strategy is not a winning strategy for the MAIDEN, DEATH can play such that $y := \{a_i : i \in \omega\} \in \mathscr{E}$. Now by construction we get that $(s, y) \geq p$ and

$$(s, y) \Vdash_{\mathbb{M}_{\mathscr{E}}} \forall k \geq n_0 \left(\underset{\sim}{m}(k) > g(k)\right)$$

which shows that m is a dominating real. q.e.d.

As a consequence of Proposition 3 we get

Corollary 7. The forcing notion $\mathbb{M}_{\mathscr{E}}$ adds splitting reals.

4.2 $\mathbb{M}_{\mathscr{E}}$ has pure decision

The following property of Mathias forcing is known as **pure decision** (*cf.* [Mat$_0$77, Proposition 4.12 (2.9)]):

Theorem 8. For every $\mathbb{M}_{\mathscr{E}}$-condition (s,x) and for every sentence of the forcing language φ, there exists an $\mathbb{M}_{\mathscr{E}}$-condition $(s,y) \geq (s,x)$ with the same stem as (s,x) such that (s,y) decides φ, *i.e.*,

$$(s,y) \Vdash_{\mathbb{M}_{\mathscr{E}}} \varphi \quad \textit{or} \quad (s,y) \Vdash_{\mathbb{M}_{\mathscr{E}}} \neg\varphi\,.$$

Pure decision is one of the main features of Mathias forcing, but before we can prove the theorem, we have to introduce some terminology and prove some auxiliary results: For $\mathbb{M}_{\mathscr{E}}$-conditions (s,x) let

$$[s,x] := \big\{ z \in [\omega]^\omega : s \subseteq z \subseteq s \cup x \big\}\,.$$

For a (fixed) open set $\mathcal{O} \subseteq M_{\mathscr{E}}$ let $\bar{\mathcal{O}} := \bigcup \big\{ [s,x] : (s,x) \in \mathcal{O} \big\}$. An $\mathbb{M}_{\mathscr{E}}$-condition (s,x) is called **good** (with respect to $\mathcal{O}$), if there is a condition $(s,y) \geq (s,x)$ such that $[s,y] \subseteq \bar{\mathcal{O}}$; otherwise it is called **bad**. Further, the condition (s,x) is called **ugly** if $\big(s \cup \{a\}, x \setminus a^+\big)$ is bad for all $a \in x$. Notice that if (s,x) is ugly, then (s,x) is bad, too. Finally, (s,x) is called **completely ugly** if $\big(s \cup \{a_0,\dots,a_n\}, x \setminus a_n^+\big)$ is bad for all $\{a_0,\dots,a_n\} \subseteq x$ with $a_0 < \dots < a_n$.

Lemma 9. If an $\mathbb{M}_{\mathscr{E}}$-condition (s,x) is bad, then there is a condition $(s,y) \geq (s,x)$ which is ugly.

Proof. We run the game $\mathcal{G}_{\mathscr{E}}$ where the Maiden plays according to the following strategy: She starts the game by playing $x_0 := x$, and then, for $i \in \omega$, she plays $x_{i+1} \subseteq (x_i \setminus a_i^+)$ such that $[s \cup \{a_i\}, x_{i+1}] \subseteq \bar{\mathcal{O}}$ if possible, and $x_{i+1} = (x_i \setminus a_i^+)$ otherwise. Strictly speaking we assume that $\mathscr{E}$ is well-ordered and that x_{i+1} is the first element of $\mathscr{E}$ with the required properties. However, since this strategy is not a winning strategy for the Maiden, Death can play such that $z := \{a_i : i \in \omega\} \in \mathscr{E}$ and let $y = \big\{ a_i \in z : [s \cup \{a_i\}, x_{i+1}] \subseteq \bar{\mathcal{O}} \big\}$. Because $\mathscr{E}$ is a free family, by Lemma 5 we get that y or $z \setminus y$ belongs to $\mathscr{E}$. If $y \in \mathscr{E}$, then $[s,y] \subseteq \bar{\mathcal{O}}$ which would imply that (s,x) is good, but this contradicts the premise of the lemma. Hence, $z \setminus y \in \mathscr{E}$, which implies that $(s, z \setminus y)$ is ugly. q.e.d.

Lemma 10. If an $\mathbb{M}_{\mathscr{E}}$-condition (s,x) is bad, then there is a condition $(s,y) \geq (s,x)$ such that (s,y) is completely ugly.

Proof. This follows by an iterative application of Lemma 9. In fact, for every $i \in \omega$, the MAIDEN can play a set $x_i \in \mathscr{E}$ such that for each $t \subseteq \{a_0, \ldots, a_{i-1}\}$, either the condition $(s \cup t, x_i)$ is ugly or $[s \cup t, x_i] \subseteq \bar{\mathcal{O}}$. Now DEATH can play such that $y := \{a_i : i \in \omega\} \in \mathscr{E}$. Assume that there exists a finite set $t \subseteq y$ such that $(s \cup t, y \setminus \max(t)^+)$ is good. Such a set cannot be empty, since (s, x) was assumed to be bad. Now let t_0 be a smallest finite subset of y such that $q_0 = (s \cup t_0, y \setminus \max(t_0)^+)$ is good and let $t_0^- = t_0 \setminus \{\max(t_0)\}$. Then by definition of t_0, the condition $q_0^- = \big(s \cup t_0^-, y \setminus \max(t_0)\big)$ is not good, and hence, by the strategy of the MAIDEN, it must be ugly, but if q_0^- is ugly, then q_0 is bad, which is a contradiction to our assumption. Thus, there is no finite set $t \subseteq y$ such that $(s \cup t, y \setminus \max(t)^+)$ is good, which implies that all these conditions are ugly, and therefore (s, y) is completely ugly. q.e.d.

Now we are ready to proof Theorem 8 (*i.e.*, that the forcing notion $\mathbb{M}_{\mathscr{E}}$ has pure decision):

Proof. Let (s, x) be an $\mathbb{M}_{\mathscr{E}}$-condition and let φ be a sentence of the forcing language. With respect to φ we define $\mathcal{O}_1 := \{q \in M_{\mathscr{E}} : q \Vdash_{\mathbb{M}_{\mathscr{E}}} \varphi\}$ and $\mathcal{O}_2 := \{q \in M_{\mathscr{E}} : q \Vdash_{\mathbb{M}_{\mathscr{E}}} \neg\varphi\}$. Clearly $\mathcal{O}_1$ and $\mathcal{O}_2$ are both open and $\mathcal{O}_1 \cup \mathcal{O}_2$ is even dense in $M_{\mathscr{E}}$. By Lemma 10 we know that for any (s, x) there exists $(s, y) \geq (s, x)$ such that either $[s, y] \subseteq \bar{\mathcal{O}}_1$ or $[s, y] \cap \bar{\mathcal{O}}_1 = \varnothing$. In the former case we have $(s, y) \Vdash_{\mathbb{M}_{\mathscr{E}}} \varphi$ and we are done. In the latter case we find $(s, y') \geq (s, y)$ such that $[s, y'] \subseteq \bar{\mathcal{O}}_2$. (Otherwise we would have $[s, y] \cap (\bar{\mathcal{O}}_1 \cup \bar{\mathcal{O}}_2) = \varnothing$, which is impossible by the density of $\mathcal{O}_1 \cup \mathcal{O}_2$.) Hence, $(s, y') \Vdash_{\mathbb{M}_{\mathscr{E}}} \neg\varphi$. q.e.d.

For the following result, which is again a consequence of Lemma 10, see also [Mat$_0$77, Corollary 4.10 (ii)] (and for a kind of reverse implication see [Mat$_0$77, Theorem 2.10]).

Proposition 11. Every infinite subset of an $\mathbb{M}_{\mathscr{E}}$-generic real is also $\mathbb{M}_{\mathscr{E}}$-generic.

Proof. Let $\mathcal{D} \subseteq M_{\mathscr{E}}$ be an arbitrary dense open subset of $M_{\mathscr{E}}$ and let $\mathcal{D}'$ be the set of all conditions $(s, z) \in M_{\mathscr{E}}$ such that for all $t \subseteq s$, $[t, z] \subseteq \bar{\mathcal{D}}$.

First we show that $\mathcal{D}'$ is dense (and by definition also open) in $M_{\mathscr{E}}$: For this take an arbitrary condition $(s, x) \in \mathcal{D}$ and let $\{t_i : 0 \leq i \leq m\}$ be an enumeration of all subsets of s. Because $\mathcal{D}$ is dense open in $M_{\mathscr{E}}$, by Lemma 10 we find a condition $(s, y) \geq (s, x)$ such that $(s, y) \in \mathcal{D}'$, which implies that $\mathcal{D}'$ is dense in $M_{\mathscr{E}}$.

Let $m \in [\omega]^\omega$ be $\mathbb{M}_{\mathscr{E}}$-generic and let m' be an infinite subset of m. Since $\mathcal{D}'$ is dense open and m is $\mathbb{M}_{\mathscr{E}}$-generic, there exists a condition $(s,x) \in \mathcal{D}'$ such that $s \subseteq m \subseteq s \cup x$. Let $t = m' \cap s$, then $t \subseteq m' \subseteq t \cup x$ and by definition of $\mathcal{D}'$ we have $[t,x] \subseteq \bar{\mathcal{D}}$. Thus, m' meets the dense open set $\mathcal{D}$, and since $\mathcal{D}$ was arbitrary, this completes the proof. q.e.d.

4.3 $\mathbb{M}_{\mathscr{E}}$ does not add Cohen reals

In Section 1 we have seen that Cohen forcing adds unbounded but not dominating reals. Now we will see that the forcing notion $\mathbb{M}_{\mathscr{E}}$, even though it adds dominating reals (*cf.* Theorem 6), it does not add Cohen reals:

Theorem 12. The forcing notion $\mathbb{M}_{\mathscr{E}}$ does not add Cohen reals.

Proof. Let $\underset{\sim}{f}$ be an $\mathbb{M}_{\mathscr{E}}$-name for a function in ${}^\omega 2$ and let m be $\mathbb{M}_{\mathscr{E}}$ generic over $\mathbf{V}$. We have to show that $\underset{\sim}{f}[m]$ is not $\mathbb{C}$-generic over $\mathbf{V}$, *i.e.*, there is a dense open set $\mathcal{D}_f \subseteq C$ in $\mathbf{V}$ such that for all $p \in \mathcal{D}_f$ we have $f[m]|_{\mathrm{dom}(p)} \not\equiv p$.

Notice that by Theorem 8, for every $\mathbb{M}_{\mathscr{E}}$-condition (s,x) and for every $k \in \omega$ there exists a condition (s,y) which decides $\underset{\sim}{f}(k)$, *i.e.*, $(s,y) \Vdash_{\mathbb{M}_{\mathscr{E}}} \underset{\sim}{f}(k) = 0$ or $(s,y) \Vdash_{\mathbb{M}_{\mathscr{E}}} \underset{\sim}{f}(k) = 1$. Consequently, for every (s,x) and every $n \in \omega$ there exists a condition (s,y) which decides $\underset{\sim}{f}(k)$ for all $k < n$.

In order to construct $\mathcal{D}_f$ we run the game $\mathcal{G}_{\mathscr{E}}$ where the MAIDEN plays according to the following strategy: For $i \in \omega$ she plays x_i such that for all $t \subseteq \{a_0, \dots, a_{i-1}\}$, the condition (t, x_i) decides $\underset{\sim}{f}(k)$ for all $k < 2^{2^i}$. Further, for $t \subseteq \{a_0, \dots, a_{i-1}\}$ let $p^i_t \in C$ be such that $\mathrm{dom}(p^i_t) = 2^{2^i}$ and $(t, x_i) \Vdash_{\mathbb{M}_{\mathscr{E}}} \underset{\sim}{f}|_{\mathrm{dom}(p^i_t)} \equiv p^i_t$. Since this strategy is not a winning strategy for the MAIDEN, DEATH can play such that $x := \{a_i : i \in \omega\} \in \mathscr{E}$. Now, let

$$\mathcal{C}_f = \left\{ q \in C : \exists i \in \omega\, \exists t \subseteq x \, (q \leq p^i_t) \right\}$$

and let $\mathcal{D}_f := C \setminus \mathcal{C}_f$.

By construction $\mathcal{D}_f$ is open in C and it remains to show that $\mathcal{D}_f$ is also dense: Firstly notice that for all $n \in \omega$, $(\varnothing, x) \Vdash_{\mathbb{M}_{\mathscr{E}}} \underset{\sim}{f}|_n \in \mathcal{C}_f$. Secondly notice that for any finite $t \subseteq x$ and for any $i \geq |t|$,

$$\left| \left\{ q \in C : q \geq p^i_t,\ \mathrm{dom}(q) = 2^{2^{i+1}} \right\} \right| = 2^{2^i} \left(2^{2^i} - 1 \right)$$

whereas

$$\left| \left\{ q \in \mathcal{C}_f : q \geq p^i_t,\ \mathrm{dom}(q) = 2^{2^{i+1}} \right\} \right| = 2^{i+1}$$

which implies that $\mathcal{D}_f$ is dense in C and completes the proof. q.e.d.

5 Properties of Silver Forcing Notions

Throughout this section let $\mathscr{E}$ be an arbitrary but fixed P-family. It will be shown that the forcing notion $\mathbb{S}_{\mathscr{E}}$ is ${}^{\omega}\omega$-bounding, *i.e.*, adds no unbounded reals (and consequently no Cohen reals), but adds splitting reals and is minimal, *i.e.*, if g is a Silver real, then every real f in the extension which does not belong to the ground model reconstructs g.

5.1 $\mathbb{S}_{\mathscr{E}}$ is ${}^{\omega}\omega$-bounding

Recall that a forcing notion is ${}^{\omega}\omega$-bounding if no function $f \in {}^{\omega}\omega$ in the generic extension is unbounded, *i.e.*, every function $f \in {}^{\omega}\omega$ in the generic extension is dominated by some function in the ground model. Before we can show that the forcing notion $\mathbb{S}_{\mathscr{E}} = (S_{\mathscr{E}}, \leq)$ is ${}^{\omega}\omega$-bounding, we have to introduce the following notation: Remember that a condition $p \in S_{\mathscr{E}}$ is a function from some $x \subseteq \omega$ to $\{0,1\}$, where $x^{c} \in \mathscr{E}$. For a condition $p \in S_{\mathscr{E}}$ and a finite set $t \subseteq \operatorname{dom}(p)$ let

$$\overline{p \mathbin{\text{-}} t} = \left\{q \in S_{\mathscr{E}} : \operatorname{dom}(q) = \operatorname{dom}(p),\ q|_{\operatorname{dom}(q)\setminus t} \equiv p|_{\operatorname{dom}(p)\setminus t}\right\}.$$

Theorem 13. The forcing notion $\mathbb{S}_{\mathscr{E}}$ is ${}^{\omega}\omega$-bounding.

Proof. Let G be $\mathbb{S}_{\mathscr{E}}$-generic over $\mathbf{V}$, let $f \in {}^{\omega}\omega$ be a function in $\mathbf{V}[G]$, and let $\underset{\sim}{f}$ be an $\mathbb{S}_{\mathscr{E}}$-name for f. In order to show that f is bounded by some function some function in the ground model it is enough to prove that for every condition $p \in S_{\mathscr{E}}$ there is a condition $q_0 \geq p$ and a function $g \in {}^{\omega}\omega \cap \mathbf{V}$ such $q_0 \Vdash_{\mathbb{S}_{\mathscr{E}}}$ "g dominates $\underset{\sim}{f}$".

We construct the condition q_0 by running the game $\mathcal{G}^{*}_{\mathscr{E}}$ where the Maiden plays according to the following strategy: Let $m_0 \in \omega$ be the least number for which there exists a condition $p_0 \geq p$ such that $p_0 \Vdash_{\mathbb{S}_{\mathscr{E}}} \underset{\sim}{f}(0) < m_0$. Then the Maiden plays $x_0 = \operatorname{dom}(p_0)^{c}$. For positive integers $i \in \omega$ let $t_i = \bigcup_{k\in i} s_k$, where $s_0, \dots, s_{i-1}$ are the moves of Death, and let $m_i \in \omega$ be the least number for which there exists a condition $p_i \geq p_{i-1}$ with $\operatorname{dom}(p_i) \supseteq \operatorname{dom}(p_{i-1}) \cup t_i$ such that for all $q \in \overline{p_i \mathbin{\text{-}} t_i}$ we have $p_i \Vdash_{\mathbb{S}_{\mathscr{E}}} \underset{\sim}{f}(i) < m_i$. Then the Maiden plays $x_i = \operatorname{dom}(p_i)^{c}$.

Since this strategy of the Maiden is not a winning strategy, Death can play such that $\bigcup_{i\in\omega} s_i \in \mathscr{E}$. Let $h = \bigcup_{i\in\omega} p_i$, then $h \in {}^{x}2$ for some $x \subseteq \omega$ (but h is not necessarily a $\mathbb{S}_{\mathscr{E}}$-condition). Now let $q_0 \in S_{\mathscr{E}}$ be such that $\operatorname{dom}(q_0) = \operatorname{dom}(h) \setminus \bigcup_{i\in\omega} s_i$ and $q_0 \equiv h|_{\operatorname{dom}(q_0)}$, and define the function $g \in {}^{\omega}\omega$ by stipulating $g(i) := m_i$. Then g belongs to the ground model $\mathbf{V}$ and by construction we have

$$q_0 \Vdash_{\mathbb{S}_{\mathscr{E}}} \forall i \in \omega \left(g(i) > \underset{\sim}{f}(i)\right)$$

which shows that f is dominated by g. q.e.d.

By Proposition 4, Theorem 13 implies that the forcing notion $\mathbb{S}_{\mathscr{E}}$ does not add Cohen reals. However, it adds splitting reals:

Proposition 14. The forcing notion $\mathbb{S}_{\mathscr{E}}$ adds splitting reals.

Proof (Sketch). Let $f \in {}^{\omega}2$ be $\mathbb{S}_{\mathscr{E}}$-generic over $\mathbf{V}$. We can identify f with the function $\bar{f} \in {}^{\omega}\omega$ by stipulating $\bar{f}(n) = k$ *iff* $f(k) = 1$ and $\big|\big\{m < k : f(m) = 1\big\}\big| = n$. Then the set

$$\sigma_f = \bigcup \Big\{ \big[\bar{f}(2n), \bar{f}(2n+1)\big) : n \in \omega \Big\}$$

splits every real in the ground model. To see this, notice that for each $x \in [\omega]^{\omega} \cap \mathbf{V}$ and for every $n \in \omega$, the set

$$D_{x,n} = \big\{ p \in S_{\mathscr{E}} : p \Vdash_{\mathbb{S}_{\mathscr{E}}} \text{“}|x \cap \underset{\sim}{\sigma_f}| > n \text{ and } |x \setminus \underset{\sim}{\sigma_f}| > n|\text{”} \big\}$$

is dense open in $S_{\mathscr{E}}$. q.e.d.

5.2 $\mathbb{S}_{\mathscr{E}}$ is minimal

A real g is **minimal** over $\mathbf{V}$ if g is not in the ground model $\mathbf{V}$ and every real f in $\mathbf{V}[g]$ is either in $\mathbf{V}$ or it reconstructs g, *i.e.*, g belongs to $\mathbf{V}[f]$ (where $\mathbf{V}[f]$ is the smallest model of ZFC which contains all sets of $\mathbf{V}$ as well as the function f). Let $\mathbb{P}$ be a forcing notion and let G be $\mathbb{P}$-generic over $\mathbf{V}$. If there is a real g such that $\mathbf{V}[g] = \mathbf{V}[G]$ and g is minimal over $\mathbf{V}$, then the forcing notion $\mathbb{P}$ is called **minimal**.

In the following we show that the forcing notion $\mathbb{S}_{\mathscr{E}}$ is minimal. The result will be a consequence of the following lemmas, but first we have to introduce some terminology (*cf.* [Gri71, p. 375 f.]): Let G be $\mathbb{S}_{\mathscr{E}}$-generic over $\mathbf{V}$ and let $\underset{\sim}{f}$ be an $\mathbb{S}_{\mathscr{E}}$-name for a function $f \in {}^{\omega}2 \cap \mathbf{V}[G]$. Two $\mathbb{S}_{\mathscr{E}}$-conditions p and q are called $\underset{\sim}{f}$**-compatible** if for all $k \in \omega$ and $\varepsilon \in \{0,1\}$ we have:

$$p \Vdash_{\mathbb{S}_{\mathscr{E}}} \underset{\sim}{f}(k) = \varepsilon \quad \Longleftrightarrow \quad q \Vdash_{\mathbb{S}_{\mathscr{E}}} \underset{\sim}{f}(k) = \varepsilon$$

For conditions p and functions $h \in {}^{u}2$, where $u \subseteq \omega$ is finite and $u \cap \mathrm{dom}(p) = \varnothing$, we write $p \cup h$ for the extension of p by h, *i.e.*, $(p \cup h)|_{\mathrm{dom}(p)} \equiv p$ and $(p \cup h)|_u \equiv h$. We say that $n \in \omega$ is $\underset{\sim}{f}$**-indifferent** to a condition p if $n \notin \mathrm{dom}(p)$ and for any $q \geq p$ we have either $n \in \mathrm{dom}(q)$ or the conditions $q \cup \langle n, 0\rangle$ and $q \cup \langle n, 1\rangle$ are $\underset{\sim}{f}$-compatible. Roughly speaking, n is $\underset{\sim}{f}$-indifferent to p if above p, n is of no use for the interpretation of $\underset{\sim}{f}$.

For any condition p, two mutually exclusive cases are possible:

(i) $\exists q \geq p\, \forall r \geq q\, \forall n \in \omega\, (n \text{ is not } \underset{\sim}{f}\text{-indifferent to } r)$

(ii) $\forall q \geq p \, \exists r \geq q \, \exists n \in \omega \, (n \text{ is } \underset{\sim}{f}\text{-indifferent to } r)$

Firstly consider the case when p satisfies (i) (*cf.* [Gri71, Lemma 4.6]):

Lemma 15. If p satisfies (i), then there is a condition $q \geq p$ such that for every $k \in \omega$ and for any distinct functions $t_1, t_2 : \mathrm{dom}(q)^{\mathrm{c}} \cap k \to 2$, the conditions $q \cup t_1$ and $q \cup t_2$ are $\underset{\sim}{f}$-incompatible.

Proof. Since by assumption p satisfies (i), there is a $q_0 \geq p$ such that for *all* $r \geq q_0$ and for *all* $n \in \mathrm{dom}(r)^{\mathrm{c}}$ we have:

$$\exists r' \geq r \left(r' \cup \langle n, 0 \rangle \text{ and } r' \cup \langle n, 1 \rangle \text{ are } \underset{\sim}{f}\text{-incompatible} \right) \qquad (\clubsuit)$$

In order to construct the condition q we run the game $\mathcal{G}^*_{\mathscr{E}}$ where the Maiden plays according to the following strategy: She begins by playing $x_0 := \mathrm{dom}(q_0)^{\mathrm{c}}$. Then, for positive integers $i \in \omega$, she plays $x_i := \mathrm{dom}(q_i)^{\mathrm{c}}$ where the condition q_i has the following properties: $q_i \geq q_{i-1}$ and any two different conditions which belong to the set $\overline{q_i \,\text{-}\, t_i}$, where $t_i = \bigcup_{j \in i} s_j$ and the s_j's are the moves of Death, are $\underset{\sim}{f}$-incompatible.

Notice that by $(\clubsuit)$ and since $q_i \geq q_{i-1} \geq q_0$, such a condition exists. Since this strategy is not a winning strategy for the Maiden, Death can play such that $x := \{s_i : i \in \omega\} \in \mathscr{E}$. Define q by stipulating $\mathrm{dom}(q) := x^{\mathrm{c}}$ and $q \equiv \bigcup_{i \in \omega} q_i|_{\mathrm{dom}(q)}$, then by construction, q belongs to $S_{\mathscr{E}}$ and has the desired properties. q.e.d.

Secondly consider the case when p satisfies (ii) (*cf.* [Gri71, Lemma 4.7]):

Lemma 16. If p satisfies (ii), then there is a condition $q \geq p$ which decides $\underset{\sim}{f}(k)$ for each $k \in \omega$.

Proof. Since by assumption p satisfies (ii), for each $p' \geq p$ there is an $r \geq p'$ such that we have:

$$\exists n \in \mathrm{dom}(r)^{\mathrm{c}} \left(n \text{ is } \underset{\sim}{f}\text{-indifferent to } r \right) \qquad (\spadesuit)$$

For conditions $p' \in S_{\mathscr{E}}$ let

$$I_{p'} = \left\{ n \in \mathrm{dom}(p')^{\mathrm{c}} : n \text{ is } \underset{\sim}{f}\text{-indifferent to } p' \right\}.$$

Claim 17. For each condition $p' \geq p$, $I_{p'}$ belongs to $\mathscr{E}$.

Proof. Assume towards a contradiction that $I_{p'} \notin \mathscr{E}$, then, since $\mathrm{dom}(p')^{\mathrm{c}} \in \mathscr{E}$, by Lemma 5 we get $\mathrm{dom}(p')^{\mathrm{c}} \setminus I_{p'} \in \mathscr{E}$. Let $r \geq p'$ be any condition with $\mathrm{dom}(r) = \mathrm{dom}(p') \cup I_{p'}$, then there is no $n \in \omega$ such that n is $\underset{\sim}{f}$-indifferent to r, since such an n would also be $\underset{\sim}{f}$-indifferent to p', but this is impossible by the definition of $I_{p'}$ and completes the proof of the claim.

q.e.d. (Claim 17)

In order to construct the condition q we run the game $\mathcal{G}^*_{\mathscr{E}}$ where the Maiden plays according to the following strategy: She starts the game by playing $x_0 := I_{p_0}$, where $p_0 \geq p$ is such that p_0 decides $\underset{\sim}{f}(0)$. In addition, she plays a condition $q_0 \geq p_0$ such that $\mathrm{dom}(q_0) = x_0^{\mathsf{c}}$. In general, for a positive integer $i \in \omega$ let $t_i = \bigcup_{j \in i} s_j$, where the s_j's are the moves of Death, and let $p_i \geq q_{i-1}$ be such that $\mathrm{dom}(p_i) \supseteq x_{i-1}^{\mathsf{c}} \cup t_i$ and every $p' \in \overline{p_i \frown t_i}$ decides $\underset{\sim}{f}(i)$. Now the Maiden plays $x_i := I_{p_i}$ and a condition $q_i \geq p_i$ such that $\mathrm{dom}(q_i) = x_i^{\mathsf{c}}$.

Notice that by (♠) and by the claim, the strategy of the Maiden is well-defined. Since her strategy is not a winning strategy, Death can play such that $x := \{s_i : i \in \omega\} \in \mathscr{E}$. Define q by stipulating $\mathrm{dom}(q) := x^{\mathsf{c}}$ and $q \equiv \bigcup_{i \in \omega} q_i|_{\mathrm{dom}(q)}$, then by construction, q belongs to $S_{\mathscr{E}}$ and has the desired properties. q.e.d. (Lemma 16)

By combining the previous two lemmas we are now able to prove that the forcing notion $\mathbb{S}_{\mathscr{E}}$ is minimal, or equivalently, that each Silver real is minimal (*cf.* [Gri71, Theorem 4.1]).

Theorem 18. Each real $g \in {}^{\omega}2$ which is $\mathbb{S}_{\mathscr{E}}$-generic over $\mathbf{V}$ is minimal over $\mathbf{V}$, *i.e.*, for every real $f \in {}^{\omega}2 \cap \mathbf{V}[g]$, either $f \in \mathbf{V}$ or $g \in \mathbf{V}[f]$.

Proof. Let G be $\mathbb{S}_{\mathscr{E}}$-generic over $\mathbf{V}$, let $g \in {}^{\omega}2$ be the Silver real which corresponds to G, and let $\underset{\sim}{f}$ be an $\mathbb{S}_{\mathscr{E}}$-name for a function $f \in {}^{\omega}2 \cap \mathbf{V}[g]$. We have to show that for $f = \underset{\sim}{f}[g]$, either $f \in \mathbf{V}$ or $g \in \mathbf{V}[f]$.

Let $D = D_1 \cup D_2$ where D_1 and D_2 are defined as follows:

$$D_1 = \big\{q \in S_{\mathscr{E}} : q \text{ as in Lemma 15 with respect to some } p \leq q\big\}$$
$$D_2 = \big\{q \in S_{\mathscr{E}} : q \text{ as in Lemma 16 with respect to some } p \leq q\big\}$$

By definition, D is obviously dense open in $S_{\mathscr{E}}$ which implies that there exists a $q_0 \in G \cap D$. We have to consider the following two cases.

"$q_0 \in D_1$": In $\mathbf{V}[f]$ define the function $g' \in {}^{\omega}2 \cap \mathbf{V}[f]$ as follows. Firstly, on $\mathrm{dom}(q_0)$ define g' such that $g'|_{\mathrm{dom}(q_0)} \equiv q_0$. Secondly, on $\mathrm{dom}(q_0)^{\mathsf{c}}$ define g' by the following induction: Suppose that the function g' is already defined on some $k \in \omega$. Let $t_k \equiv g'|_{k \cap \mathrm{dom}(q_0)}$ and let $m = \min(\mathrm{dom}(q_0)^{\mathsf{c}} \setminus k)$. Then, by the definition of q_0, the conditions $p^0_m := q_0 \cup t_k \cup \langle m, 0\rangle$ and $p^1_m := q_0 \cup t_k \cup \langle m, 1\rangle$ are $\underset{\sim}{f}$-incompatible, *i.e.*, there is an $n \in \omega$ such that $p^0_m \Vdash_{\mathbb{S}_{\mathscr{E}}} \underset{\sim}{f}(n) = \varepsilon$ and $p^1_m \Vdash_{\mathbb{S}_{\mathscr{E}}} \underset{\sim}{f}(n) = 1 - \varepsilon$ (for some $\varepsilon \in \{0,1\}$). Take the least such n and define $g'(m) := f(n)$. Notice that this can be done since we work in the model $\mathbf{V}[f]$ in which we know the value $f(n)$. Notice also that since $f = \underset{\sim}{f}[g]$, for $i \in \{0,1\}$ the condition p^i_m belongs to G *iff* $p^i_m \Vdash_{\mathbb{S}_{\mathscr{E}}} \underset{\sim}{f}(n) = g'(m)$. Thus, $g'(m)$ decides which of the two incompatible condition, p^0_m or p^1_m, belongs to G. Now, because q_0 is in G, we see inductively how the function g' reconstructs the $\mathbb{S}_{\mathscr{E}}$-generic filter G or equivalently the function g, and since $g' \in \mathbf{V}[f]$ we consequently have $g \in \mathbf{V}[f]$.

"$q_0 \in D_2$": By definition, q_0 decides $\underset{\sim}{f}(k)$ for each $k \in \omega$, which shows that the function f belongs to $\mathbf{V}$.

Hence, we have either $g \in \mathbf{V}[f]$ (if $q_0 \in D_1$) or $f \in \mathbf{V}$ (if $q_0 \in D_2$), which completes the proof. q.e.d.

6 Happy Families and Their Relatives

Firstly we recall Mathias' notion of a happy family (*cf.* [Mat$_0$77]): Let $[\omega]^{<\omega}$ be the set of all finite subsets of ω, and for $s \in [\omega]^{<\omega}$, let $\bar{s}^+ := (\max s) + 1$. A set $x \subseteq \omega$ is said to **diagonalize** the set $\{x_s : s \in [\omega]^{<\omega}\} \subseteq [\omega]^\omega$, if $x \subseteq x_\varnothing$ and for all $s \in [\omega]^{<\omega}$, if $(\max s) \in x$, then $x \setminus \bar{s}^+ \subseteq x_s$. For $\mathscr{A} \subseteq [\omega]^\omega$ we write $\operatorname{fil}\mathscr{A}$ for the filter generated by the members of $\mathscr{A}$, *i.e.*, $\operatorname{fil}\mathscr{A}$ consists of all subsets of ω which are supersets of intersections of finitely many members of $\mathscr{A}$. A free family $\mathscr{E}$ is called a **happy family** if whenever $\operatorname{fil}\{x_s : s \in [\omega]^{<\omega}\} \subseteq \mathscr{E}$, then there is an $x \in \mathscr{E}$ which diagonalizes the set $\{x_s : s \in [\omega]^{<\omega}\}$.

An obvious example of a happy family is the set $[\omega]^\omega$, and it is not hard to see that all Ramsey ultrafilters are happy (*cf.* [Mat$_0$77, Section 0]). Other examples of happy families are Ramsey families:

Fact 19. Every Ramsey family is happy.

Proof. Let $\mathscr{E}$ be a free family which is not happy and let $\{x_s : s \in [\omega]^{<\omega}\} \subseteq \mathscr{E}$ be such that $\operatorname{fil}\{x_s : s \in [\omega]^{<\omega}\} \subseteq \mathscr{E}$ but there is no $x \in \mathscr{E}$ which diagonalizes the set $\{x_s : s \in [\omega]^{<\omega}\}$. We leave it as an exercise to the reader to construct with the set $\{x_s : s \in [\omega]^{<\omega}\}$ a winning strategy for the MAIDEN for the game $\mathcal{G}_{\mathscr{E}}$. q.e.d.

More examples of happy families we obtain by maximal almost disjoint families: An infinite family $\mathscr{A} \subseteq [\omega]^\omega$ is called **maximal almost disjoint**, abbreviated *m.a.d.*, if any two distinct sets of $\mathscr{A}$ have a finite intersection and for every $y \in [\omega]^\omega \setminus \mathscr{A}$ there is an $x \in \mathscr{A}$ such that $x \cap y$ is infinite. Now, let $\mathscr{A} \subseteq [\omega]^\omega$ be a *m.a.d.* family and let $\mathscr{F} = \mathrm{fil}\{\omega \setminus x : x \in \mathscr{A}\}$, then $\mathscr{E}(\mathscr{A}) := \mathscr{F}^+$ is a happy family (*cf.* [Mat$_0$77, Proposition 0.7]). This example of a happy family leads to the following:

Proposition 20. Not every happy family is Ramsey.

Proof. Let $S = \{s_i : i \in \omega\}$ be the set of all finite sequences of ω, which is partially ordered by the extension relation, denoted "$\prec$". For infinite sequences $f \in {}^\omega\omega$, let $x_f := \big\{i \in \omega : \exists n \in \omega(f|_n = s_i)\big\}$. Obviously, for any distinct sequences $f, g \in {}^\omega\omega$ we have that $x_f \cap x_g$ is finite. Now, let $\mathscr{A}_0 := \{x_f : f \in {}^\omega\omega\}$, then $\mathscr{A}_0 \subseteq [\omega]^\omega$ is a set of pairwise almost disjoint sets which can be extended to a *m.a.d.* family, say $\mathscr{A}$.

We show that $\mathscr{E}(\mathscr{A})$ is not a Ramsey family: Let $x_0 := \omega$ be the first move of the MAIDEN and let a_0 be DEATH' response. In general, for $i \in \omega$ she plays

$$x_{i+1} = \{i \in \omega : s_{a_n} \prec s_i\}\,.$$

For every $n \in \omega$, x_n is the union of infinitely many members of $\mathscr{A}_0$, and therefore is an element of $\mathscr{E}(\mathscr{A})$. Now, by the MAIDEN's strategy, $s_{a_0} \prec s_{a_1} \prec \cdots$ corresponds to an infinite sequence $f \in {}^\omega\omega$ and therefore, $\{a_n : n \in \omega\} \subseteq x_f$ for some $x_f \in \mathscr{A}_0$, which implies that $\{a_n : n \in \omega\} \notin \mathscr{E}(\mathscr{A})$ and that DEATH loses the game. Thus, the MAIDEN has a winning strategy for the game $\mathcal{G}_{\mathscr{E}(\mathscr{A})}$, and hence, $\mathscr{E}(\mathscr{A})$ is not Ramsey. q.e.d.

A *m.a.d.* family $\mathscr{A}$ is called **strongly maximal almost disjoint** if given countably many members of $\mathscr{E}(\mathscr{A})$, then there is a member of $\mathscr{A}$ that meets each of them in an infinite set.

For a free family $\mathscr{E}$, consider the following game $\bar{\mathcal{G}}_{\mathscr{E}}$: The moves of the MAIDEN are members of $\mathscr{E}$ and DEATH responses like in the game $\mathcal{G}_{\mathscr{E}}$. Further, DEATH wins if and only if the set of integers played by DEATH belongs to $\mathscr{A}$, but has infinite intersection with each set played by the MAIDEN.

If $\mathscr{A}$ is a *m.a.d.* family, then obviously the MAIDEN has a winning strategy for $\bar{\mathcal{G}}_{\mathscr{E}(\mathscr{A})}$ if and only if $\mathscr{A}$ is not strongly *m.a.d.*, which motivates the following:

Question 21. Is it the case that for a *m.a.d.* family $\mathscr{A}$, $\mathscr{E}(\mathscr{A})$ is Ramsey if and only if $\mathscr{A}$ is strongly *m.a.d.*?

Related to happy families are the so-called moderately happy families introduced by Mathias in [Mat$_0$77, Section 9]: A free family $\mathscr{E}$ is **moderately happy** if whenever $\mathrm{fil}\{x_n : n \in \omega\} \subseteq \mathscr{E}$, then there is an $x \in \mathscr{E}$ such that for all $n \in \omega$, $x \setminus x_n$ is finite.

On the one hand, it is not hard to verify that every P-family is moderately happy. On the other hand, by similar arguments as in the proof of Proposition 20, one can show that there exist moderately happy families which are not P-families.

Now, since the results of Section 4 are also valid for Mathias forcing restricted to happy families, it is natural to ask whether something similar holds for Silver forcing with respect to moderately happy families:

Question 22. Are the results of Section 5 also valid for Silver forcing restricted to moderately happy families?

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