Journées de Metz 2009
University of Metz, France.

Anisotropic singular perturbations method for solving some integro-differential problems

Senoussi Guesmia
Angewandte Mathematik, Universität Zürich
E-mail: senoussi.guesmia@math.uzh.ch

This is joint work with Prof. Michel Chipot

WHAT HAPPENS IN THE LINEAR CASE

Let $\Omega = (0,1) \times (0,1)$, $\varepsilon > 0$, $x = (x_1, x_2)$

$$\begin{cases} -\varepsilon^2 \partial^2_{x_1} u_\varepsilon (x_1, x_2) - \partial^2_{x_2} u_\varepsilon (x_1, x_2) = f(x_1, x_2) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

What happens when $\varepsilon \longrightarrow 0$?

The natural limit is u_0 solution to

$$\begin{cases} -\partial^2_{x_2} u_0 (x_1, \cdot) = f(x_1, \cdot) & \text{in } \omega_2 = (0,1), \\ u_0 (x_1, \cdot) = 0 & \text{on } \partial\omega_2. \end{cases}$$

Convergence results

Let $\Omega_a = (a, 1-a) \times (0,1)$, $a > 0$.

Diagonal structure $\longrightarrow A_\varepsilon = \begin{pmatrix} a_{11} = \varepsilon^2 & a_{12} = 0 \\ a_{21} = 0 & a_{22} = 1 \end{pmatrix}$

Theorem 1 *We have*

$$\left. \begin{array}{c} |u_\varepsilon - u_0|_{L^2(\Omega_a)}, \quad |\partial_{x_2}(u_\varepsilon - u_0)|_{L^2(\Omega_a)} = o(\varepsilon), \\ \partial_{x_1} u_\varepsilon \to \partial_{x_1} u_0 \ \ \textit{in } L^2(\Omega_a) \end{array} \right\} \Longrightarrow u_\varepsilon \to u_0 \ \ \textit{in } H^1(\Omega_a)$$

If f **is independent of** $\mathrm{x_1}$ *then* $\exists \alpha, C > 0$

$$|u_\varepsilon - u_0|_{H^1(\Omega_a)} \le C e^{-\frac{\alpha}{\varepsilon}}$$

Non-diagonal structure $\longrightarrow A_\varepsilon = \begin{pmatrix} a_{11} = \varepsilon^2 & a_{12} = \varepsilon \\ a_{21} = \varepsilon & a_{22} = 1 \end{pmatrix}$

Theorem 2 *We have*

$$|u_\varepsilon - u_0|_{L^2(\Omega_a)}, \quad |\partial_{x_2} u_\varepsilon - \partial_{x_2} u_0|_{L^2(\Omega_a)} = O(\varepsilon),$$
$$\partial_{x_1} u_\varepsilon \rightharpoonup \partial_{x_1} u_0 \quad \textit{weakly in } L^2(\Omega_a).$$

Remark 3 *We have*

$$\frac{1}{\varepsilon}(u_\varepsilon - u_0) \rightharpoonup 0 \quad \textit{in} \quad L^2(\Omega_a) \Longleftrightarrow |\partial_{x_2}(u_\varepsilon - u_0)|_{L^2(\Omega_a)} = o(\varepsilon), \ \forall a > 0$$
$$\Updownarrow$$
$$\int_\Omega a_{12} \partial_{x_2} u_0 \cdot \partial_{x_1} v \, dx + \int_\Omega a_{21} \partial_{x_1} u_0 \cdot \partial_{x_2} v \, dx = 0 \quad \forall v \in H_0^1(\Omega).$$

In the whole domain

Theorem 4 *We have*

$$u_\varepsilon \longrightarrow u_0, \quad \partial_{x_2} u_\varepsilon \longrightarrow \partial_{x_2} u_0, \quad \varepsilon \partial_{x_1} u_\varepsilon \longrightarrow 0 \quad in \quad L^2(\Omega). \quad (1)$$

If $$f = f(x_2) \neq 0 \implies u_0 = u_0(x_2) \notin H_0^1(\Omega)$$

$$\Downarrow$$

$$u_\varepsilon \nrightarrow u_0 \quad \textit{in} \quad H^1(\Omega)$$

REFERENCES

M. Chipot, “Elliptic Equations: An Introductory Course," Birkhäuser, 2009.

M. Chipot and S. Guesmia, On the asymptotic behavior of elliptic, anisotropic singular perturbations problems, Commun. Pure Appl. Anal. 8(1), (2009), 179–193.

M. Chipot and S. Guesmia, Correctors for some asymptotic problems, (In press).

PHYSICAL MOTIVATION

We consider the linear transport equation for the neutron angular flux $\psi\left(r,\hat{\Omega}\right)$

$$\hat{\Omega}\cdot\nabla_r\psi\left(r,\hat{\Omega}\right)+\sigma\left(r\right)\psi\left(r,\hat{\Omega}\right)-\sigma_s\left(r\right)\phi\left(r\right)=s\left(r,\hat{\Omega}\right) \tag{2}$$

coupled with some boundary conditions.
$\hat{\Omega}$: the traveling direction of a neutron,
r : the spatial variable.
The scalar flux $\phi\left(r\right)$ is given by

$$\phi\left(r\right)=\int_{\hat{\Omega}}\psi\left(r,\hat{\Omega}\right)d\hat{\Omega}.$$

Vladimirov method

To derive the even-parity transport equation let us decompose $\psi\left(r,\hat{\Omega}\right)$ as

$$\psi\left(r,\hat{\Omega}\right)=\psi^{+}\left(r,\hat{\Omega}\right)+\psi^{-}\left(r,\hat{\Omega}\right),$$

where

$$\psi^{+}\left(r,\hat{\Omega}\right)=\frac{1}{2}\left(\psi\left(r,\hat{\Omega}\right)+\psi\left(r,-\hat{\Omega}\right)\right),\quad \psi^{-}\left(r,\hat{\Omega}\right)=\frac{1}{2}\left(\psi\left(r,\hat{\Omega}\right)-\psi\left(r,-\hat{\Omega}\right)\right).$$

Then we have

$$\phi\left(r\right)=\int_{\hat{\Omega}}\psi\left(r,\hat{\Omega}\right)d\hat{\Omega}=\int_{\hat{\Omega}}\psi^{+}\left(r,\hat{\Omega}\right)d\hat{\Omega}.$$

Rewriting (2) for $-\hat{\Omega}$ we derive

$$-\hat{\Omega}\cdot\nabla_{r}\psi\left(r,-\hat{\Omega}\right)+\sigma\left(r\right)\psi\left(r,-\hat{\Omega}\right)-\sigma_{s}\left(r\right)\phi\left(r\right)=s\left(r,-\hat{\Omega}\right). \qquad (3)$$

Summing (2) and (3) term by term, we get

$$\hat{\Omega}\cdot\nabla_{r}\psi^{-}\left(r,\hat{\Omega}\right)+\sigma\left(r\right)\psi^{+}\left(r,\hat{\Omega}\right)-\sigma_{s}\left(r\right)\phi\left(r\right)=s^{+}\left(r,\hat{\Omega}\right), \qquad (4)$$

then subtracting them leads to

$$\hat{\Omega}\cdot\nabla_{r}\psi^{+}\left(r,\hat{\Omega}\right)+\sigma\left(r\right)\psi^{-}\left(r,\hat{\Omega}\right)=s^{-}\left(r,\hat{\Omega}\right), \qquad (5)$$

whence

$$\psi^{-}\left(r,\hat{\Omega}\right)=\frac{1}{\sigma\left(r\right)}\left[s^{-}\left(r,\hat{\Omega}\right)-\hat{\Omega}\cdot\nabla_{r}\psi^{+}\left(r,\hat{\Omega}\right)\right],$$

where $\sigma\left(r\right)$ is assumed different from 0. Dropping the term $\psi^{-}\left(r,\hat{\Omega}\right)$ in (4) yields

$$-\hat{\Omega}\cdot\nabla_{r}\left(\frac{1}{\sigma\left(r\right)}\hat{\Omega}\cdot\nabla_{r}\psi^{+}\left(r,\hat{\Omega}\right)\right)+\sigma\left(r\right)\psi^{+}\left(r,\hat{\Omega}\right)=$$
$$\sigma_{s}\left(r\right)\int_{\hat{\Omega}}\psi^{+}\left(r,\hat{\Omega}\right)d\hat{\Omega}+s^{+}\left(r,\hat{\Omega}\right)-\hat{\Omega}\cdot\nabla_{r}\frac{s^{-}\left(r,\hat{\Omega}\right)}{\sigma\left(r\right)}.$$

REFERENCES

E. Lewis, W. Miller Jr., "Computational Methods of Neutron Transport," John Wiley & Sons, London, 1984.

V. S. Vladimirov, "Mathematical problems in one-speed particle transport theory," Trudy Mat. Inst. Akad. Nauk SSSR, 61 (1961).

Motived by the model above we consider in the following some nonlocal problems.

POSITION OF THE PROBLEM

$\Omega = \omega_1 \times \omega_2$ such that $\omega_1 \subset \mathbb{R}^m$ and $\omega_2 \subset \mathbb{R}^n$

$$x = (X_1, X_2) \in \mathbb{R}^{m+n}, \qquad X_1 = (x_1, \ldots, x_m) \quad \text{and} \quad X_2 = \left(x_1', \ldots, x_n'\right).$$

With this notation we set

$$\nabla u = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix} = \begin{pmatrix} (\partial_{x_1} u, \ldots, \partial_{x_m} u)^T \\ (\partial_{x_1'} u, \ldots, \partial_{x_n'} u)^T \end{pmatrix}.$$

Let us consider the integro-differential problem defined by

$$\begin{cases} -\nabla_{X_2}\left(A \nabla_{X_2} u\right) + \chi u = a(l\,(u)) \quad \text{in} \quad \Omega, \\ u(X_1, \cdot) = 0 \quad \text{on} \quad \partial\omega_2 \quad a.e.\ X_1 \in \omega_1. \end{cases} \tag{6}$$

For some function $h \in L^{\infty}(\omega_1 \times \Omega)$,

$$l(u) = \int_{\omega_1} h\left(X_1, X_1', X_2\right) u\left(X_1', X_2\right) dX_1'. \tag{7}$$

Let

$$\begin{aligned} A &= (a_{ij}(x)) \quad n \times n - matrix, \quad a_{ij} \in L^{\infty}(\Omega) \quad \forall\, i, j = 1, \ldots, n, & (8)\\ A\xi \cdot \xi &\geq \lambda|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n, \ \text{ a.e. } x \in \Omega, \quad (\lambda > 0), & (9) \end{aligned}$$

$a \in C(\mathbb{R})$ a continuous function satisfies

$$a(r) = O(r) \quad \text{when } |r| \to \infty, \tag{10}$$

$\chi > 0$ large enough. We set

$$V := \left\{ v \in L^2(\Omega) \,|\partial_{x_i'} v \in L^2(\Omega),\ i = 1, \cdots, n \text{ and } v \in H_0^1(\omega_2) \text{ a.e. } X_1 \in \omega_1 \right\}.$$

Let us equipped V with the norm

$$|u|_V^2 = \left|\nabla_{X_2} u\right|_{L^2(\Omega)}^2 + |u|_{L^2(\Omega)}^2 .$$

$u_0 \in V$ is a weak solution of the problem (6) if we have

$$\int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} v + \chi u_0 v dx = \int_\Omega a(l\,(u_0)) v dx, \qquad \forall v \in V.$$

Note that

$$V \hookrightarrow L^2\,(\Omega) \text{ is not compact.}$$

To show the existence of the solution u_0, we introduce the following anisotropic singular perturbation problem ($\boldsymbol{\varepsilon \to 0}$)

$$\begin{cases} -\varepsilon^2 \Delta_{X_1} u_\varepsilon - \nabla_{X_2}\left(A\nabla_{X_2} u_\varepsilon\right) + \chi u_\varepsilon = a(l\,(u_\varepsilon)) \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{11}$$

The plan of the study is as follows.

$i)$ Existence of the solutions of the problem (11) when a **is bounded**.

$ii)$ Existence of the solutions of the problem (11) when a **satisfies (10)**.

$iii)$ Asymptotic behavior of u_ε solution to (11) when $\boldsymbol{\varepsilon \to 0}$. $(\mathbf{u}_\varepsilon \to \mathbf{u}_0)$.

Step 1. Nonlocal elliptic problems with bounded data

We assume that

$$a : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{is bounded.} \tag{12}$$

Theorem 5 *Under the assumptions above, there exists at least one weak solution to the problem (11).*

Proof. ε is fixed. For $w \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be solution to the linear elliptic problem

$$\begin{cases} -\varepsilon^2 \Delta_{X_1} u - \nabla_{X_2}\left(A \nabla_{X_2} u\right) = a(l(w)) \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{13}$$

Since a is bounded the mapping

$$T: \begin{array}{lcl} L^2(\Omega) & \rightarrow & L^2(\Omega) \\ w & \rightarrow & u = T(w) \end{array} \tag{14}$$

is continuous and $\exists\ B \subset H_0^1(\Omega)$ (bounded in $H_0^1(\Omega)$) such that

$$T(B) \subset B.$$

By the Schauder fixed point theorem $\exists\ \ u \in B$

$$T(u) = u \Longrightarrow\ u \text{ is a solution of (11).}$$

Step 2. More general nonlocal elliptic problems

a satisfies (10). Then we have

Theorem 6 *Under the assumptions above, there exists at least one weak solution to the problem (11).*

Proof. ε is fixed. Let us introduce a sequence of functions $\theta_n : \mathbb{R} \to \mathbb{R}$ defined as

$$\theta_n(r) = \begin{cases} r \text{ if } |r| \leq n \\ n \text{ if } r \geq n, \\ -n \text{ if } r \leq -n, \end{cases}$$

then a_n defined as

$$a_n = a \circ \theta_n$$

is a continuous and bounded function. According to the previous step there exists $u^n \in H_0^1(\Omega)$ solution to

$$\begin{cases} -\varepsilon^2 \Delta_{X_1} u^n - \nabla_{X_2}\left(A\nabla_{X_2} u^n\right) + \chi u^n = a_n(l(u^n)) \text{ in } \Omega, \\ u^n = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{15}$$

Testing with u^n we deduce, by (10), that

$$u^n \text{ is bounded in } H_0^1(\Omega).$$

Then -up to a subsequence- we have

$$\begin{aligned} u^{n'} &\rightharpoonup u \quad \text{in } H^1(\Omega), \\ u^{n'} &\to u \quad \text{in } L^2(\Omega), \\ u^{n'} &\to u \quad \text{a.e. in } \Omega. \end{aligned}$$

Passing to the limit in (15) we get

$$\begin{aligned}\int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + A\nabla_{X_2} u \cdot \nabla_{X_2} v + \chi u v \, dx &= \lim_{n' \to \infty} \int_\Omega a_{n'}\left(l\left(u^{n'}\right)\right) v dx \\ &= \int_\Omega a\left(l\left(u\right)\right) v dx, \qquad \forall v \in H_0^1\left(\Omega\right),\end{aligned}$$

since we have

$$a_{n'}\left(l\left(u^{n'}\right)\right) \to a\left(l\left(u\right)\right) \quad \text{a.e. in } \Omega.$$

Step 3. Anisotropic singular perturbations method ($\varepsilon \to 0$)

Lemma 7 *Let $w_n \in V$ be a weakly converging sequence to w in V. Then we have*

$$l(w_n) \rightharpoonup l(w) \quad \textit{in } H^1(\Omega), \qquad l(w_n) \to l(w) \quad \textit{in } L^2(\Omega).$$

Proof. We can easily show

$$|l(w_n)|_{H^1(\Omega)} \leq C\,|w_n|_V \Longrightarrow l(w_n) \text{ is bounded in } H^1(\Omega).$$

$\exists\, W \in H^1(\Omega)$, -up to a subsequence

$$l(w_{n'}) \rightharpoonup W \text{ in } H^1(\Omega), \tag{16}$$

$$l(w_{n'}) \to W \text{ in } L^2(\Omega). \tag{17}$$

On the other hand we have for every $v \in \mathcal{D}(\Omega)$

$$\begin{aligned}
\int_{\Omega} l(w_{n'})\, v dx &= \int_{\omega_1} \left(\int_{\Omega} h\left(X_1, X_1', X_2\right) v(X_1, X_2)\, w_{n'}\left(X_1', X_2\right) dX_1' dX_2 \right) dX_1 \\
&\rightarrow \int_{\omega_1} \int_{\Omega} h\left(X_1, X_1', X_2\right) v(X_1, X_2)\, w\left(X_1', X_2\right) dX_1' dX_2 \\
&= \int_{\Omega} l(w)\, v dx \qquad \Rightarrow W = l(w).
\end{aligned}$$

Definition 8 ($\varepsilon - \mathbf{nets}$) *Given a metric space $(X; d)$, a subset Y of X is said to be an $\varepsilon - net$, if for all $x \in X$, there exists an $a \in Y$ such that*

$$d(x, a) < \varepsilon.$$

Theorem 9 *Under the assumptions above, the set of solutions of (6) is not empty. Moreover if we consider the metric structure of V corresponding to the norm of V, then for every $r > 0$, there exists $\varepsilon_0 > 0$ such that the set of solutions of (6) consists in a $r-$net of the set*

$$A_{\varepsilon_0} = \{u_\varepsilon \text{ solution to (11) for } \varepsilon < \varepsilon_0\},$$

and we have

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad in \quad L^2(\Omega).$$

Corollary 10 *If the problem (6) has only one solution u_0 then we have*

$$u_\varepsilon \longrightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0 \quad \text{and} \quad \varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad in \quad L^2(\Omega).$$

Proof. Let us take $v = u_\varepsilon$ in the weak formulation

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + A \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v + \gamma u_\varepsilon v \, dx = \int_\Omega a\left(l\left(u_\varepsilon\right)\right) v dx. \quad (18)$$

Then we get

$$u_\varepsilon, \quad |\varepsilon\nabla_{X_1}u_\varepsilon|, \quad |\nabla_{X_2}u_\varepsilon| \quad \text{are bounded in } L^2(\Omega). \tag{19}$$

It follows that there exists $u_0 \in L^2(\Omega)$ such that – up to a subsequence

$$u_\varepsilon \rightharpoonup u_0, \qquad \nabla_{X_2}u_\varepsilon \rightharpoonup \nabla_{X_2}u_0, \qquad \varepsilon\nabla_{X_2}u_\varepsilon \rightharpoonup 0 \quad \text{in } L^2(\Omega).$$

Using Lemma 7,

$$l(u_\varepsilon) \to l(u_0) \quad \text{a.e. in } \Omega.$$

The continuity of a gives

$$a(l(u_\varepsilon)) \to a(l(u_0)) \quad \text{a.e. in } \Omega \Longrightarrow a(l(u_\varepsilon)) \to a(l(u_0)) \quad in \quad L^2(\Omega) \tag{20}$$

Passing to the limit in (18) we derive

$$\int_\Omega A\nabla_{X_2}u_0 \cdot \nabla_{X_2}v + \gamma u_0 v\, dx = \int_\Omega a(l(u_0))\, v dx. \tag{21}$$

Taking $v = u_\varepsilon$ in (21) and passing to the limit we get

$$\int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 \, dx = \int_\Omega a\left(l\left(u_0\right)\right) u_0 dx. \tag{22}$$

We set

$$I_\varepsilon := \int_\Omega \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} u_\varepsilon \, dx + \int_\Omega A\nabla_{X_2}(u_\varepsilon - u_0) \cdot \nabla_{X_2}(u_\varepsilon - u_0) dx.$$

Using (18) we derive

$$\begin{aligned} I_\varepsilon = & \int_\Omega a\left(l\left(u_\varepsilon\right)\right) u_\varepsilon dx \\ & - \int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_\varepsilon \, dx - \int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_\varepsilon \, dx \\ & + \int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx. \end{aligned}$$

Passing to the limit in I_ε we get

$$\lim_{\varepsilon\longrightarrow 0} I_\varepsilon = \int_\Omega a\,(l\,(u_0))\,u_0 dx - \int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0\, dx = 0,$$

since we have (22). Using the coerciveness assumption we get

$$\int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + \lambda |\nabla_{X_2}(u_\varepsilon - u_0)|^2\, dx \le I_\varepsilon.$$

It follows that

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0, \qquad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0, \qquad u_\varepsilon \longrightarrow u_0 \quad \text{in } L^2(\Omega).$$

It follows that for almost every X_1

$$\int_{\omega_2} |\nabla_{X_2}(u_\varepsilon - u_0)|^2\, dX_2 \longrightarrow 0.$$

Since

$$\left\{ \int_{\omega_2} |\nabla_{X_2} v|^2 \, dX_2 \right\}^{\frac{1}{2}}$$

is a norm on $H_0^1(\omega_2)$ and $u_\varepsilon(X_1, \cdot) \in H_0^1(\omega_2)$ we have

$$u_0(X_1, \cdot) \in H_0^1(\omega_2)$$

for almost every X_1. Then u_0 is a solution of the intergo-differential problem. Since the only possible limits are the solutions of our problem the proof is complete.

Thank you for your attention.