5. Normal Subgroups

Before we define the notion of a normal subgroup, let us prove the following:

**Fact 5.1.** Let $G$ be a group. If $H \subseteq G$ and $x \in G$, then
$$xHx^{-1} = \{xhx^{-1} : h \in H\}$$
is a subgroup of $G$.

*Proof.* Let $xh_1x^{-1}$ and $xh_2x^{-1}$ be in $xHx^{-1}$. Then $(xh_2x^{-1})^{-1} = xh_2^{-1}x^{-1}$ and $(xh_1x^{-1})(xh_2^{-1}x^{-1}) = x(h_1h_2^{-1})x^{-1} \in xHx^{-1}$. So, by definition, $xHx^{-1} \subseteq G$. \(\dashv\)

This leads to the following definition.

**Definition.** Suppose that $G$ is a group and that $N \leq G$, then $N$ is called a normal subgroup of $G$ if for all $x \in G$ we have
$$xNx^{-1} = N,$$
or equivalently, if for all $x \in G$, $xN = Nx$.

In particular, the trivial subgroups are normal and all subgroups of an abelian group are normal.

**Notation.** If $N \leq G$ ($N < G$) is a normal subgroup of $G$, then we write $N \trianglelefteq G$ ($N \lhd G$).

The following is just a consequence of Corollary 3.10:

**Fact 5.2.** If $H < G$ and $|G : H| = 2$, then $H \lhd G$.

*Proof.* By Corollary 3.10 we know that if $|G : H| = 2$, then for all $x \in G$ we have $xH = Hx$, and therefore $H \lhd G$. \(\dashv\)

**Proposition 5.3.** If $N \leq G$, then $N \trianglelefteq G$ if and only if for all $x \in G$ and all $n \in N$ we have
$$xnx^{-1} \in N.$$

*Proof.* If $N \leq G$, then $xNx^{-1} = N$ (for all $x \in G$), thus, $xnx^{-1} \in N$ for all $x \in G$ and $n \in N$.

On the other hand, if $xnx^{-1} \in N$ for all $x \in G$ and $n \in N$, then $xNx^{-1} \subseteq N$ (for all $x \in G$). Further, replacing $x$ by $x^{-1}$ we get
$$N = x \underbrace{(x^{-1}Nx)}_{\subseteq N} x^{-1} \subseteq xNx^{-1}.$$Hence, $xNx^{-1} = N$ (for all $x \in G$). \(\dashv\)

The following Fact is similar to Proposition 3.2:

**Fact 5.4.** If $K, H \leq G$, then $(K \cap H) \leq G$.

*Proof.* If $K, H \leq G$, then, by Proposition 5.3, for all $x \in G$ and $n \in K \cap H$ we have $xnx^{-1} \in K$ (since $K \leq G$) and $xnx^{-1} \in H$ (since $H \leq G$), and therefore, $xnx^{-1} \in K \cap H$ (for all $x \in G$ and $n \in K \cap H$). \(\dashv\)
Notice that if \( H \triangleleft K \triangleleft G \), then \( H \) is not necessarily a normal subgroup of \( G \). To see this, let \( T \) be the tetrahedron-group, let \( \rho_1, \rho_2 \) and \( \rho_3 \) be the three elements of \( T \) of order 2, and let \( \iota \) be the neutral element of \( T \). Further, let \( H = \{ \iota, \rho_1 \} \) and \( K = \{ \iota, \rho_1, \rho_2, \rho_3 \} \). Since the group \( K \) is isomorphic to \( C_2 \times C_2 \), it is abelian and therefore we get \( H \triangleleft K \). Further, for each \( \tau \in T \) and \( \rho \in K \), \( \tau \rho \tau^{-1} \) has either order 1 or 2. Thus, \( \tau \rho \tau^{-1} \in K \), which implies by Proposition 5.3 that \( K \triangleleft T \). Finally, it is not hard to see that \( H \) is not a normal subgroup of \( T \).

Let us now give some examples of normal subgroups:

1. \( T \triangleleft C \) (since \( |C : T| = 2 \)).
2. For \( n \geq 3 \), \( C_n \triangleleft D_n \) (since \( |D_n : C_n| = 2 \)).
3. For \( n \geq 1 \), \( \text{SO}(n) \triangleleft \text{O}(n) \) (since \( |\text{O}(n) : \text{SO}(n)| = 2 \)).
4. As we have seen above, \( T \) contains a normal subgroup which is isomorphic to \( C_2 \times C_2 \).
5. For \( n \geq 1 \), \( \text{SL}(n) \triangleleft \text{GL}(n) \): For all \( B \in \text{GL}(n) \) and \( A \in \text{SL}(n) \) we have \( \det(BAB^{-1}) = \det(A) = 1 \), thus, \( BAB^{-1} \in \text{SO}(n) \).

**Definition.** Suppose that \( G \) is a group. We define the **centre** \( Z(G) \) of \( G \) by

\[
Z(G) := \{ a \in G : \forall x \in G(ax = xa) \}
\]

In other words, \( Z(G) \) consists of those elements of \( G \) which commute with every element of \( G \).

**Fact 5.5.** \( Z(G) = G \) if and only if \( G \) is abelian.

**Proof.** If \( G \) is abelian, then for all \( a \in G \) and for all \( x \in G \) we have \( ax = xa \), thus, \( Z(G) = G \). On the other hand, \( Z(G) = G \) implies that for all \( a \in G \) and for all \( x \in G \), \( ax = xa \), thus, \( G \) is abelian. \( \square \)

**Fact 5.6.**

(a) \( Z(G) \trianglelefteq G \) (see Hw7.Q31.a).

(b) \( Z(G) \trianglelefteq G \) (see Hw7.Q31.b).

(c) \( Z(G) \) is abelian (see Hw7.Q31.c).

(d) If \( H \leq Z(G) \), then \( H \leq G \) (see Hw7.Q31.d).

It is possible that the centre of a group is just the neutral element, e.g., \( Z(T) = \{ \iota \} \).

**Definition.** Let \( G \) be a group and let \( H \) and \( K \) be subgroups of \( G \). If \( G = HK \), then we say that \( G \) is the **inner product** of \( H \) and \( K \).

**Proposition 5.7.** Let \( G \) be a finite group and let \( H, K \leq G \). Then

\[
|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.
\]

**Proof.** First notice that \( HK = \bigcup_{h \in H} hK \) and that \( (H \cap K) \leq H \).

Now, for \( h_1, h_2 \in H \) we have

\[
h_1K = h_2K \iff h_1h_2^{-1} \in K,
\]

and further we have

\[
h_1(H \cap K) = h_2(H \cap K) \iff h_1h_2^{-1} \in (H \cap K) \iff h_1h_2^{-1} \in K.
\]
Therefore, 

\[ |HK| = \sum_{h \in H} hK| = |H : (H \cap K)| \cdot |K| = \frac{|H|}{|H \cap K|} \cdot |K| = \frac{|H| \cdot |K|}{|H \cap K|}. \]

Notice that if \( H \) and \( K \) are subgroups of a group \( G \), then \( HK \) is not necessarily a subgroup of \( G \) (see Hw7.Q34). On the other hand, if at least one of these two subgroups is a normal subgroup, then \( HK \) is a subgroup of \( G \):

**Theorem 5.8.** If \( K \leq G \) and \( N \unlhd G \), then \( KN = NK \leq G \).

**Proof.** Let us first show that \( KN = NK \): Let \( k \in K \) and \( n \in N \), and let \( n_1 = knk^{-1} \) and \( n_2 = k^{-1}nk \). Then, since \( N \unlhd G \), \( n_1, n_2 \in N \), and further we have

\[ kn = n_1k \quad \text{and} \quad nk = kn_2, \]

which shows that \( KN = NK \). To see that \( KN \leq G \), pick two elements \( (k_1n_1) \) and \( (k_2n_2) \) of \( KN \). We have to show that \( (k_1n_1)(k_2n_2)^{-1} \in KN \):

\[ (k_1n_1)(k_2n_2)^{-1} = k_1n_1k_2^{-1} = k_1k_2^{-1}k_2n_2^{-1} = kn \in KN. \]

Let us give an example for Theorem 5.8: Consider the cube-group \( C \). Let \( a, b, \) and \( c \) be the three axes joining centres of opposite faces and let \( \rho_a, \rho_b, \rho_c \in C \) be the rotations about the axes \( a, b, \) and \( c \) respectively through \( \pi \) and let \( \delta \in C \) be the rotation about the axis \( a \) through \( \pi /2 \). Now, let \( N = \langle \{\rho_a, \rho_b, \rho_c\} \rangle \) and let \( K = \langle \delta \rangle \). It is easy to see that \( K \) and \( N \) are both subgroups of \( C \) of order 4. Notice that \( K \cong C_4 \) and that \( N \cong C_2 \times C_2 \), so, \( K \) and \( N \) are not isomorphic, but they are both abelian. Let us now show that \( N \) is a normal subgroup of \( C \): For this, we consider the set of axes \( \{a, b, c\} \). Now, every \( x \in C \) corresponds to a permutation \( \tau_x \) on \( \{a, b, c\} \), and \( n \in N \) if and only if \( \tau_n(a) = a, \tau_n(b) = b, \) and \( \tau_n(c) = c \), or in other words, \( n \in N \) if \( n \) corresponds to the identity permutation on \( \{a, b, c\} \). For any \( x \in C \) and \( n \in N \), the permutation \( \tau_{xn}^{-1} = \tau_x \tau_n \tau_{x^{-1}} \) is the identity permutation on \( \{a, b, c\} \), and hence, \( xn \in N \), which shows that \( N \lhd C \). Thus, by Theorem 5.8, \( KN \leq C \).

Since \( |K \cap N| = 2 \), by Proposition 5.7 we have \( |KN| = \frac{|K| |N|}{|K \cap N|} = 8 \) and it is not hard to see that \( KN \cong D_4 \).

**Proposition 5.9.** If \( K \) and \( H \) are subgroups of the finite group \( G \), \( |H \cap K| = 1 \) and \( |H| \cdot |K| = |G| \), then \( HK = G = KH \).

**Proof.** Let us just prove that \( HK = G \) (to show that \( KH = G \) is similar). Since \( HK = \{hk : h \in H \text{ and } k \in K\} \subseteq G \), \( HK = G \) if and only if \( |HK| = |G| \), which implies that \( h_1k_1 = h_2k_2 \) if and only if \( h_1 = h_2 \) and \( k_1 = k_2 \). So, let us assume that \( h_1k_1 = h_2k_2 \), then \( h_1^{-1}(h_1k_1)k_2^{-1} = h_1^{-1}(h_1k_2)k_2^{-1} \), and hence, \( k_1k_2^{-1} = h_1^{-1}h_2 \in H \cap K \), but since \( H \cap K = \{e\} \), this implies that \( h_1 = h_2 \) and \( k_1 = k_2 \).

The following proposition shows that if \( K \) and \( H \) are normal subgroups of \( G \) such that \( |H \cap K| = 1 \), then the elements of \( H \) commute with the elements of \( K \) and vice versa. Notice that this is stronger than just saying \( KH = HK \).
Proposition 5.10. If $K$ and $H$ are normal subgroups of $G$ and $|H \cap K| = 1$, then for all $h \in H$ and all $k \in K$, $hk = kh$.

Proof. Let $h \in H$ and $k \in K$. Consider the element $hkh^{-1}k^{-1}$: On the one hand we have

\[
\{hkh^{-1}k^{-1} \in H \mid e \in H\}
\]

and on the other hand we have

\[
\{hkh^{-1}k^{-1} \in K \mid e \in K\}
\]

Thus, $hkh^{-1}k^{-1} \in H \cap K$, and since $|H \cap K| = 1$, $hkh^{-1}k^{-1} = e$, which implies $kh = hkh^{-1}k^{-1}(kh) = hk$. \[\]

Proposition 5.11. If $K$ and $H$ are normal subgroups of $G$, then $KH \trianglelefteq G$.

Proof. For any $x \in G$, $xkhx^{-1} = (xkx^{-1})(xhx^{-1}) \in KH$, thus, $xKHx^{-1} = KH$. \[\]

Definition. A group $G$ is called simple if it does not contain any non-trivial normal subgroup.

In particular, any abelian group which has a non-trivial subgroup cannot be simple, but there are also simple abelian groups, e.g., the cyclic groups $C_p$, where $p$ is prime (see Hw7.Q35). An example of a simple group which is not abelian is the dodecahedron-group $D$ (as we will see later). On the other hand, there are many non-abelian groups which are not simple groups:

1. The cube-group $C$, because $T \triangleleft C$.
2. $D_n$ for $n \geq 3$, because $C_n \triangleleft D_n$.
3. $O(n)$ for $n \geq 2$, because $SO(n) \triangleleft O(n)$.
4. The tetrahedron-group $T$, because $T$ contains a normal subgroup which is isomorphic to $C_2 \times C_2$.
5. $GL(n)$ for $n \geq 2$, because $SL(n) \triangleleft GL(n)$.