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ASYMPTOTIC BEHAVIOR OF SOLUTION OF HYPERBOLIC PROBLEMS ON A CYLINDRICAL DOMAIN

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INTRODUCTION

The asymptotic behavior of the hyperbolic evolution problems of order two, on a cylindrical domain, with coefficients dependent on a parameter is examined. The convergence of the solution of such problems towards a solution of a problem of the same type defined in ω is proved, and the rate of convergence estimates is given. One can see work like a singular perturbation of the hyperbolic problems in some directions.

u_θ solution of hyperbolic problem defined on $[0, T] \times \Delta \times \omega \subset \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^{n-p}$ $\searrow$

u solution of hyperbolic problem defined on $[0, T] \times \omega \subset \mathbb{R}^{n-p}$ $\swarrow$

$$u_\theta(t, X_1, X_2) \approx u(t, X_2)$$

INTEREST OF APPLICATION

The interest of this work for the applications of the resolution of the hyperbolic problems is due to the fact that if one of the solutions is given by a simple resolution, the approximation neglects the difficulty of the other solution, therefore several application are possible.

✓ If the resolution of u_θ is complicated, we can approach it by u for which, may be, a numerical or analytical calculation is possible.

✓✓ The finite element method for u (for example defined on $\mathbb{R}^2$) more accessible than for u_θ(for example defined on $\mathbb{R}^3$).

OBJECT OF VERY MANY STUDIES

Many works were realized in this domain, under the name of "singular perturbation", and different questions worked out intil now. We find in the book J.L Lions and his bibliography several work in this field. In this work, we base on some ideas given in the works of M. Chipot and A. Rougirel, and the book of J.L. Lions and E. Magenes.

TECHNICAL TOOLS

We give here some technical tools, which one will use them in the continuation.

Definition : Let X and Y two separable Hilbert spaces, such that the injection $X \hookrightarrow Y$ is continuous. We define the spaces $W(0,T;X,Y)$ and $W^{\infty}(0,T;X,Y)$ by

$$\begin{aligned} W(0,T;X,Y) &= \left\{ u \in L^2(0,T;X),\ u' \in L^2(0,T;Y) \right\}, &(1)\\ W^{\infty}(0,T;X,Y) &= \left\{ u \in L^{\infty}(0,T;X),\ u' \in L^{\infty}(0,T;Y) \right\}, &(2) \end{aligned}$$

equipped with the norm

$$\begin{aligned} \|u\|_W^2 &= \|u\|^2_{L^2(a,b;X)} + \left\| u' \right\|^2_{L^2(a,b;Y)}, &(3)\\ \|u\|_{W^{\infty}} &= \|u\|_{L^{\infty}(a,b;X)} + \left\| u' \right\|_{L^{\infty}(a,b;Y)}. &(4) \end{aligned}$$

Lemma 1 *Let* $v \in H_0^1(\Delta \times \omega)$, *then*

$$v(X_1, .) \in H_0^1(\omega) \;\; a.e. \;\; X_1 \in \Delta$$

Lemma 2 *Friedrichs lemma In a Hilbert space* $\mathcal{H}$, *a family of operators* $\mathcal{C}(t) \in L(\mathcal{H}, \mathcal{H})$, *such that the function* $t \longrightarrow (\mathcal{C}(t)u, v) \;\; \forall u, v \in \mathcal{H}$ *is once continuously differentiable, with*

$$\left| \left(\frac{d}{dt} \mathcal{C}(t)u, v \right) \right| \leqslant c\, |u|\, |v| \,. \tag{5}$$

Then, for $u \in L^2(-\infty, +\infty; \mathcal{H})$, *we have*

$$\frac{d}{dt} \left[\mathcal{C}(t) \left(\rho_n * u \right) - \rho_n * \left(\mathcal{C}(t)u \right) \right] \underset{n \to +\infty}{\longrightarrow} 0 \quad \textit{in } L^2(-\infty, +\infty; \mathcal{H}) \,. \tag{6}$$

POSITION OF THE PROBLEM

Let Ω be a cylindrical domain of $\mathbb{R}^n$ defined by

$$\Omega = \Delta \times \omega$$

where Δ, ω are bounded Lipschitz domains of $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$ respectively, n and p integers with $n > p \geqslant 1$. For $x \in \mathbb{R}^n$, we set

$$X_1 = (x_1, ..., x_p), \qquad X_2 = (x_{p+1}, ..., x_n). \tag{7}$$

We denote

$$Q = [0, T] \times \Omega, \quad Q_\infty = [0, T] \times \omega$$

where $T > 0$ is a positive constant. For a positive parameter θ, we consider the two evolutions problems defined by

$$\begin{cases} u'' - \sum\limits_{0 \leqslant i,j \leqslant n} \partial_{x_i}(a_{ij}^{\theta}(t,x)\partial_{x_j}u) + c(t,x)u = f & \text{in } \ Q \\ u = 0 & \text{on } \ [0,T] \times \partial\Omega \\ u(0,.) = u_{\theta,0}, \quad u'(0,.) = u_{\theta,1} & \text{in } \Omega \end{cases} \tag{8}$$

$$\begin{cases} u' - \sum\limits_{p+1 \leqslant i,j \leqslant n} \partial_{x_i}(a_{ij}^{\theta}(t,x)\partial_{x_j}u) + c(t,x)u = f & \text{in } \ Q_\infty \\ u = 0 & \text{on } \ [0,T] \times \partial\omega \\ u(0,.) = u_0, \quad u'(0,.) = u_1 & \text{in } \omega \end{cases} \tag{9}$$

such that $f : Q \longrightarrow \mathbb{R}$, u_0, $u_1 : \omega \longrightarrow \mathbb{R}$, $u_{\theta,0}$, $u_{\theta,1} : \Omega \longrightarrow \mathbb{R}$, a_{ij}^{θ}, $a : Q \longrightarrow \mathbb{R}$.

ASSUMPTIONS

For the initial conditions and the source term f, we suppose

$$f \in L^2(Q_\infty), \tag{10}$$

$$\begin{cases} u_0 \in H_0^1(\omega), \quad u_1 \in L^2(\omega), \\ u_{\theta,0} \in H_0^1(\Omega), \qquad u_{\theta,1} \in L^2(\Omega). \end{cases} \tag{11}$$

We put the assumptions on the coefficients a_{ij}^θ , c

$$a_{ij}^\theta \in \mathcal{C}^1(\overline{Q}),\ c \in \mathcal{C}(\overline{Q}), \ \ a_{ij}^\theta = a_{ji}^\theta, \tag{12}$$

for all $i, j = \widehat{1, n}$, moreover we put the assumptions (59). The hyperbolicity of the problem is supposed so that there are two constants $\lambda > 0$

and $\lambda' > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}^{\theta}(t,x)\xi_i\xi_j \geqslant \lambda\theta \left|\boldsymbol{\xi}^1\right|^2 + \lambda' \left|\boldsymbol{\xi}^2\right|^2, \text{ for } a.e.\ (t,x) \in Q \text{ and } \forall\xi \in \mathbb{R}^n, \quad (13)$$

$\xi^1 = \left(\xi_1, ..., \xi_p\right)$ and $\xi^2 = \left(\xi_{p+1}, ..., \xi_n\right)$, Under the conditions (10), (11), (12) and (13), the problems (8) and (9) admit weak solutions u_θ and u_∞ respectively, and in particular verifies

$$\frac{d^2}{dt^2}\int_\Omega u_\theta v dx + \int_\Omega \left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^{\theta}(t,x)\partial_{x_j}u_\theta\partial_{x_i}v + c(t,x)u_\theta v\right) dx = \int_\Omega f v dx$$

$$u_\theta(0,.) = u_{\theta,0}, \quad u_\theta'(0,.) = u_{\theta,1} \quad (14)$$

$$\forall v \in H_0^1(\Omega) \text{ and for } a.e.\ t \in [0,T],$$

$$
\frac{d^2}{dt^2}\int_\omega u_\infty v dx + \int_\omega \left(\sum_{p+1\leqslant i,j\leqslant n} a_{ij}(t,x)\partial_{x_j}u_\infty \partial_{x_i} v + c(t,x)u_\infty v \right) dx = \int_\omega f v dx
$$

$$
u_\infty(0,.) = u_0, \quad u'_\infty(0,.) = u_1 \tag{15}
$$

$$
\forall v \in H_0^1(\omega), \text{ and for } a.e.\ t \in [0,T]\,.
$$

$$
u_\theta \in C^0(0,T;H_0^1(\Omega)), \quad u'_\theta \in C^0(0,T;L^2(\Omega)), \tag{16}
$$

$$
u_\infty \in C^0(0,T;H_0^1(\omega)), \quad u'_\infty \in C^0(0,T;L^2(\omega)). \tag{17}
$$

The injection $H_0^1(\omega) \hookrightarrow H^1(\Omega)$ is continuous, then we obtain

$$
u_\infty \in \mathcal{D}'(0,T;H^1(\Omega)), \quad u'_\infty \in \mathcal{D}'\left(0,T;L^2(\Omega)\right) \tag{18}
$$

$$
u_\infty \in C^0(0,T;H^1(\Omega)), \quad u'_\infty \in C^0(0,T;L^2(\Omega)). \tag{19}
$$

OBJECT

We will give in this work, the asymptotic behaviour for the solution u_θ when $\theta \to 0$, and more precisely,

✓ We justify convergence $u_\theta \to u_\infty$ when $\theta \to 0$.

✓✓ We give the rate of convergence (the estimate of $u_\theta - u_\infty$).

EQUALITY OF THE ENERGY TYPE

We take in (15) $v \in H_0^1(\Omega)$ and we integrate on Δ, we obtain

$$\frac{d^2}{dt^2}\int_\Omega u_\infty v dx+\int_\Omega \left(\sum_{p+1\leqslant i,j\leqslant n} a_{ij}(t,x)\partial_{x_j}u_\infty \partial_{x_i}v + c(t,x)u_\infty v\right) dx = \int_\Omega f v dx, \tag{20}$$

using (14), we find

$$\begin{aligned} &\frac{d^2}{dt^2}\int_\Omega u_\theta v dx + \int_\Omega \left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(t,x)\partial_{x_j}u_\theta \partial_{x_i}v + c(t,x)u_\theta v\right) dx \\ &= \frac{d^2}{dt^2}\int_\Omega u_\infty v dx + \int_\Omega \left(\sum_{p+1\leqslant i,j\leqslant n} a_{ij}(t,x)\partial_{x_j}u_\infty \partial_{x_i}v + c(t,x)u_\infty v\right) dx. \end{aligned} \tag{21}$$

The coefficients $a_{ij}^{\theta}(t,x)$ for $p+1 \leqslant j \leqslant n$ and the solution u_{∞} depend only on X_2 and t, we deduce

$$\frac{d^2}{dt^2}\int_{\Omega}(u_\theta - u_\infty)\, v dx + \int_{\Omega}\left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^{\theta}(t,x)\partial_{x_j}(u_\theta - u_\infty)\,\partial_{x_i}v + c(t,x)(u_\theta - u_\infty)\, v\right)dx =$$

$$-\int_{\Omega}\left(\sum_{\substack{p+1\leqslant j\leqslant n,\\ 1\leqslant i\leqslant p}} a_{ij}^{\theta}(t,x)\partial_{x_j}u_\infty \partial_{x_i}v\right)dx = -\int_{\Omega}\sum_{\substack{p+1\leqslant j\leqslant n,\\ 1\leqslant i\leqslant p}}\partial_{x_i}\left(a_{ij}^{\theta}(t,x)\partial_{x_j}u_\infty v\right)dx$$

$$= -\int_{\partial\Omega}\left(\sum_{\substack{p+1\leqslant j\leqslant n,\\ 1\leqslant i\leqslant p}} a_{ij}(t,x)\partial_{x_j}u_\infty v \nu_i\right)dx$$

v is null on the boundary of Ω, we obtain

$$\frac{d^2}{dt^2}\int_\Omega w_\theta v dx + a_\theta\left(t, w_\theta, v\right) = 0 \quad \forall v \in H_0^1(\Omega). \tag{22}$$

where

$$w_\theta \overset{Def}{=} u_\theta - u_\infty,$$
$$a_\theta\left(t, u, v\right) \overset{Def}{=} \int_\Omega \left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(t,x)\partial_{x_j}u\partial_{x_i}v + c(t,x)uv\right) dx \quad \forall u, v \in H^1(\Omega). \tag{23}$$

The idea to have an estimate of w_θ, generally in the hyperbolic problems, on the way by the equality of the energy type, therefore we replace by w'_θ in (22) when we have the regularity of the solutions. But in this case we have

- $w'_\theta(t) \in L^2(\Omega)$ and $v \in H^1(\Omega)$ in (22).
- v in (22) is null on the boundary of Ω.

To avoid these obstacles, the method is based on the idea to approach $u_\theta - u_\infty$ by functions with value in $H^1_0(\Omega)$, which is done by triple regularization.

$$
\begin{array}{ccccc}
w'_\theta \in L^2(0,T;L^2(\Omega)) & \dashrightarrow & \begin{array}{c}\gamma_n * w'_\theta \in \\ L^2(\mathbb{R};H^1(\Omega))\end{array} & & \\
\searrow & & & & \\
\begin{array}{c}(\vartheta w_\theta)' \in \\ L^2(\mathbb{R};L^2(\Omega))\end{array} & \longrightarrow & \begin{array}{c}\gamma_n * (\vartheta w_\theta)' \\ = \gamma'_n * \vartheta w_\theta\end{array} \in L^2(\mathbb{R};H^1(\Omega)) & \longrightarrow & \begin{array}{c}\rho_\epsilon \gamma_n * (\vartheta w_\theta)' \in \\ L^2(\mathbb{R};H^1_0(\Omega))\end{array}
\end{array}
$$

For $\epsilon > 0$, we pose

$$\Delta_\epsilon = \{x \in \Delta,\ d(\partial\Delta, x) > \epsilon\}\,. \tag{24}$$

Let ρ_ϵ be a family of indefinitely derivable functions with compact support in Δ, such that

$$\text{supp}\rho_\epsilon \subset \Delta_{\frac{\epsilon}{2}}, \quad \left(\Delta_\epsilon \subset \Delta_{\frac{\epsilon}{2}}\right),$$

for all x from Δ_ϵ, we have

$$\rho_\epsilon(x) = 1,$$

and for all x from Δ

$$0 \leqslant \rho_\epsilon(x) \leqslant 1. \tag{25}$$

we start with this equality of the energy type.

Theorem 3 *Let u_θ be the solution to (8) and u_∞ be the solution to (9).*

Then for any t

$$\int_{\Delta_{\epsilon/2}\times\omega}\left(\rho_\epsilon w_\theta'\right)^2 dx + b_\epsilon\left(t, w_\theta, w_\theta\right) = \int_{\Delta_{\epsilon/2}\times\omega}\left|\rho_\epsilon\left(u_{\theta,1}-u_1\right)\right|^2 dx +$$
$$-2\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left(\sum_{\substack{1\leqslant j\leqslant n\\ 1\leqslant i\leqslant p}} a_{ij}^\theta(\sigma,x)\partial_{x_j}w_\theta(\sigma)\partial_{x_i}\rho_\epsilon\rho_\epsilon w_\theta'(\sigma)\right)dx d\sigma + \qquad (26)$$
$$b_\epsilon\left(0, u_{\theta,0}-u_0, u_{\theta,0}-u_0\right) + \int_0^t b_\epsilon'\left(\sigma, w_\theta, w_\theta\right)d\sigma + \int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2 c(t,x) w_\theta w_\theta' dx dt$$

where

$$b_\epsilon(t,u,v) \overset{Def}{=} \int_{\Delta_{\epsilon/2}\times\omega} \left(\sum_{1\leqslant i,j\leqslant n} \rho_\epsilon^2 a_{ij}^\theta \partial_{x_j} u \partial_{x_i} v \right) dx,\ \forall u,v \ \in \ H^1(\Omega) \quad (27)$$

$$b'_\epsilon(t,u,v) \overset{Def}{=} \int_{\Delta_\epsilon\times\omega} \left(\sum_{1\leqslant i,j\leqslant n} \rho_\epsilon^2 \left(a_{ij}^\theta\right)' \partial_{x_j} u \partial_{x_i} v \right) dx,\ \forall u,v \ \in \ H^1(\Omega) \quad (28)$$

Proof For any $\delta > 0$, we introduce the function ϑ_δ defined on $\mathbb{R}$, for $\delta > 0$ by

$$\vartheta_\delta = \begin{cases} 1 & \text{on } [\delta, t_0-\delta] \\ 0 & \text{out } [0,t_0] \\ \frac{t}{\delta} & \text{on } [0,\delta] \\ \frac{t_0-t}{\delta} & \text{on } [t_0-\delta, t_0] \end{cases} \quad (29)$$

ϑ_δ being continuous on $\mathbb{R}$, and we pose

$$\vartheta_0 = \begin{cases} 1 & \text{on } [0, t_0] \\ 0 & \text{out } [0, t_0] . \end{cases} \tag{30}$$

We have

$$\vartheta'_\delta = \begin{cases} 0 & \text{on }]\delta, t_0 - \delta[\text{ and out } [0, t_0] \\ \frac{1}{\delta} & \text{on }]0, \delta[\\ \frac{-1}{\delta} & \text{on }]t_0 - \delta, t_0[. \end{cases}$$

Let γ_n be a regularization sequence of even functions defined on $\mathbb{R}$, check

$$\int_{\mathbb{R}} \gamma_n(t) dt = 1. \tag{31}$$

Supposing that the functions a_{ij}^θ and c are defined for any $t \in \mathbb{R}$, with same properties of regularity on $\mathbb{R}$ as on $[0,T]$ (prolongation by continuity), and the same for w_θ (for example prolongation by reflexion). It is clear that

$$\rho_\epsilon^2\gamma_n*\left(\vartheta_\delta w_\theta'\right) = \rho_\epsilon^2\gamma_n'*(\vartheta_\delta w_\theta)-\rho_\epsilon^2\gamma_n*\left(\vartheta_\delta' w_\theta\right) \in L^2\left(-\infty,+\infty;H_0^1(\Omega)\right). \quad (32)$$

The sketch of the proof uses the equality (22), replacing v by $\rho_\epsilon^2\gamma_n * \left(\vartheta_\delta w_\theta'\right)$, so that

$\lozenge$ we leave $\delta \to 0$,
$\lozenge\lozenge$ then $n \to +\infty$

For the first step $\delta \to 0$, we obtain the following lemma.

Lemma 4 *Under the assumptions of the theorem, we obtain for any* t_0 *from* $[0,T]$

$$\int_{-\infty}^{+\infty} b_\epsilon'\left(t, \overline{\rho}_n * \left(\vartheta_0 w_\theta\right), \gamma_n * \left(\vartheta_0 w_\theta\right)\right) dt + 2\int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \gamma_n * \gamma_n * \left(\vartheta_0 w_\theta\right)(0) w_\theta'(0) dx$$

$$+2\int_{\Delta_{\epsilon/2}\times\omega} \gamma_n * \gamma_n * \left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)(0)\rho_\epsilon^2 \partial_{x_i} w_\theta'(0) dx$$

$$-2\int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \gamma_n * \gamma_n * \left(\vartheta_0 w_\theta'\right)(t_0) w_\theta'(t_0) dx$$

$$-2\int_{\Delta_{\epsilon/2}\times\omega} \gamma_n * \gamma_n * \left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)(t_0)\, \rho_\epsilon^2 \partial_{x_i} w_\theta\left(t_0\right) dx + \Upsilon_n$$

$$= 2\int_{-\infty}^{+\infty}\int_{\Delta_{\epsilon/2}\times\omega} \gamma_n * \left(c(.,x)\vartheta_0 w_\theta\right) \rho_\epsilon^2 \left(\gamma_n * \left(\vartheta_0 w_\theta'\right)\right) dx dt +$$

$$+2\int_{-\infty}^{+\infty}\int_{\Delta_{\epsilon/2}\times\omega} \sum_{1\leqslant j\leqslant n,\ 1\leqslant i\leqslant p} \left(\gamma_n * \left(a_{ij}^\theta(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)\right) \partial_{x_i}\rho_\epsilon \rho_\epsilon \left(\overline{\rho}_n * \left(\vartheta_0 w_\theta'\right)\right) dx dt. \tag{33}$$

where

$$\Upsilon_n = \int_{-\infty}^{+\infty} \int_{\Delta_{\epsilon/2}\times\omega} \sum_{1\leqslant i,j\leqslant n} \left(a_{ij}^{\theta}(t,x)\partial_{x_j}\gamma_n * (\vartheta_0 w_\theta) - \gamma_n * \left(a_{ij}^{\theta}(.,x)\partial_{x_j}(\vartheta_0 w_\theta)\right)(t)\right)$$
$$\rho_\epsilon^2 \partial_{x_i}\gamma_n' * (\vartheta_0 w_\theta)\, dxdt$$

The prove is very technic and it is based on these two results ϑ

$$\vartheta_\delta \underset{\delta\to 0}{\longrightarrow} \vartheta_0 \text{ in } L^2(\mathbb{R}),$$
$$\int_{\mathbb{R}} \left|\vartheta_\delta'(t)\right| dt = 2.$$

In this step, we take $n \to +\infty$ in the above formula of the lemma, and we give the convergence by term in (33), we start by Υ_n

$$\Upsilon_n = \int_{-\infty}^{+\infty} \int_{\Delta_{\epsilon/2}\times\omega} \sum_{1\leqslant i,j\leqslant n} a_{ij}^{\theta}(t,x)\partial_{x_j}\gamma_n * (\vartheta_0 w_\theta)\, \rho_\epsilon^2 \left(\partial_{x_i}\gamma_n * (\vartheta_0 w_\theta)\right)' \, dxdt$$
$$- \int_{-\infty}^{+\infty} \int_{\Delta_{\epsilon/2}\times\omega} \sum_{1\leqslant i,j\leqslant n} a_{ij}^{\theta}(t,x)\partial_{x_j} (\vartheta_0 w_\theta)\, \rho_\epsilon^2 \partial_{x_i}\gamma_n * \left(\gamma_n * (\vartheta_0 w_\theta)\right)' \, dxdt. \quad (34)$$

The bilinear forms $b_\epsilon(t,.,.)$ given by (27) are continuous on $H^1(\Omega)$, therefore there exists a family of operators $\mathcal{B}_\epsilon(t) \in \mathcal{L}\left(H^1(\Omega), H^1(\Omega)\right)$, such that

$$b_\epsilon(t,u,v) = ((\mathcal{B}_\epsilon(t)u, v)) \quad \forall u,v \in H^1(\Omega). \quad (35)$$

The application $t \longrightarrow ((\mathcal{B}_\epsilon(t)u, v))$ $\forall u, v \in H^1(\Omega)$, is once continuously differentiable according to (12) . The notation $((,))$ represents the scalar product in $H^1(\Omega)$. Replacing in (34) and with a simple calculation, it comes

$$\Upsilon_n = -\int_{-\infty}^{+\infty} ((\frac{d}{dt}\left[\mathcal{B}_\epsilon(t)\gamma_n * (\vartheta_0 w_\theta) - \gamma_n * \mathcal{B}_\epsilon(t)\,(\vartheta_0 w_\theta)\right], \gamma_n * (\vartheta_0 w_\theta)))dt. \quad (36)$$

Using the Friedrichs lemma 2 and the fact that

$$\int_{\mathbb{R}} \|\gamma_n * (\vartheta_0 w_\theta)\|^2 \, dt \leqslant C \int_{\mathbb{R}} |\vartheta_0|^2 \, dt,$$

C does not depend on n, we deduced that

$$\Upsilon_n \longrightarrow 0.$$

Since γ_n is a regularization sequence, therefore

$$\gamma_n * (\vartheta_0 w_\theta) \underset{n\to+\infty}{\longrightarrow} (\vartheta_0 w_\theta) \quad \text{in } L^2\left(-\infty,+\infty; H^1(\Omega)\right), \tag{37}$$

$$\gamma_n * \left(\vartheta_0 w'_\theta\right) \underset{n\to+\infty}{\longrightarrow} \left(\vartheta_0 w'_\theta\right) \quad \text{in } L^2\left(-\infty,+\infty; L^2(\Omega)\right). \tag{38}$$

Then we obtain the convergence

$$\int_{-\infty}^{+\infty} b'_\epsilon\left(t, \gamma_n * (\vartheta_0 w_\theta), \gamma_n * (\vartheta_0 w_\theta)\right) dt \underset{n\to+\infty}{\longrightarrow} \int_0^{t_0} b'_\epsilon\left(t, w_\theta, w_\theta\right) dt,$$

$$\int_{-\infty}^{+\infty} \int_\Omega \gamma_n * (c(.,x)\vartheta_0 w_\theta)\, \rho_\epsilon^2 \left(\gamma_n * \left(\vartheta_0 w'_\theta\right)\right) dxdt \underset{n\to+\infty}{\longrightarrow} \int_0^{t_0} \int_\Omega \rho_\epsilon^2 c(t,x) w_\theta w'_\theta dxdt$$

and

$$\int_{-\infty}^{+\infty}\int_{\Omega}\sum_{\substack{1\leqslant j\leqslant n\\1\leqslant i\leqslant p}}\left(\gamma_n*\left(a_{ij}^{\theta}(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)\right)\partial_{x_i}\rho_\epsilon\rho_\epsilon\left(\gamma_n*\left(\vartheta_0 w_\theta'\right)\right)dxdt$$

$$\underset{n\rightarrow+\infty}{\longrightarrow}\int_{0}^{t_0}\int_{\Omega}\sum_{\substack{1\leqslant j\leqslant n\\1\leqslant i\leqslant p}}a_{ij}^{\theta}(t,x)\partial_{x_j}w_\theta\partial_{x_i}\rho_\epsilon\rho_\epsilon w_\theta' dxdt.$$

For the remainder of the terms in (33)

$$\int_{\Omega}\gamma_n*\gamma_n*\left(\sum_{1\leqslant i,j\leqslant n}a_{ij}(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)(t_0)\,\rho_\epsilon^2\partial_{x_i}w_\theta\,(t_0)\,dx,$$

and

$$\int_{\Delta_{\epsilon/2}\times\omega}\gamma_n*\gamma_n*\left(\sum_{1\leqslant i,j\leqslant n}a_{ij}^{\theta}(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)(0)\rho_\epsilon^2\partial_{x_i}w_\theta(0)dx,$$

$$\int_{\Omega}\rho_\epsilon^2\gamma_n*\gamma_n*\left(\vartheta_0 w_\theta'(.)\right)w_\theta'(t_0)dx,\ \ \int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\gamma_n*\gamma_n*\left(\vartheta_0 w_\theta'\right)(0)w_\theta'(0)dx.$$

We pose

$$\sigma_n=\gamma_n*\gamma_n,$$

then

$$\int_{-t_0}^{0}\sigma_n(t)dt=\int_{0}^{t_0}\sigma_n(t)dt=\frac{1}{2}. \quad (39)$$

Thus, for the first term, we obtain

$$\Lambda_n \overset{def}{=} 2\int_\Omega \gamma_n * \gamma_n * \left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(.,x)\partial_{x_j}(\vartheta_0 w_\theta)\right)(t_0)\,\rho_\epsilon^2 \partial_{x_i} w_\theta(t_0)\,dx$$

$$- \int_\Omega \sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(t_0,x)\partial_{x_j} w_\theta(t_0)\,\rho_\epsilon^2 \partial_{x_i} w_\theta(t_0)\,dx =$$

$$2\int_0^{t_0} \sigma_n(t)\int_\Omega \left(\sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(.,x)\partial_{x_j} w_\theta\right)(t_0 - t)\,\rho_\epsilon^2 \partial_{x_i} w_\theta(t_0)\,dx$$

$$- 2\int_0^{t_0} \sigma_n(t)\int_\Omega \sum_{1\leqslant i,j\leqslant n} a_{ij}^\theta(t_0,x)\partial_{x_j} w_\theta(t_0)\,\rho_\epsilon^2 \partial_{x_i} w_\theta(t_0)\,dx$$

$$= 2\int_{supp\sigma_n\cap[0,t_0]}\sigma_n(t)\int_\Omega\Bigg(\sum_{1\leqslant i,j\leqslant n}a_{ij}^\theta(t_0-t,x)\partial_{x_j}w_\theta\left(t_0-t\right)$$

$$-\sum_{1\leqslant i,j\leqslant n}a_{ij}^\theta(t_0,x)\partial_{x_j}w_\theta\left(t_0\right)\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)\Bigg)\,dx,$$

from the continuity of a_{ij}^θ and w_θ, we deduce that $\Lambda_n \to 0$, i.e.

$$2\int_\Omega\gamma_n*\gamma_n*\left(\sum_{1\leqslant i,j\leqslant n}a_{ij}^\theta(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)\left(t_0\right)\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)dx$$

$$\underset{n\to+\infty}{\longrightarrow}\int_\Omega\sum_{1\leqslant i,j\leqslant n}a_{ij}^\theta(t_0,x)\partial_{x_j}w_\theta\left(t_0\right)\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)dx.$$

Same way for the other terms, we obtain

$$2\int_{\Omega}\gamma_n*\gamma_n*\left(\sum_{1\leqslant i,j\leqslant n}a^{\theta}_{ij}(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)(0)\,\rho_\epsilon^2\partial_{x_i}w_\theta\,(0)\,dx$$

$$\underset{n\to+\infty}{\longrightarrow}\int_{\Omega}\sum_{1\leqslant i,j\leqslant n}a^{\theta}_{ij}(0,x)\partial_{x_j}\left(u_{\theta,0}-u_0\right)\rho_\epsilon^2\partial_{x_i}\left(u_{\theta,0}-u_0\right)dx.$$

$$\int_{\Omega}\rho_\epsilon^2\gamma_n*\gamma_n*\left(\vartheta_0 w'_\theta(.)\right)(t_0)w'_\theta(t_0)dx\underset{n\to+\infty}{\longrightarrow}\int_{\Omega}\rho_\epsilon^2 w'_\theta(t_0)w'_\theta(t_0)dx$$

$$\int_{\Omega}\rho_\epsilon^2\gamma_n*\gamma_n*\left(\vartheta_0 w'_\theta\right)(0)w'_\theta(0)dx\underset{n\to+\infty}{\longrightarrow}\int_{\Omega}\rho_\epsilon^2\left(u_{\theta,1}-u_1\right)\left(u_{\theta,1}-u_1\right)dx$$

The theorem is proved. ■

ESTIMATION

We estimate the terms of the second member of (26), using the properties of the family of the functions ρ_ϵ and (59), we obtain for the first term

$$\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left(\sum_{\substack{1\leqslant j\leqslant n \\ 1\leqslant i\leqslant p}} a_{ij}^{\theta}(\sigma,x)\partial_{x_j} w_\theta(\sigma)\partial_{x_i}\rho_\epsilon \rho_\epsilon w'_\theta(\sigma) \right) dx d\sigma \leqslant$$

$$C\theta^{1/2+\alpha}\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_1} w_\theta\right| \left|\rho_\epsilon w'_\theta(\sigma)\right| dx d\sigma + C\theta^{\alpha}\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_2} w_\theta\right| \left|\rho_\epsilon w'_\theta(\sigma)\right| dx d\sigma$$

C independent of θ (depends of ϵ). Applying Young inequality $ab \leqslant \mu a^2 + \frac{b^2}{\mu}$, for the last terms, with $\mu = \theta^{1/2}$ and $\mu = 1$, we deduce

$$\left|\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left(\sum_{\substack{1\leqslant j\leqslant n\\ 1\leqslant i\leqslant p}} a_{ij}^{\theta}(\sigma,x)\partial_{x_j} w_\theta(\sigma)\partial_{x_i}\rho_\epsilon \rho_\epsilon w'_\theta(\sigma)\right) dx d\sigma\right| \leqslant$$

$$C\theta^{\alpha}\left(\theta \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_1} w_\theta\right|^2 dx d\sigma + \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_2} w_\theta\right|^2 dx d\sigma\right) \qquad (40)$$

$$+C\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\rho_\epsilon w'_\theta(\sigma)\right|^2 dx d\sigma. \quad (\theta^{\alpha} \leqslant 1)$$

We estimate the remainder terms of (26) with same technics, and we use the hyperbolicity condition, then (26) gives

$$\int_{\Delta_{\epsilon/2}\times\omega}\left(\rho_\epsilon w_\theta'\right)^2 dx+\theta\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_1}w_\theta\right|^2 dx+\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_2}w_\theta\right|^2 dx\leqslant$$
$$C\theta^\alpha\left(\theta\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1}w_\theta\right|^2 dxd\sigma+\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2}w_\theta\right|^2 dxd\sigma\right)$$
$$+C\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left|\rho_\epsilon w_\theta'(\sigma)\right|^2 dxd\sigma+\int_{\Delta_{\epsilon/2}\times\omega}\left|u_{\theta,1}-u_1\right|^2 dx+C\theta^{2\alpha}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1}u_{\theta,0}\right|^2 dx \tag{41}$$
$$+C\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2}\left(u_{\theta,0}-u_0\right)\right|^2 dx+C\theta\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_1}w_\theta\right|^2 dxd\sigma+$$
$$C\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_2}w_\theta\right|^2 dxd\sigma+C\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}|\rho_\epsilon w_\theta|^2 dxd\sigma.$$

Applying the Poincaré inequality on $\int_{\omega} |\rho_\epsilon w_\theta|^2 \, dX_2$, we obtain

$$\int_{\Delta_{\epsilon/2}\times\omega} \left(\rho_\epsilon w'_\theta\right)^2 dx + \theta \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \left|\nabla_{X_1} w_\theta\right|^2 dx + \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \left|\nabla_{X_2} w_\theta\right|^2 dx \leqslant$$

$$C\theta^\alpha \left(\theta \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_1} w_\theta\right|^2 dx d\sigma + \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_2} w_\theta\right|^2 dx d\sigma\right) +$$

$$C \int_0^t \left(\int_{\Delta_{\epsilon/2}\times\omega} \left|\rho_\epsilon w'_\theta(\sigma)\right|^2 dx + \theta \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \left|\nabla_{X_1} w_\theta\right|^2 dx + \right.$$

$$\left. \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \left|\nabla_{X_2} w_\theta\right|^2 dx\right) d\sigma + \int_{\Delta_{\epsilon/2}\times\omega} \left|u_{\theta,1} - u_1\right|^2 dx +$$

$$C\theta^{2\alpha} \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_1} u_{\theta,0}\right|^2 dx + C \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_2} \left(u_{\theta,0} - u_0\right)\right|^2 dx,$$

since $\rho_\epsilon w_\theta(t, X_1, .)$ is null on the boundary of on ω for a.e. (t, X_1). Using the Gronwall inequality, we deduce that

$$\int_{\Delta_{\epsilon/2}\times\omega} \left(\rho_\epsilon w'_\theta(t)\right)^2 dx + \theta \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \left|\nabla_{X_1} w_\theta(t)\right|^2 dx + \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 \left|\nabla_{X_2} w_\theta(t)\right|^2 dx \leqslant$$
$$C\theta^\alpha \left(\theta \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_1} w_\theta\right|^2 dx d\sigma + \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_2} w_\theta\right|^2 dx d\sigma\right)$$
$$+ \int_{\Delta_{\epsilon/2}\times\omega} \left|u_{\theta,1} - u_1\right|^2 dx + C\theta^{2\alpha} \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_1} u_{\theta,0}\right|^2 dx$$
$$+ C \int_{\Delta_{\epsilon/2}\times\omega} \left|\nabla_{X_2}\left(u_{\theta,0} - u_0\right)\right|^2 dx,$$

since $\rho_\epsilon = 1$ on Δ_ϵ, we obtain $\left(\Delta_\epsilon \subset \Delta_{\epsilon/2}\right)$

$$\int_{\Delta_\epsilon\times\omega}\left(w_\theta'(t)\right)^2 dx + \theta\int_{\Delta_\epsilon\times\omega}\left|\nabla_{X_1} w_\theta(t)\right|^2 dx + \int_{\Delta_\epsilon\times\omega}\left|\nabla_{X_2} w_\theta(t)\right|^2 dx \leqslant$$
$$C\theta^\alpha\left(\theta\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1} w_\theta\right|^2 dx d\sigma + \int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2} w_\theta\right|^2 dx d\sigma\right) +$$
$$\int_{\Delta_{\epsilon/2}\times\omega}\left|u_{\theta,1} - u_1\right|^2 dx + C\theta^{2\alpha}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1} u_{\theta,0}\right|^2 dx$$
$$+ C\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2}\left(u_{\theta,0} - u_0\right)\right|^2 dx,$$

the inequality is checked for any t of $(0, T)$, then, by (16) and (19), this implies that

$$\sup_{0\leqslant t\leqslant T}\int_{\Delta_\epsilon\times\omega}\left(w_\theta'(t)\right)^2dx+\theta\sup_{0\leqslant t\leqslant T}\int_{\Delta_\epsilon\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx+$$
$$\sup_{0\leqslant t\leqslant T}\int_{\Delta_\epsilon\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\leqslant C\theta^\alpha\Bigg(\theta\sup_{0\leqslant t\leqslant T}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx+$$
$$\sup_{0\leqslant t\leqslant T}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\Bigg)+\int_{\Delta_{\epsilon/2}\times\omega}\left|u_{\theta,1}-u_1\right|^2dx+$$
$$C\theta^{2\alpha}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1}u_{\theta,0}\right|^2dx+C\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2}\left(u_{\theta,0}-u_0\right)\right|^2dx. \tag{42}$$

We take $\epsilon = \frac{\varepsilon}{2^k}$ for $k = \widehat{0, \tau - 1}$, in (42) and we pose

$$\chi_{\epsilon}^{\theta} = \int_{\Delta_{\epsilon/2} \times \omega} \left| (u_{\theta,1} - u_1) \right|^2 dx + C\theta^{2\alpha} \int_{\Delta_{\epsilon/2} \times \omega} \left| \nabla_{X_1} u_{\theta,0} \right|^2 dx + c \int_{\Delta_{\epsilon/2} \times \omega} \left| \nabla_{X_2} \left(u_{\theta,0} - u_0 \right) \right|^2 dx,$$

thus, one can write

$$\theta \sup_{0 \leqslant t \leqslant T} \int_{\Delta_{\frac{\varepsilon}{2^k}} \times \omega} \left| \nabla_{X_1} w_\theta (t) \right|^2 dx + \sup_{0 \leqslant t \leqslant T} \int_{\Delta_{\frac{\varepsilon}{2^k}} \times \omega} \left| \nabla_{X_2} w_\theta (t) \right|^2 dx \leqslant$$

$$C\theta^{\alpha} \left(\theta \sup_{0 \leqslant t \leqslant T} \int_{\Delta_{\frac{\varepsilon}{2^{k+1}}} \times \omega} \left| \nabla_{X_1} w_\theta (t) \right|^2 dx + \sup_{0 \leqslant t \leqslant T} \int_{\Delta_{\frac{\varepsilon}{2^{k+1}}} \times \omega} \left| \nabla_{X_2} w_\theta (t) \right|^2 dx \right) + \chi_{\frac{\varepsilon}{2^k}}^{\theta},$$

if we vary k from 0 to $\tau-1$, we obtain

$$\sup_{0\leqslant t\leqslant T}\int_{\Delta_\epsilon\times\omega}\left(w'_\theta(t)\right)^2dx+\theta\sup_{0\leqslant t\leqslant T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx+$$
$$\sup_{0\leqslant t\leqslant T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\leqslant C\theta^\alpha\left(\theta\sup_{0\leqslant t\leqslant T}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx+\right.$$
$$\left.\sup_{0\leqslant t\leqslant T}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\right)+\chi_\varepsilon^\theta\leqslant\ldots\leqslant$$
$$C\theta^{\tau\alpha}\left(\theta\sup_{0\leqslant t\leqslant T}\int_{\Delta_{\frac{\varepsilon}{2^\tau}}\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx+\sup_{0\leqslant t\leqslant T}\int_{\Delta_{\frac{\varepsilon}{2^\tau}}\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\right)+ \tag{43}$$
$$+C(\chi_\varepsilon^\theta+\theta^\alpha\chi_{\frac{\varepsilon}{2}}^\theta+\ldots+\theta^{(\tau-1)\alpha}\chi_{\frac{\varepsilon}{2^{\tau-1}}}^\theta),$$

Thus, the estimate of $u_\theta - u_\infty$ depends on the proximity on the initial conditions, and the estimate of the quantity

$$\sup_{0 \leqslant t \leqslant T} \int_{\Delta_{\frac{\varepsilon}{2^T}} \times \omega} \left| \nabla \left(u_\theta - u_\infty \right) (t) \right|^2 dx. \tag{44}$$

The following lemma gives a estimation of u_θ.

Lemma 5 *Under the preceding conditions, we have*

$$\sup_{0\leqslant t\leqslant T}\int_{\Delta\times\omega}\left|u_\theta'(t)\right|^2dx \leqslant C\left(\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx+1\right),$$

$$\sup_{0\leqslant t\leqslant T}\int_{\Delta\times\omega}\left|\nabla_{X_2}u_\theta(t)\right|^2dx \leqslant C\left(\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx+1\right),$$

$$\sup_{0\leqslant t\leqslant T}\int_{\Delta\times\omega}\left|\nabla_{X_1}u_\theta(t)\right|^2dx \leqslant \frac{C}{\theta}\left(\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx+1\right),$$

where C independent of θ.

To show this lemma one needs the following lemma.

Lemma 6 *Let u_θ be solution of* (8) , *then*

$$\frac{1}{2}\int_{\Delta\times\omega}\left|u'_\theta(t)\right|^2 dx+\frac{1}{2}\int_{\Delta\times\omega}\left(\sum_{1\leqslant i,j\leqslant n} a^\theta_{ij}(\sigma,x)\partial_{x_j}u_\theta(t)\,\partial_{x_i}u_\theta(t)\right)dx$$
$$=\frac{1}{2}\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2 dx+\frac{1}{2}\int_{\Delta\times\omega}\left(\sum_{1\leqslant i,j\leqslant n} a^\theta_{ij}(0,x)\partial_{x_j}u_{\theta,0}\partial_{x_i}u_{\theta,0}\right)dx$$
$$-\int_0^t\int_{\Delta\times\omega}\left(\sum_{1\leqslant i,j\leqslant n}\left(a^\theta_{ij}\right)'(\sigma,x)\partial_{x_j}u_\theta(\sigma)\,\partial_{x_i}u_\theta(\sigma)\right)dxd\sigma \tag{45}$$
$$-\int_0^t\int_{\Delta\times\omega}c(t,x)u_\theta(\sigma)\,u'_\theta(\sigma)\,dxd\sigma+\int_0^t\int_{\Delta\times\omega}f(\sigma)\,u'_\theta(\sigma)\,dxd\sigma,$$

We find the proof detailed of the lemma 6 in the book of J.L Lions and E. Magenes. For the proof of lemma 5, and to give the estimation of u_θ, one uses the same technics as before.

Using the lemma 5, (43) implies

$$\sup_{0\leqslant t\leqslant T}\int_{\Delta_\epsilon\times\omega}\left(w'_\theta\left(t\right)\right)^2dx+\theta\sup_{0\leqslant t\leqslant T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_1}w_\theta\left(t\right)\right|^2dx+\sup_{0\leqslant t\leqslant T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_2}w_\theta\left(t\right)\right|^2dx\leqslant C\left(\theta^{\tau\alpha}+\Xi_{\theta,\tau}\right),$$

where C is a constant independent of θ, and $\Xi_{\theta,\tau}$ is given by

$$\Xi_{\theta,\tau}=\theta^{\tau\alpha}\left(\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx\right)+\chi^\theta_\varepsilon+\theta^\alpha\chi^\theta_{\frac{\varepsilon}{2}}+\ldots+\theta^{(\tau-1)\alpha}\chi^\theta_{\frac{\varepsilon}{2^{\tau-1}}}. \quad (46)$$

Then, for $r > 0$, we take τ such that $\tau\alpha > r$, thus, it is enough to take

$$\tau = \left[\frac{r}{\alpha}\right] + 1. \tag{47}$$

to write

$$\theta \sup_{0\leqslant t\leqslant T} \int_{\Delta_\varepsilon\times\omega} \left|\nabla_{X_1} w_\theta (t)\right|^2 dx + \sup_{0\leqslant t\leqslant T} \int_{\Delta_\varepsilon\times\omega} \left|\nabla_{X_2} w_\theta (t)\right|^2 dx \leqslant C \left(\theta^r + \Xi_{\theta,r}\right),$$

$$\sup_{0\leqslant t\leqslant T} \int_{\Delta_{2\varepsilon}\times\omega} \left(w'_\theta (t)\right)^2 dx \leqslant C \left(\theta^r + \Xi_{\theta,r}\right),$$

with $\Xi_{\theta,r} = \Xi_{\theta,\tau}$, τ is given by (47). Thus, we proved a basic theorem to study the asymptotic behavior.

Theorem 7 *Under the conditions(10)-(12), (13) and (59)* $r > 0$, $\varepsilon > 0$, *there exists a constant* $C > 0$ *independent of* θ, *such that*

$$\sup_{0 \leqslant t \leqslant T} \int_{\Delta_\varepsilon \times \omega} \left| \nabla_{X_1} w_\theta (t) \right|^2 dx \leqslant \frac{C}{\theta} \left(\theta^r + \Xi_{\theta,r} \right), \tag{48}$$

$$\sup_{0 \leqslant t \leqslant T} \int_{\Delta_\varepsilon \times \omega} \left| \nabla_{X_2} w_\theta (t) \right|^2 dx \leqslant C \left(\theta^r + \Xi_{\theta,r} \right), \tag{49}$$

$$\sup_{0 \leqslant t \leqslant T} \int_{\Delta_{2\varepsilon} \times \omega} \left(w'_\theta (t) \right)^2 dx \leqslant C \left(\theta^r + \Xi_{\theta,r} \right), \tag{50}$$

where u_θ *and* u_∞ *are solutions of (8) and (9) respectively, and* $\Xi_{\theta,r} = \Xi_{\theta,\tau}$, *is given by (47).*

STUDY OF THE ASYMPTOTIC BEHAVIOR

Let Φ be an open set of Ω with boundary disjoined of $\partial\Delta \times \omega$, we take $\varepsilon = dist(\Phi, \partial\Delta \times \omega)$, thus, we have $\Phi \subset \Delta_\varepsilon \times \omega$. In other way, it is clear, according to the results (48), (49) and (50), that the initial conditions play a sufficient role to estimate w_θ, therefore if we suppose

$$\chi^{\theta}_{\frac{\varepsilon}{2^k}} = O(\theta^{r-k\alpha}) \quad k = \widehat{0.\tau - 1}, \tag{51}$$

$$\int_{\Delta\times\omega} \left|u_{\theta,1}\right|^2 dx + \int_{\Delta\times\omega} \left|\nabla u_{\theta,0}\right|^2 dx = O(1) \tag{52}$$

τ is given by (47). The result of convergence is given.

Corollary 8 *Let* Φ *be an open set of* Ω *with boundary disjoined of* $\partial\Delta\times\omega$, *we suppose that (51), (52) and the conditions in theorem 7 are checked, then for any* $r' = r - \alpha > 0$, u_θ *goes to* u_∞ *in* $W^\infty(0, T; H^1(\Phi), L^2(\Phi))$, *and there exists a constant* $C > 0$ *independent of* θ, *such that the estimates*

$$\sup_{0\leqslant t\leqslant T} \left|(u_\theta - u_\infty)'(t)\right|_{L^2(\Phi)} \leqslant C\theta^{r'} \tag{53}$$

$$\sup_{0\leqslant t\leqslant T} \|(u_\theta - u_\infty)(t)\|^2_{H^1(\Phi)} \leqslant C\theta^{r'} \tag{54}$$

are verified for any $\theta > 0$.

Proof *The assumptions (51) and (52) implies that*

$$\Xi_{\theta,r} = O\left(\frac{1}{\ell^r}\right),$$

thus, if we use (48), (49) and (50), the corollary is proved. ■

Remark 9 *According to the definition of* Φ *in the precedent corollary, we can give estimates (53) and (54) in the neighborhood of any point of* Ω*, and we can also give them for all compact in* Ω.

NECESSARY CONDITIONS

The conditions of the type (51) and (52), are they necessary for convergence? For the space $W^{\infty}(0,T;H^1(\Delta_\varepsilon\times\omega),L^2(\Delta_\varepsilon\times\omega))$, we have

$$\sup_{0\leqslant t\leqslant T}\left|(u_\theta-u_\infty)'(t)\right|_{L^2(\Delta_\varepsilon\times\omega)}+\sup_{0\leqslant t\leqslant T}\|(u_\theta-u_\infty)(t)\|_{H^1(\Delta_\varepsilon\times\omega)}\geqslant$$
$$\int_{\Delta_\varepsilon\times\omega}\left|(u_{\theta,1}-u_1)\right|^2dx+\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_1}u_{\theta,0}\right|^2dx+c\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_2}\left(u_{\theta,0}-u_0\right)\right|^2dx \tag{55}$$

what justifies the need for such conditions. But this inequality does not show the need for the norm of $W(0,T;H^1(\Delta_\varepsilon\times\omega),L^2(\Delta_\varepsilon\times\omega))$.

Then, we strart by (26), and using (13) and (59), we obtain

$$\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|u_{\theta,1}-u_1\right|^2dx+\theta\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_1}u_{\theta,0}\right|^2dx$$
$$+\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_2}\left(u_{\theta,0}-u_0\right)\right|^2dx\leqslant C\left(H'(t)+H(t)\right) \tag{56}$$

where

$$H(t)=\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left(w_\theta'\right)^2dxd\sigma+C\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla w_\theta\right|^2dxd\sigma.$$

Multiplying (56) by e^t and integrating from 0 to T, and using

$\left\{e^t\left(H'(t)+H(t)\right)=\left(e^tH(t)\right)'\right\}$ we obtain

$$\int_{\Delta_\varepsilon\times\omega}\rho_\epsilon^2\left|u_{\theta,1}-u_1\right|^2dx+\theta\int_{\Delta_\varepsilon\times\omega}\rho_\epsilon^2\left|\nabla_{X_1}u_{\theta,0}\right|^2dx+$$
$$\int_{\Delta_\varepsilon\times\omega}\rho_\epsilon^2\left|\nabla_{X_2}\left(u_{\theta,0}-u_0\right)\right|^2dx\leqslant C\int_0^T\int_{\Delta_\varepsilon\times\omega}\left(w_\theta'\right)^2dxd\sigma+C\int_0^T\int_{\Delta_\varepsilon\times\omega}\left|\nabla w_\theta\right|^2dxd\sigma.$$

where $\epsilon=2\varepsilon$. For any $\varepsilon'<\varepsilon$, we can choose $\rho_{2\varepsilon}$ such that $\rho_{2\varepsilon}>0$ on $\overline{\Delta}_{\varepsilon+\varepsilon'}$, then, if we pose $c=\min_{\overline{\Delta}_{\varepsilon+\varepsilon'}}\rho_{2\varepsilon}$, we obtain

$$\int_{\Delta_{\varepsilon+\varepsilon'}\times\omega} \left|u_{\theta,1} - u_1\right|^2 dx + \theta \int_{\Delta_{\varepsilon+\varepsilon'}\times\omega} \left|\nabla_{X_1} u_{\theta,0}\right|^2 dx +$$

$$\int_{\Delta_{\varepsilon+\varepsilon'}\times\omega} \left|\nabla_{X_2}\left(u_{\theta,0} - u_0\right)\right|^2 dx \leqslant C \int_0^T \int_{\Delta_\varepsilon\times\omega} \left(w_\theta'\right)^2 dx d\sigma$$

$$+ C \int_0^T \int_{\Delta_\varepsilon\times\omega} \left|\nabla w_\theta\right|^2 dx d\sigma.$$

Thus, we can state the theorem,

Theorem 10 *A necessary condition to have convergence* $u_\theta \longrightarrow u_\infty$

- *in* $W^{\infty}(0,T;H^1(\Delta_\varepsilon \times \omega), L^2(\Delta_\varepsilon \times \omega))$ *is*

$$\begin{array}{lll} u_{\theta,0} & \longrightarrow & u_0, \quad \text{in } H^1(\Delta_\varepsilon \times \omega) \\ u_{\theta,1} & \longrightarrow & u_1 \quad \text{in } L^2(\Delta_\varepsilon \times \omega)\ . \end{array}$$

-*in* $W(0,T;H^1(\Delta_\varepsilon \times \omega), L^2(\Delta_\varepsilon \times \omega))$ *is*

$$\nabla_{X_2} u_{\theta,0} \longrightarrow \nabla_{X_2} u_0,\ u_{\theta,1} \longrightarrow u_1 \quad \text{in } L^2(\Delta' \times \omega), \tag{57}$$

$$\int_{\Delta' \times \omega} \left|\nabla_{X_1} u_{\theta,0}\right|^2 dx = o(\frac{1}{\theta}), \tag{58}$$

for any compact Δ' *of* Δ_ε.

For a positive number α with, $0 < \alpha \leqslant \frac{1}{2}$

	$j = 1$ … $j = p$	$j = p+1$ … $j = n$
$i = 1$ … $i = p$	$1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant p$ $a_{ij}^{\theta}(t, X_1, X_2)$ $\left\|a_{ij}^{\theta}(t,x)\right\| \leqslant C\theta^{1/2+\alpha},$ $\left\|\partial_t a_{ij}^{\theta}(t,x)\right\| \leqslant C\theta$ $\theta,\ t,\ X_1,\ X_2$	$1 \leqslant i \leqslant p,\ p+1 \leqslant j \leqslant n$ $a_{ij}^{\theta}(t, X_1, X_2) = a_{ij}^{\theta}(t, X_1)$ $\left\|a_{ij}^{\theta}(t,x)\right\| \leqslant C\theta^{\alpha},$ $\left\|\partial_t a_{ij}^{\theta}(t,x)\right\| \leqslant C\theta^{1/2}$ $\theta,\ t,\ X_2$
$i = p+1$ … $i = n$	$p+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant p$ $a_{ij}^{\theta}(t, X_1, X_2) = a_{ij}^{\theta}(t, X_1)$ $\left\|a_{ij}^{\theta}(t,x)\right\| \leqslant C\theta^{\alpha}$ $\left\|\partial_t a_{ij}^{\theta}(t,x)\right\| \leqslant C\theta^{1/2}$ $\theta,\ t,\ X_2$	$p+1 \leqslant i \leqslant n,\ p+1 \leqslant j \leqslant n$ $a_{ij}^{\theta}(t, X_1, X_2) = a_{ij}(t, X_2)$ $t,\ X_2$

(59)