SINGULAR PERTURBATIONS OF SOME NONLINEAR PROBLEMS

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Dedicated to Prof. V.V. Zhikov on the occasion of his 70th birthday

Abstract. In this paper we deal with singular perturbations of nonlinear problems depending on a small parameter $\varepsilon > 0$. First we consider the abstract theory of singular perturbations of variational inequalities involving some nonlinear operators, defined in Banach spaces, and describe the asymptotic behaviour of these solutions when $\varepsilon \to 0$. Then these abstract results are applied to some boundary value problems.

1. Introduction

The goal of this paper is to study the asymptotic behaviour of singular perturbations problems when a parameter ε goes towards 0. Our results are very general but we have more particularly in mind anisotropic cases where ε only acts on some variables of a domain $\Omega \subset \mathbb{R}^n$ (n is an integer) where we consider the partial differential equations. To be more precise we can take,

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as a model, the diffusion problem defined in the unit square $\Omega = (0,1)\times(0,1)$

$$\begin{cases} -\varepsilon^2 \partial_{x_1}^2 u_\varepsilon - \partial_{x_2}^2 u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ and f represents the source term. We assume that the diffusion in the x_1-direction is negligible with respect to the other direction when $\varepsilon \to 0$. Formally the natural limit of u_ε is a function u_0 defined on the sections $\{x_1\} \times (0,1)$ for a.e. $x_1 \in (0,1)$ as a solution of

$$\begin{cases} -\partial_{x_2}^2 u_0 (x_1, \cdot) = f (x_1, \cdot) & \text{in } (0,1), \\ u_0 (x_1, \cdot) = 0 & \text{on } \{0,1\}. \end{cases} \tag{1.2}$$

Note that the variable x_1 plays a role of a parameter. It is clear that if f (not identically equal to 0) is independent of x_1, i.e. $f = f(x_2)$, then $u_0 \notin H_0^1(\Omega)$. This prevents the convergence $u_\varepsilon \to u_0$ to occur in $H^1(\Omega)$. From this remark we may discuss many issues concerning this convergence.

In this note we begin by dealing with abstract singular perturbations problems of variational inequalities. Our approach has the advantage to include in a short theory a wide class of problems spread in the literature. We give then some applications of it.

In the literature, linear elliptic, parabolic and hyperbolic problems defined on arbitrary domains are analyzed in different contexts and the convergence $u_\varepsilon \to u_0$ is obtained in different norms. A boundary layer may occur at the lateral boundary of cylindrical domains ($\{0,1\} \times (0,1)$ for the above example). The convergence in Sobolev spaces may be shown in regions far from this lateral boundary. We may see this clearly when our perturbed problem satisfies some cylindrical symmetries. This means that $f = f(x_2)$ in the above example. In this case u_ε converges towards u_0 at an exponential rate. For more details we refer the reader to [1, 2, 3, 5, 6, 7, 8, 9, 10, 11].

An abstract approach to this theory was also given in [14, 16] where the following operator equation is considered

$$\varepsilon A u_\varepsilon + B u_\varepsilon = f, \tag{1.3}$$

with A and B linear operators defined on Hilbert spaces. This approach covers diagonal structure problems as problem (1.1). The authors also showed, as in the case of partial differential equations, that u_ε converges towards u_0 solution to

$$Bu_0 = f,$$

when $\varepsilon \to 0$. There are also some previous works on singular perturbations of variational inequalities, i.e. when (1.3) is replaced by

$$(\varepsilon A u_\varepsilon, v - u_\varepsilon) + (B u_\varepsilon, v - u_\varepsilon) \geq (f, v - u_\varepsilon)\,, \quad \forall v \in K \tag{1.4}$$

where K is some nonempty closed convex set (cf. [12, 13, 15]). In [15] this abstract approach is established to investigate the isotropic singular perturbations problems.

In order to cover a larger class of problems by an abstract theory, we will deal with the variational inequality (1.4) when A and B are nonlinear operators defined on different Banach spaces V and W respectively, which in particular applies to the anisotropic singular perturbations problems. This is what we will see in the next section. In the last section, the first example is devoted to show that these results also cover the isotropic case. Then some examples of anisotropic singular perturbations problems are introduced in order to illustrate some points of the theory as, for instance, the lack of compactness.

2. Abstract singular perturbations problems

Let V and W be two reflexive separable Banach spaces equipped with the norms $|\cdot|_V$ and $|\cdot|_W$ respectively. We suppose that the space $V \cap W$ is dense in V and W, and is equipped with the norm

$$|\cdot|_{V\cap W} = |\cdot|_V + |\cdot|_W\,.$$

Of course the $V \cap W$ is a Banach space equipped with the previous norm. For any space X, we denote by $\langle \cdot,\cdot \rangle_X$ the duality pairing between X' and X where X' is the dual of X. It is clear that

$$V \cap W \subset V, W \quad \text{and} \quad V', W' \subset (V \cap W)'\,.$$

Moreover one can check that $(V \cap W)' = V' + W'$. We consider two nonlinear operators A and B such that

$$A : V \to V', \qquad B : W \to W'.$$

We suppose that A, B are monotone, that is to say that

$$\langle Au - Av, u - v\rangle_V \geq 0, \qquad \forall u, v \in V, \tag{2.1}$$

$$\langle Bu - Bv, u - v\rangle_W \geq 0, \qquad \forall u, v \in W. \tag{2.2}$$

We denote by $K \neq \varnothing$ a closed convex set of $V \cap W$ and for A, B we make the following coerciveness assumption. We suppose that for some $v_0 \in K$

one has

$$\frac{\langle Au - Av_0, u - v_0\rangle_V}{|u - v_0|_V} \to +\infty \text{ when } |u - v_0|_V \to +\infty, \quad u \in K, \tag{2.3}$$

$$\frac{\langle Bu - Bv_0, u - v_0\rangle_W}{|u - v_0|_W} \to +\infty \text{ when } |u - v_0|_W \to +\infty, \quad u \in K. \tag{2.4}$$

Remark 1. *If K is bounded in V (resp. in W) we will not need the assumption (2.3) (resp. (2.4)). Note also that for some $v_0 \in K$ they are equivalent with*

$$\frac{\langle Au, u - v_0\rangle_V}{|u - v_0|_V} \to +\infty \text{ when } |u - v_0|_V \to +\infty, \quad u \in K, \tag{2.5}$$

$$\frac{\langle Bu, u - v_0\rangle_W}{|u - v_0|_W} \to +\infty \text{ when } |u - v_0|_W \to +\infty, \quad u \in K. \tag{2.6}$$

In addition we assume that

$$A \text{ sends bounded sets of } V \text{ in bounded sets of } V', \tag{2.7}$$

$$B \text{ sends bounded sets of } W \text{ in bounded sets of } W', \tag{2.8}$$

$$A, B \text{ are hemicontinuous on } V \text{ and } W \text{ respectively.} \tag{2.9}$$

This last assumption means that - for instance for A -

$$t \mapsto \langle A(u + tv), w\rangle_V \text{ is continuous on } \mathbb{R}, \quad \forall u, v, w \in V.$$

Under the assumptions above we have:

Theorem 1. *For $f \in (V \cap W)'$ and $\varepsilon > 0$ there exists u_ε solution to*

$$\begin{cases} \varepsilon \langle Au_\varepsilon, v - u_\varepsilon\rangle_V + \langle Bu_\varepsilon, v - u_\varepsilon\rangle_W \geq \langle f, v - u_\varepsilon\rangle_{V\cap W}, \ \forall v \in K, \\ u_\varepsilon \in K. \end{cases} \tag{2.10}$$

Moreover if A or B is strictly monotone (i.e. if one of the inequalities (2.1), (2.2) is strict for $u \neq v$) the solution is unique.

Proof. We consider A_ε the operator defined by

$$\begin{aligned} A_\varepsilon : V \cap W &\to (V \cap W)' = V' + W', \\ v &\mapsto \varepsilon Av + Bv. \end{aligned}$$

This operator is monotone, hemicontinuous and coercive on K. For this last point, by the coerciveness assumptions of A and B, for every $M > 0$ there exist $\delta_1(M), \delta_2(M) \geq 1$ such that

$$|u - v_0|_V \geq \delta_1(M) \Rightarrow \frac{\langle \varepsilon Au, u - v_0\rangle_V}{|u - v_0|_V} \geq M, \tag{2.11}$$

$$|u-v_0|_W \geq \delta_2(M) \Rightarrow \frac{\langle Bu, u-v_0\rangle_W}{|u-v_0|_W} \geq M. \tag{2.12}$$

Since A, B are bounded there exist constants C_A, C_B such that

$$|u-v_0|_V \leq \delta_1(M) \Rightarrow |\langle \varepsilon Au, u-v_0\rangle_V| \leq C_A(M),$$
$$|u-v_0|_W \leq \delta_2(M) \Rightarrow |\langle Bu, u-v_0\rangle_W| \leq C_B(M).$$

Choose

$$\begin{aligned} |u-v_0|_V + |u-v_0|_W \quad &\geq \quad 2\delta_1(M) + 2\delta_2(M) + \delta_1(2M+2C_B(M)) \\ &\quad +\delta_2(2M+2C_A(M)). \end{aligned}$$

Of course one has either $|u-v_0|_V \geq \delta_1(M)$ or $|u-v_0|_W \geq \delta_2(M)$. Suppose for instance that $|u-v_0|_V \geq \delta_1(M)$, the other case being the same. If moreover $|u-v_0|_W \geq \delta_2(M)$, from (2.11), (2.12) one has

$$\begin{aligned} &\frac{\langle \varepsilon Au, u-v_0\rangle_V + \langle Bu, u-v_0\rangle_W}{|u-v_0|_V + |u-v_0|_W} \\ &= \frac{|u-v_0|_V}{|u-v_0|_V + |u-v_0|_W} \cdot \frac{\langle \varepsilon Au, u-v_0\rangle_V}{|u-v_0|_V} \\ &+ \frac{|u-v_0|_W}{|u-v_0|_V + |u-v_0|_W} \cdot \frac{\langle Bu, u-v_0\rangle_W}{|u-v_0|_W} \geq M. \end{aligned}$$

If $|u-v_0|_W \leq \delta_2(M)$ then one has

$$|u-v_0|_V \geq \delta_2(M),\ \delta_1(2M+2C_B(M)),$$

so that

$$\begin{aligned} &\frac{\langle \varepsilon Au, u-v_0\rangle_V + \langle Bu, u-v_0\rangle_W}{|u-v_0|_V + |u-v_0|_W} \\ \geq\ & \frac{|u-v_0|_V}{|u-v_0|_V + |u-v_0|_W} \{2M+2C_B(M)\} - C_B(M) \\ \geq\ & \frac{1}{2}\{2M+2C_B(M)\} - C_B(M) \geq M. \end{aligned}$$

This shows the coerciveness of A_ε. The existence of u_ε follows from the classical theory of variational inequalities. □

Remark 2. *When $K = V \cap W$ one sees by taking $v = u_\varepsilon \pm w, w \in K$ that u_ε is solution to*

$$\begin{cases} \varepsilon Au_\varepsilon + Bu_\varepsilon = f, \\ u_\varepsilon \in V \cap W. \end{cases} \tag{2.13}$$

We are now interested in studying the behaviour of u_ε when $\varepsilon \to 0$. Note that this is not possible in general. Indeed, taking for instance V a Hilbert space, $A =$ the identity, $B = 0, f \in V' = V$ we can see that the solution of (2.13) is given by $u_\varepsilon = f/\varepsilon$ and $(u_\varepsilon)_\varepsilon$ has no limit. In what follows we will assume that

$$f \in W'. \tag{2.14}$$

The essential convergences are given as follows:

Theorem 2. *Suppose that $f \in W'$ and let u_ε be solution to* (2.10). *Then we have when $\varepsilon \to 0$*

$$(i)\ u_\varepsilon \text{ is bounded in } W \text{ independently of } \varepsilon, \tag{2.15}$$

$$(ii)\ \varepsilon u_\varepsilon \to 0 \text{ in } V, \tag{2.16}$$

$$(iii)\ \varepsilon A u_\varepsilon \to 0 \text{ in } V', \tag{2.17}$$

$$(iv)\ \langle \varepsilon A u_\varepsilon, u_\varepsilon \rangle_V \to 0. \tag{2.18}$$

Proof. Proof of **(i).** Choose $v_0 \in K$, such that (2.5) and (2.6) hold. Suppose that $|u_\varepsilon - v_0|_W$ is unbounded. For some sequence $\varepsilon_k \to 0$ one has then

$$|u_{\varepsilon_k} - v_0|_W \to +\infty.$$

Taking $v = v_0$ in (2.10) we derive

$$\begin{aligned} \varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_V + \langle B u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_W &\leq \langle f, u_{\varepsilon_k} - v_0 \rangle_W \\ &\leq |f|_{W'} |u_{\varepsilon_k} - v_0|_W . \end{aligned}$$

It follows that

$$\frac{\varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_V}{|u_{\varepsilon_k} - v_0|_W} + \frac{\langle B u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_W}{|u_{\varepsilon_k} - v_0|_W} \leq |f|_{W'} . \tag{2.19}$$

If $|u_{\varepsilon_k} - v_0|_V$ is bounded then

$$\frac{\varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_V}{|u_{\varepsilon_k} - v_0|_W} \to 0$$

else by the coerciveness of A this term is nonnegative for some k large enough. In both cases, due to the coerciveness of B, the left hand side of (2.19) is unbounded which is impossible. This proves (2.15).

Proof of **(ii).** Since u_ε is bounded in W, and by consequence $B u_\varepsilon$ is bounded in W', we derive from (2.10) written for $v = v_0$ that

$$\varepsilon \langle A u_\varepsilon, u_\varepsilon - v_0 \rangle_V \leq C \tag{2.20}$$

for some constant C independent of ε. If $(u_\varepsilon - v_0)$ is bounded in V it is clear that $\varepsilon u_\varepsilon = \varepsilon (u_\varepsilon - v_0) + \varepsilon v_0 \to 0$. Else we have from (2.5), (2.20) -up to a subsequence-

$$\varepsilon |u_\varepsilon - v_0|_V \leq C \frac{|u_\varepsilon - v_0|_V}{\langle Au_\varepsilon, u_\varepsilon - v_0\rangle_V} \to 0$$

and the result follows as in the previous case.

Proof of **(iii)** and **(iv).** We first show that $\varepsilon Au_\varepsilon \rightharpoonup 0$ in V'. Let $v \in$ V. From the monotonicity of A we have

$$\varepsilon \langle Au_\varepsilon - Av, u_\varepsilon - v\rangle_V \geq 0, \tag{2.21}$$

whence

$$\varepsilon \langle Au_\varepsilon, v\rangle_V \leq \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V + \langle Av, \varepsilon (v - u_\varepsilon)\rangle_V . \tag{2.22}$$

For $v_0 \in K$ we derive from (2.20) that

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V \leq \langle \varepsilon Au_\varepsilon, v_0\rangle_V + C$$

and thus, using (2.22), we get

$$\varepsilon \langle Au_\varepsilon, v - v_0\rangle_V \leq C + \langle Av, \varepsilon (v - u_\varepsilon)\rangle_V , \tag{2.23}$$

where C is a constant independent of ε. Choosing $v \in v_0 + \mathcal{B}_1$, where $\mathcal{B}_1$ is the unit ball of V, we arrive to

$$\varepsilon \langle Au_\varepsilon, v_1\rangle_V \leq C', \quad \forall v_1 \in \mathcal{B}_1,$$

where C' is independent of ε. Thus $\varepsilon A_\varepsilon$ is bounded in V' and -for some subsequence-

$$\varepsilon Au_\varepsilon \rightharpoonup \psi \text{ in } V'.$$

Passing to the limit in (2.23) we derive

$$\langle \psi, v - v_0\rangle_V \leq C, \qquad \forall v \in V$$

and thus $\psi = 0$. By the uniqueness of the possible limits we have shown that

$$\varepsilon Au_\varepsilon \rightharpoonup 0 \text{ in } V'.$$

For any $v \in K$ we have by (2.10) and the monotonicity of B

$$\begin{aligned} \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V &\leq \langle \varepsilon Au_\varepsilon, v\rangle_V + \langle f, u_\varepsilon - v\rangle_W + \langle Bu_\varepsilon, v - u_\varepsilon\rangle_W \\ &\leq \langle \varepsilon Au_\varepsilon, v\rangle_V + \langle f, u_\varepsilon - v\rangle_W + \langle Bv, v - u_\varepsilon\rangle_W . \end{aligned} \tag{2.24}$$

Let $(\varepsilon_k)_k$ be a sequence such that

$$\varepsilon_k \langle Au_{\varepsilon_k}, u_{\varepsilon_k}\rangle_V \to \limsup_{\varepsilon\to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V .$$

Since u_{ε_k} is bounded in W - extracting if necessary another subsequence - one can suppose that

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \text{ in } W.$$

Then passing to the limit in (2.24) written for ε_k we get

$$\limsup_{\varepsilon\to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V \leq \langle f, \tilde{u} - v\rangle_W + \langle Bv, v - \tilde{u}\rangle_W, \forall v \in K. \tag{2.25}$$

It is clear that $\tilde{u}$ belongs to $\bar{K}^W$, the weak closure of K in W which coincides with its strong closure since K is convex. Thus, there exists a sequence $v_n \in K$ such that

$$v_n \to \tilde{u} \text{ in } W.$$

Taking $v = v_n$ in (2.25) and passing to the limit we derive

$$\limsup_{\varepsilon\to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V \leq 0.$$

Passing to the limit in (2.22) we also have

$$\liminf_{\varepsilon\to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V \geq 0,$$

which proves (iv).

To complete the proof, going back to (2.22) one has for every $v_1 \in \mathcal{B}_1$

$$\begin{aligned} \varepsilon \langle Au_\varepsilon, v_1\rangle_V &\leq \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V + |Av_1|_{V'} (\varepsilon + |\varepsilon u_\varepsilon|_V) \\ &\leq \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V + C(\varepsilon + |\varepsilon u_\varepsilon|_V) \to 0 \end{aligned}$$

where C is independent of v_1. This completes the proof of the theorem. □

Remark 3. *In the case where* $K = V \cap W$, *from the equation* (2.13) *one derives that*

$$Bu_\varepsilon - f \to 0 \text{ in } V'. \tag{2.26}$$

In addition we have

Theorem 3. *Suppose that for some sequence* $\varepsilon_k \to 0$ *one has*

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \text{ in } W. \tag{2.27}$$

Then $\tilde{u}$ *is a solution to the variational inequality*

$$\begin{cases} \langle B\tilde{u}, v - \tilde{u}\rangle_W \geq \langle f, v - \tilde{u}\rangle_W, & \forall v \in \bar{K}^W, \\ \tilde{u} \in \bar{K}^W. \end{cases} \tag{2.28}$$

Moreover one has

$$Bu_{\varepsilon_k} \rightharpoonup B\tilde{u} \text{ in } W', \qquad \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W \to \langle B\tilde{u}, \tilde{u}\rangle_W. \tag{2.29}$$

Proof. Up to a subsequence - still labelled ε_k - one can assume that

$$Bu_{\varepsilon_k} \rightharpoonup \chi \text{ in } W'.$$

Passing to the limit in (2.10) written for ε_k we obtain (see Theorem 2)

$$\lim_{\varepsilon_k \to 0} \sup \langle Bu_{\varepsilon_k}, u_{\varepsilon_k} \rangle_W \leq \langle \chi, v \rangle_W + \langle f, \tilde{u} - v \rangle_W \,, \quad \forall v \in K. \tag{2.30}$$

Considering a sequence $v = v_n \to \tilde{u}$ as above we obtain

$$\lim_{\varepsilon_k \to 0} \sup \langle Bu_{\varepsilon_k}, u_{\varepsilon_k} \rangle_W \leq \langle \chi, \tilde{u} \rangle_W \,.$$

From the monotonicity of B we have

$$\langle Bu_{\varepsilon_k}, u_{\varepsilon_k} \rangle_W \geq \langle Bu_{\varepsilon_k}, v \rangle_W + \langle Bv, u_{\varepsilon_k} - v \rangle_W \,, \quad \forall v \in W.$$

Then

$$\lim_{\varepsilon_k \to 0} \inf \langle Bu_{\varepsilon_k}, u_{\varepsilon_k} \rangle_W \geq \langle \chi, v \rangle_W + \langle Bv, \tilde{u} - v \rangle_W \,, \quad \forall v \in W. \tag{2.31}$$

It follows - taking $v = \tilde{u}$ - that

$$\lim_{\varepsilon_k \to 0} \langle Bu_{\varepsilon_k}, u_{\varepsilon_k} \rangle_W = \langle \chi, \tilde{u} \rangle_W \,.$$

From (2.31) we derive

$$\langle \chi - Bv, \tilde{u} - v \rangle_W \geq 0, \quad \forall v \in W.$$

Replacing v by $\tilde{u} + tw$ and letting $t \to 0$ we obtain

$$\langle \chi - B\tilde{u}, w \rangle_W \geq 0, \quad \forall w \in W,$$

i.e. $\chi = B\tilde{u}$. It follows that the whole sequence Bu_{ε_k} converges toward $B\tilde{u}$. Moreover (2.30) becomes

$$\langle B\tilde{u}, v - \tilde{u} \rangle_W \geq \langle f, v - \tilde{u} \rangle_W \,, \quad \forall v \in K.$$

Since $\bar{K}^W$ is closed - weakly closed - one has $\tilde{u} \in \bar{K}^W$ and the above inequality holds also for every $v \in \bar{K}^W$. This completes the proof of the theorem. □

Remark 4. (i) *We have proved that the only possible limits for the subsequences of* $(u_\varepsilon)_\varepsilon$ *are solutions of the variational inequality* (2.28). *In particular if the solution is unique one has*

$$u_\varepsilon \rightharpoonup \tilde{u} \text{ in } W, \quad Bu_\varepsilon \rightharpoonup B\tilde{u} \text{ in } W'.$$

This is the case when B is strictly monotone.

(ii) *In the case where $K = V \cap W$ then $\bar{K}^W = W$ and $\tilde{u}$ is solution to the equation*

$$B\tilde{u} = f.$$

As a corollary we have

Corollary 1. (i) *Suppose that A is strongly coercive in the sense that*

$$\langle Av, v\rangle_V \geq \lambda \left| v \right|_V^\alpha, \qquad \forall v \in V, \tag{2.32}$$

for some constants $\lambda > 0$ and $\alpha > 1$, then one has

$$\varepsilon^{1/\alpha} u_\varepsilon \to 0 \ \text{ in } V. \tag{2.33}$$

(ii) *If B is strongly monotone in the sense that for some $\delta > 0$ and $\beta > 1$*

$$\langle Bu - Bv, u - v\rangle_W \geq \delta \left| u - v \right|_W^\beta, \qquad \forall v, u \in W \tag{2.34}$$

then the solution $\tilde{u}$ of (2.28) *is unique and one has*

$$u_\varepsilon \to \tilde{u} \ \ \textit{in } W.$$

Proof. (i) follows directly from

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_V \geq \lambda\varepsilon \left| u_\varepsilon \right|_V^\alpha$$

and Theorem 2-(iv).

For (ii) one has by (2.34) and since $u_\varepsilon \in K$

$$\begin{aligned} \delta \left| \tilde{u} - u_\varepsilon \right|_W^\beta &\leq \langle B\tilde{u} - Bu_\varepsilon, \tilde{u} - u_\varepsilon\rangle_W \\ &\leq \langle f, \tilde{u} - u_\varepsilon\rangle_W - \langle Bu_\varepsilon, \tilde{u} - u_\varepsilon\rangle_W \\ &= \langle f, \tilde{u} - u_\varepsilon\rangle_W + \langle Bu_\varepsilon, u_\varepsilon\rangle_W - \langle Bu_\varepsilon, \tilde{u}\rangle_W \to 0 \end{aligned}$$

by (2.29). □

Remark 5. *Assuming only the basic coerciveness* (2.3) *of A, the convergence result* (2.16) *is sharp since if α approaches* 1 *in* (2.33) *the exponent of ε tends to* 1.

In the following corollary some monotonicity property of $(u_\varepsilon)_\varepsilon$ is shown.

Corollary 2. *Let $\varepsilon > \varepsilon' > 0$ then*

$$\langle Au_\varepsilon, u_\varepsilon\rangle_V \leq \langle Au_\varepsilon, u_{\varepsilon'}\rangle_V. \tag{2.35}$$

Proof. Indeed, set $v = u_\varepsilon$ (resp. $v = u_{\varepsilon'}$) in (2.10), written for ε (resp. ε'), we get

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V - \varepsilon' \langle Au_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_V + \langle Bu_\varepsilon - Bu_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_W \leq 0.$$

Using the monotonicity of A and B, it comes

$$\begin{aligned} \varepsilon \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V &\leq \varepsilon' \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V - \varepsilon' \langle Au_\varepsilon - Au_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_V \\ &\leq \varepsilon' \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V. \end{aligned}$$

Then (2.35) follows, since $\varepsilon > \varepsilon'$. □

Remark 6. *The above characterization is more clear if A is linear. For instance if V is a Hilbert space and $A = I_d$ then* (2.35) *yields*

$$|u_\varepsilon|_V \le |u_{\varepsilon'}|_V\,, \text{ for } \varepsilon' < \varepsilon.$$

Next we pay attention to more regular problems, i.e. when some solutions of (2.28) are in V.

Corollary 3. *If the variational inequality* (2.28) *has a solution $\hat{u} \in K$ satisfying*

$$\liminf \langle Au, u - \hat{u}\rangle_V > 0 \quad \text{when } |u|_V \to +\infty,\ u \in K, \tag{2.36}$$

then u_ε is bounded in V and there exists always a sequence u_{ε_k} such that

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \quad \text{in } V \text{ and } W, \tag{2.37}$$

where $\tilde{u} \in K$ is solution to (2.28)*, i.e. the accumulation points of $(u_\varepsilon)_\varepsilon$ are all in K and solutions to* (2.28)*.*

In addition if B satisfies (2.34)*, one has*

$$|u_\varepsilon - \tilde{u}|_W = o\left(\varepsilon^{1/\beta}\right). \tag{2.38}$$

Proof. Taking $v = \hat{u}$ in (2.10) we derive

$$\begin{aligned} \varepsilon \langle Au_\varepsilon, u_\varepsilon - \hat{u}\rangle_V &\le \langle f, u_\varepsilon - \hat{u}\rangle_W - \langle Bu_\varepsilon, u_\varepsilon - \hat{u}\rangle_W \\ &\le -\langle Bu_\varepsilon - B\hat{u}, u_\varepsilon - \hat{u}\rangle_W \le 0. \end{aligned} \tag{2.39}$$

Thus $\langle Au_\varepsilon, u_\varepsilon - \hat{u}\rangle_V \le 0$ for all $\varepsilon > 0$, and

$$\lim_{\varepsilon\to 0}\sup \langle Au_\varepsilon, u_\varepsilon - \hat{u}\rangle_V \le 0.$$

By (2.36), u_ε must be bounded in V and one can find a sequence ε_k such that

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \quad \text{in } W,\ V \text{ and } V \cap W.$$

In fact, since u_{ε_k} is bounded in V, W and $W \cap V$ one can assume that -up to a subsequence-

$$u_{\varepsilon_k} \rightharpoonup u \text{ in } V, \quad u_{\varepsilon_k} \rightharpoonup u' \text{ in } W, \quad u_{\varepsilon_k} \rightharpoonup u'' \text{ in } V \cap W.$$

If $h \in V' \subset V' + W'$ one has

$$\langle h, u_{\varepsilon_k}\rangle_{V\cap W} \to \langle h, u\rangle_{V\cap W}\,, \quad \langle h, u_{\varepsilon_k}\rangle_{V\cap W} \to \left\langle h, u''\right\rangle_{V\cap W}$$

whence

$$\langle h, u\rangle_V = \left\langle h, u''\right\rangle_V\,, \quad \forall h \in V'.$$

Similarly one can show that

$$\left\langle h, u' \right\rangle_W = \left\langle h, u'' \right\rangle_W, \quad \forall h \in W'.$$

It follows that

$$u = u' = u'' = \tilde{u}, \tag{2.40}$$

and $\tilde{u}$ is necessarily a solution to (2.28).

For the last part of the theorem, since by the uniqueness of the solution of (2.28), $\hat{u} = \tilde{u}$, one has from (2.39)

$$\begin{aligned} \delta \left| \tilde{u} - u_\varepsilon \right|_W^\beta &\leq \left\langle B\tilde{u} - Bu_\varepsilon, \tilde{u} - u_\varepsilon \right\rangle_W \\ &\leq -\varepsilon \left\langle Au_\varepsilon, u_\varepsilon - \tilde{u} \right\rangle_V \\ &= -\varepsilon \left\langle Au_\varepsilon - A\tilde{u}, u_\varepsilon - \tilde{u} \right\rangle_V + \varepsilon \left\langle A\tilde{u}, u_\varepsilon - \tilde{u} \right\rangle_V \\ &\leq \varepsilon \left\langle A\tilde{u}, u_\varepsilon - \tilde{u} \right\rangle_V = o\left(\varepsilon\right) \end{aligned}$$

and the result follows. □

Remark 7. *If we assume that* $f = 0$, $B(0) = 0, 0 \in K$ *and* B *satisfies a hypothesis as* (2.32) *then*

$$u_\varepsilon \to 0 \quad \text{in } W.$$

Indeed, taking $v = 0$ *in* (2.10)

$$\varepsilon \left\langle Au_\varepsilon, u_\varepsilon \right\rangle_V + \left\langle Bu_\varepsilon, u_\varepsilon \right\rangle_W \leq 0,$$

and by the monotonicity of A *we have*

$$\lambda \left| u_\varepsilon \right|_W^\beta \leq \varepsilon \left\langle Au_\varepsilon - A(0), u_\varepsilon \right\rangle_V + \left\langle Bu_\varepsilon, u_\varepsilon \right\rangle_W \leq -\varepsilon \left\langle A(0), u_\varepsilon \right\rangle_V.$$

The convergence follows by Theorem 2.

3. Some applications

It is interesting to note that, using a priori estimates in the previous section, there is no need to have some compactness assumptions to pass to the limit in the nonlinear terms. In order to illustrate this we will consider here three nonlinear elliptic boundary value problems as examples of the abstract theory above. We will apply the theory to some anisotropic singular perturbations problems in the last two examples. To also see the power of our abstract analysis in general, we consider a very classical case of nonlinear obstacle problems.

3.1. **Nonlinear obstacle problems.** We denote by $a(\xi) = (a_i(\xi))$ a continuous vector field in $\mathbb{R}^n$. We suppose that a is such that for some $\lambda, \Lambda > 0$ and $c \in \mathbb{R}$

$$a(\xi) \cdot \xi \geq \lambda |\xi|^2 + c, \quad |a(\xi)| \leq \Lambda |\xi|, \quad \forall \xi \in \mathbb{R}^n \tag{3.1}$$

and in addition that

$$(a(\xi) - a(\zeta)) \,.\, (\xi - \zeta) \geq 0, \quad \forall \xi, \zeta \in \mathbb{R}^n. \tag{3.2}$$

Then, for $f \in L^2(\Omega)$ there exists a unique u_ε solution to

$$\begin{cases} u_\varepsilon \in K_0 = \left\{ v \in H_0^1(\Omega) \,|\, v(x) \geq 0, \ a.e. \ x \in \Omega \right\}, \\ \varepsilon \displaystyle\int_\Omega a(\nabla u_\varepsilon) \cdot \nabla (v - u_\varepsilon)\, dx + \int_\Omega u_\varepsilon (v - u_\varepsilon)\, dx \\ \geq \displaystyle\int_\Omega f(v - u_\varepsilon)\, dx, \quad \forall v \in K_0, \end{cases} \tag{3.3}$$

where Ω is a bounded open subset in $\mathbb{R}^n$. Then setting

$$V = H_0^1(\Omega), \quad W = L^2(\Omega), \quad Au = -div(a(\nabla u)), \quad B = I_d,$$

our results apply and we get that

$$u_\varepsilon \to f^+ \ \text{ in } L^2(\Omega)$$

where f^+ (resp. f^-) denotes the positive (resp. negative) part of f. Indeed, thanks to Theorems 2, 3 and Corollary 1 we see that $u_\varepsilon \to \tilde{u}$ in $L^2(\Omega)$ where $\tilde{u}$ is the unique solution to

$$\begin{cases} \tilde{u} \in \bar{K}_0 = \left\{ v \in L^2(\Omega) \,|\, v(x) \geq 0, \ a.e. \ x \in \Omega \right\}, \\ \displaystyle\int_\Omega \tilde{u}(v - \tilde{u})\, dx \geq \int_\Omega f(v - \tilde{u})\, dx, \quad \forall v \in \bar{K}_0. \end{cases} \tag{3.4}$$

But clearly

$$\begin{aligned} \int_\Omega f^+ \left(v - f^+\right) dx &= \int_\Omega \left(f + f^-\right)\left(v - f^+\right) dx \\ &= \int_\Omega f\left(v - f^+\right) dx + \int_\Omega f^- v dx \\ &\geq \int_\Omega f\left(v - f^+\right) dx, \quad \forall v \in \bar{K}_0 \end{aligned}$$

and $\tilde{u} = f^+$. As a corollary of Theorems 2, 3 and Corollary 1 we can state the following.

Corollary 4. *When* $\varepsilon \to 0$, *we have*

$$u_\varepsilon \to f^+ \quad in\ L^2(\Omega), \quad \varepsilon u_\varepsilon \to 0 \ in\ H_0^1(\Omega),$$
$$-\varepsilon \partial_{x_i}(a(\nabla u_\varepsilon)) \to 0 \quad in\ H^{-1}(\Omega), \quad i = 1, \cdots, n.$$
$$\varepsilon \int_\Omega a(\nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \to 0.$$

Remark 8. *Note that, as in* (3.1), *we may add a constant* $c \in \mathbb{R}$ *in (2.32) since it will be neglected once it is multiplied by* ε *i.e.*

$$\langle Av, v\rangle_V \geq \lambda |v|_V^\alpha + c, \qquad \forall v \in V.$$

Of course, here the strong convergence of $\sqrt{\varepsilon}\nabla u_\varepsilon$ *comes from the last convergence in the above corollary, i.e.*

$$\sqrt{\varepsilon}\nabla u_\varepsilon \to 0 \quad in\ L^2(\Omega).$$

3.2. **Semilinear elliptic problems.** Let Ω be a bounded open subset of $\mathbb{R}^n$ with sufficiently smooth boundary. We split the components of a point $x \in \mathbb{R}^n$ into the q first components and the $n - q$ last ones i.e.

$$X_1 = (x_1, \ldots, x_q) \quad \text{and} \quad X_2 = (x_{q+1}, \ldots, x_n),$$

where q is a positive integer such that $q < n$. We denote by Π_{X_1} (resp. Π_{X_2}) the orthogonal projection from $\mathbb{R}^n$ onto the space $X_2 = 0$ (resp. $X_1 = 0$). For any $X_1 \in \Pi_1 := \Pi_{X_1}(\Omega)$ and $X_2 \in \Pi_2 := \Pi_{X_2}(\Omega)$ we denote by Ω_{X_1} (resp. Ω_{X_2}) the section of Ω above X_1 (resp. X_2) i.e.

$$\Omega_{X_1} = \{ X_2 \mid (X_1, X_2) \in \Omega \}, \qquad \Omega_{X_2} = \{ X_1 \mid (X_1, X_2) \in \Omega \}.$$

With this notation we set

$$\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)^T = \begin{pmatrix} (\partial_{x_1} u, \ldots, \partial_{x_q} u)^T \\ (\partial_{x_{q+1}} u, \ldots, \partial_{x_n} u)^T \end{pmatrix} = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}.$$

We consider the following semilinear elliptic problem

$$\begin{cases} -\varepsilon \Delta_{X_1} u_\varepsilon - \Delta_{X_2} u_\varepsilon + g(x, u_\varepsilon) = f \quad \text{in } \Omega, \\ u_\varepsilon \in H_0^1(\Omega) \cap L^p(\Omega), \end{cases} \tag{3.5}$$

where

$$\Delta_{X_1} = \sum_{i=1}^{i=q} \frac{\partial^2}{\partial^2 x_i}, \ \Delta_{X_2} = \sum_{i=q+1}^{i=n} \frac{\partial^2}{\partial^2 x_i},$$
$$p > 1, \ f \in L^2(\Omega) + L^{p'}(\Omega),$$

where p' is the conjugate of p. In order to apply the abstract approach we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathédeory function and nondecreasing in the second variable i.e.

$$\begin{aligned} x &\mapsto g(x,t) \text{ is measurable on } \Omega, \quad \forall t \in \mathbb{R}, \\ t &\mapsto g(x,t) \text{ is continuous and nondecreasing on } \mathbb{R} \text{ for a.e. } x \in \Omega \end{aligned}$$

and there exist $c, c' \geq 0$, such that

$$|g(x,t)| \leq c\,|t|^{p-1} + c', \qquad \forall t \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \tag{3.6}$$

$$g(x,t)\,t \geq |t|^p, \qquad \forall t \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \tag{3.7}$$

It is clear that if $u \in L^p(\Omega)$ then $g(\cdot, u(\cdot)) \in L^{p'}(\Omega)$. So g defines an operator (still labelled by g) from $L^p(\Omega)$ into $L^{p'}(\Omega)$ by

$$u \mapsto g(\cdot, u(\cdot)), \tag{3.8}$$

which is bounded, monotone and hemicontinuous. Then we choose the suitable Banach spaces

$$V = \left\{ u \in L^2(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_1} u \in \left[L^2(\Omega)\right]^q, \\ u(\cdot, X_2) \in H_0^1(\Omega_{X_2}), \text{ a.e. } X_2 \in \Pi_2 \end{array} \right\}, \tag{3.9}$$

equipped with the norm

$$|v|_V := |\nabla_{X_1} v|_{L^2(\Omega)}$$

and

$$W = \left\{ u \in L^2(\Omega) \cap L^p(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_2} u \in \left[L^2(\Omega)\right]^{n-q}, \\ u(X_1, \cdot) \in H_0^1(\Omega_{X_1}), \text{ a.e. } X_1 \in \Pi_1 \end{array} \right\}, \tag{3.10}$$

equipped with the norm

$$|v|_W := |\nabla_{X_2} v|_{L^2(\Omega)} + |v|_{L^p(\Omega)}.$$

We can easily check that V and W are separable reflexive Banach spaces. Next we set

$$A = -\Delta_{X_1} \quad \text{and} \quad B = -\Delta_{X_2} + g(x, \cdot).$$

Then the operator $A : V \to V'$ is linear, bounded and coercive. Since the operator $B : W \to W'$ is a sum of a linear operator, satisfying the same properties as A, and the operator defined in (3.8), it is bounded, monotone and coercive. In this example the limit problem is defined for a.e. $X_1 \in \Pi_1$ as

$$\begin{cases} -\Delta_{X_2} \tilde{u}(X_1, \cdot) + g((X_1, \cdot), \tilde{u}(X_1, \cdot)) = f(X_1, \cdot) & \text{in } \Omega_{X_1}, \\ \tilde{u}(X_1, \cdot) = 0 & \text{on } \partial\Omega_{X_1}. \end{cases} \tag{3.11}$$

Then it remains to precise the connection between the boundary conditions, which is the subject of the following proposition.

Proposition 1. *Let* V *and* W *be the spaces defined in* (3.9) *and* (3.10) *respectively, then if the boundary of* Ω *is smooth we have*

$$V \cap W = H_0^1(\Omega) \cap L^p(\Omega).$$

Proof. The first inclusion $H_0^1(\Omega) \cap L^p(\Omega) \subset V \cap W$ is easy. For $u \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists a sequence $(u_n)_n \subset \mathcal{D}(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega) \cap L^p(\Omega)$. In particular we have

$$|\nabla(u_n - u)|_{L^2(\Omega)} \to 0.$$

By the Lebesgue theorem we get - up to a subsequence - for a.e. $X_1 \in \Pi_1$ and $X_2 \in \Pi_2$

$$\begin{aligned} |\nabla(u_n(X_1,\cdot) - u(X_1,\cdot))|_{L^2(\Omega_{X_1})} &\to 0, \\ |\nabla(u_n(\cdot,X_2) - u(\cdot,X_2))|_{L^2(\Omega_{X_2})} &\to 0. \end{aligned}$$

This means that $u \in V$ and $u \in W$.

For the converse inclusion, let $u \in V \cap W$ and consider the following elliptic problem

$$\begin{cases} -\varepsilon\Delta v_\varepsilon + v_\varepsilon = u & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Since Ω is sufficiently regular and of course $V \cap W \subset H^1(\Omega) \cap L^p(\Omega)$, we have $v_\varepsilon \in H^2(\Omega)$. According to Corollary 1, we derive

$$v_\varepsilon \to u \text{ in } L^2(\Omega). \tag{3.13}$$

Then applying the Laplace operator to the first equation in (3.12) and taking $-v_\varepsilon$ as a test function, we obtain

$$\varepsilon \left\langle \Delta^2 v_\varepsilon, v_\varepsilon \right\rangle_{H_0^1(\Omega)} - \int_\Omega \Delta v_\varepsilon v_\varepsilon dx = -\left\langle \Delta u, v_\varepsilon \right\rangle_{H_0^1(\Omega)}.$$

It is clear that $\Delta u \in H^{-1}(\Omega)$, $\Delta^2 v_\varepsilon \in H^{-1}(\Omega)$ since

$$-\Delta v_\varepsilon = \frac{u - v_\varepsilon}{\varepsilon} \in H^1(\Omega). \tag{3.14}$$

It follows that

$$-\varepsilon \int_\Omega \nabla(\Delta v_\varepsilon) \cdot \nabla v_\varepsilon dx + |\nabla v_\varepsilon|_{L^2(\Omega)}^2 = \int_\Omega \nabla u \cdot \nabla v_\varepsilon dx,$$

whence

$$-\varepsilon \int_{\Pi_1} \int_{\Omega_{X_2}} \nabla_{X_1} (\Delta v_\varepsilon) \cdot \nabla_{X_1} v_\varepsilon dX_1 dX_2$$

$$- \varepsilon \int_{\Pi_2} \int_{\Omega_{X_1}} \nabla_{X_2} (\Delta v_\varepsilon) \cdot \nabla_{X_2} v_\varepsilon dX_2 dX_1 + |\nabla v_\varepsilon|^2_{L^2(\Omega)}$$

$$= \int_\Omega \nabla u \cdot \nabla v_\varepsilon dx \leq \frac{1}{2} |\nabla u|^2_{L^2(\Omega)} + \frac{1}{2} |\nabla v_\varepsilon|^2_{L^2(\Omega)} . \qquad (3.15)$$

Since $v_\varepsilon \in H^1_0(\Omega)$ and $u \in V \cap W$ in (3.14), we deduce for a.e. $X_1 \in \Pi_1$ and a.e. $X_2 \in \Pi_2$ (see [5])

$$\Delta v_\varepsilon (X_1, \cdot) \in H^1_0 (\Omega_{X_2}), \ \Delta v_\varepsilon (\cdot, X_2) \in H^1_0 (\Omega_{X_1}) .$$

Thus we can rewrite (3.15) as

$$2\varepsilon \int_{\Pi_1} \int_{\Omega_{X_2}} \Delta v_\varepsilon \Delta_{X_1} v_\varepsilon dX_1 dX_2$$

$$+ 2\varepsilon \int_{\Pi_2} \int_{\Omega_{X_1}} \Delta v_\varepsilon \Delta_{X_2} v_\varepsilon dX_2 dX_1 + |\nabla v_\varepsilon|^2_{L^2(\Omega)} \leq |\nabla u|^2_{L^2(\Omega)} ,$$

whence

$$2\varepsilon |\Delta v_\varepsilon|^2_{L^2(\Omega)} + |\nabla v_\varepsilon|^2_{L^2(\Omega)} \leq |\nabla u|^2_{L^2(\Omega)} . \qquad (3.16)$$

It follows that v_ε is bounded in $H^1_0(\Omega)$, then -up to a subsequence- its weak limit is in $H^1_0(\Omega)$ and due to (3.13) this limit is u. Thus $u \in H^1_0(\Omega)$, which ends the proof of the proposition. □

As it is known, we need a pointwise convergence to pass to the limit in the nonlinear term $g(\cdot, u_\varepsilon)$. But the estimates that one has, i.e.

$$|\nabla_{X_2} u_\varepsilon|_{L^2(\Omega)} , |u_\varepsilon|_{L^p(\Omega)} \quad \text{are bounded,}$$

are not sufficient to get the pointwise limit of $(u_\varepsilon)_\varepsilon$ since the embedding

$$W \subset L^2(\Omega)$$

is not compact. So in this case the monotonicity hypothesis is necessary and as an obvious consequence of Theorems 2, 3 and Corollary 1 we have

Corollary 5. *When $\varepsilon \to 0$, we have*

$$u_\varepsilon \to \tilde{u}, \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \text{ and } \sqrt{\varepsilon} \nabla_{X_1} u_\varepsilon \to 0 \quad \text{in } L^2(\Omega)$$

where $\tilde{u}$ and u_ε are the solutions of (3.11) *and* (3.5) *respectively. Moreover if g is strongly monotone then we obtain.*

$$u_\varepsilon \to \tilde{u} \quad \text{in } L^p(\Omega).$$

Remark 9. *Note that, even if B is not strongly monotone, the first two convergences hold strongly. This is due to the following monotone type inequality*

$$\begin{aligned} \langle \Delta_{X_2} v - \Delta_{X_2} u, v-u \rangle_W + \int_\Omega \left(g(x,v) - g(x,u) \right)(v-u)\, dx \\ \geq \left| \nabla_{X_2}(v-u) \right|^2_{L^2(\Omega)}, \quad \forall u, v \in W. \end{aligned}$$

3.3. $p-$**Laplacian type problem.** The second application of the abstract theory, in the anisotropic case, is the following quasilinear elliptic equation

$$\begin{cases} -\varepsilon \Delta_{p_1,X_1} u_\varepsilon - \Delta_{p_2,X_2} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{3.17}$$

where $p_1, p_2 > 1$ are real constants and Δ_{p_1,X_1}, Δ_{p_2,X_2} are the p_i-Laplace operators in X_1 and X_2 respectively i.e.

$$\begin{aligned} \Delta_{p_1,X_1} \cdot &= \nabla_{X_1} \cdot \left(\left| \nabla_{X_1} \cdot \right|^{p_1-2} \nabla_{X_1} \cdot \right), \\ \Delta_{p_2,X_2} \cdot &= \nabla_{X_2} \cdot \left(\left| \nabla_{X_2} \cdot \right|^{p_2-2} \nabla_{X_2} \cdot \right). \end{aligned}$$

We assume that

$$f \in L^{p_2'}(\Omega),$$

(p_2' is the conjugate of p_2). In this case we set

$$V = \left\{ u \in L^{p_1}(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_1} u \in \left[L^{p_1}(\Omega) \right]^q, \\ u(\cdot, X_2) \in W_0^{1,p_1}(\Omega_{X_2}), \text{ a.e. } X_2 \in \Pi_2 \end{array} \right\},$$

equipped with the norm

$$|v|_V = |\nabla_{X_1} v|_{L^{p_1}(\Omega)}$$

and

$$W = \left\{ u \in L^{p_2}(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_2} u \in \left[L^{p_2}(\Omega) \right]^{n-q}, \\ u(X_1, \cdot) \in W_0^{1,p_2}(\Omega_{X_1}), \text{ a.e. } X_1 \in \Pi_1 \end{array} \right\},$$

equipped with the norm

$$|v|_W = |\nabla_{X_2} v|_{L^{p_2}(\Omega)}.$$

We can easily show that V and W are separable reflexive Banach spaces. Then we define the operators $A : V \to V'$ and $B : W \to W'$ as

$$A = -\Delta_{p_1,X_1}, \quad B = -\Delta_{p_2,X_2}.$$

It is easy to see that A and B are coercive, bounded and hemicontinuous. The monotonicity of A and B is shown by the following lemma (see [2, 12]).

Lemma 1. *For all $p > 1$ and $\xi, \eta \in \mathbb{R}^n$, we have, for a constant $c_p > 0$,*

$$\left(|\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right) \cdot (\xi - \eta) \geq c_p \left\{|\xi| + |\eta|\right\}^{p-2} |\xi - \eta|^2 .$$

If $p \geq 2$, then

$$\left(|\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right) \cdot (\xi - \eta) \geq c_p \, |\xi - \eta|^p ,$$

where $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^n$ and "$\cdot$" is the scalar product.

Thus the operator A (resp. B) is strictly monotone for all $p_1 > 1$ (resp. $p_2 > 1$) and strongly monotone if $p_1 \geq 2$ (resp. $p_2 \geq 2$). The limit problem is defined, for a.e. $X_1 \in \Pi_1$, as

$$\begin{cases} -\Delta_{p_2,X_2}\tilde{u}\,(X_1,\cdot) = f\,(X_1,\cdot) & \text{in } \Omega_{X_1}, \\ \tilde{u}\,(X_1,\cdot) = 0 & \text{on } \partial\Omega_{X_1}. \end{cases} \tag{3.18}$$

Finally as in the previous subsection we can show that

$$(V \cap W) \subset W_0^{1,\min(p_1,p_2)}\,(\Omega)\,.$$

More precisely we have

$$V \cap W = \left\{ u \in L^{\max(p_1,p_2)}\,(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_1} u \in [L^{p_1}\,(\Omega)]^q\,,\ \nabla_{X_2} u \in [L^{p_2}\,(\Omega)]^{n-q}\,, \\ u|_{\partial\Omega} = 0 \end{array} \right\},$$

which gives a sense to the boundary conditions. Then by Theorem 2, 3 and Corollary 1 we have

Corollary 6. *For all $p_1, p_2 > 1$,*

$$\begin{array}{rcl} u_\varepsilon \rightharpoonup \tilde{u} & \text{in} & W, \\ \varepsilon \nabla_{X_1} u_\varepsilon \to 0 & \text{in} & L^{p_1}\,(\Omega)\,, \\ \varepsilon \Delta_{p_1,X_1} u_\varepsilon \to 0 & \text{in} & V', \\ \Delta_{p_2,X_2} u_\varepsilon \rightharpoonup f & \text{in} & W', \end{array} \tag{3.19}$$

where u_ε and $\tilde{u}$ are the solutions of (3.17) *and* (3.18) *respectively. Moreover if $p_1 \geq 2$ then*

$$\varepsilon^{1/p_1} \nabla_{X_1} u_\varepsilon \to 0 \quad \textit{in } L^{p_1}\,(\Omega)\,, \tag{3.20}$$

and if $p_2 \geq 2$ *then*

$$u_\varepsilon \to \tilde{u}, \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \quad in\ L^{p_2}(\Omega). \tag{3.21}$$

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