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ASYMPTOTIC BEHAVIOR OF ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH SOME SMALL COEFFICIENTS

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Abstract. The aim of this paper is to analyze the asymptotic behavior of the solutions to elliptic boundary-value problems where some coefficients become negligible on a cylindrical part of the domain. We show that the dimension of the space can be reduced and find estimates of the rate of convergence. Some applications to elliptic boundary-value problems on domains becoming unbounded are also considered.

1. Introduction

We study the asymptotic behavior of the solutions of elliptic boundary-value problems, posed on bounded domains of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$ with cylindrical part, where the coefficients and the domains depend on a parameter θ. We show under certain conditions on the coefficients that the solution of such problems converges towards a solution of another elliptic problem in $\mathbb{R}^{n-p}$, faster than any power of θ on the cylindrical part. More specifically, we are interested in problems invariant by translations (cylindrical symmetry) arbitrary in p directions, and we compare the solution of our problem with that of an ideal problem independent of the coordinates associated with these p directions. This study was inspired to us, on one hand by the theory of "Singular Perturbation" of boundary problem, which is the framework of this paper, and on the other hand by the ideas and the tools given in some works of Chipot and Rougirel (see [3], [5]) where another study of the asymptotic behavior of elliptic boundary-value problems on domains becoming unbounded is given. We would like to note that is difficult to locate similar studies in the literature, except some examples studied in [8] and recently some cases have been considered in [1] and [7].

The paper is organized as follows: In the second section, we give some useful lemmas which will be used in the following sections. We show the main theorem in the third section where we investigate the rate of convergence estimates. Next, in the fourth section, we apply this result to the asymptotic behavior of the solutions of elliptic problems on domains becoming unbounded in one or several directions and we extend some results of [3] and [5] for more general domains. In the last section, we give the rate of convergence according to the size of the domain in all directions.

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Let $(\Omega_\theta)_{\theta>0}$ be a family of bounded Lipschitz domains of $\mathbb{R}^n$, satisfying

$$\Delta \times \omega \subset \Omega_\theta, \quad \Delta \times \partial\omega \subset \partial\Omega_\theta, \quad P_{X_2}\Omega_\theta \subset \omega_0, \tag{1.1}$$

where ω_0 and ω are two bounded Lipschitz domains of $\mathbb{R}^{n-p}$, Δ is a bounded Lipschitz domain of $\mathbb{R}^p$, n and p two positive integers with $n > p \geq 1$ and P_{X_i} the projection on the X_i axis, such that for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we set $X_1 = (x_1, \dots, x_p)$ and $X_2 = (x_{p+1}, \dots, x_n)$.

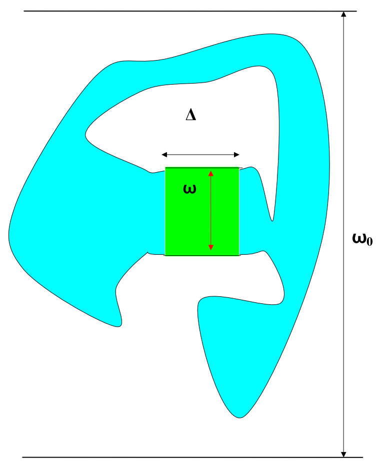

FIGURE 1. The domain Ω_θ.

We would like to consider the following three boundary-value problems

$$\begin{gathered}\sum_{i,j=1}^{n} -\partial_i(a_{ij}^\theta \partial_j u) + a_0 u = f \quad \text{in } \Omega_\theta \\ u = 0 \quad \text{on } \partial\Omega_\theta,\end{gathered} \tag{1.2}$$

$$\begin{gathered}\sum_{i,j=p+1}^{n} -\partial_i(a_{ij} \partial_j u) + a_0 u = f \quad \text{in } \omega \\ u = 0 \quad \text{on } \partial\omega,\end{gathered} \tag{1.3}$$

and

$$\begin{gathered}\sum_{i,j=p+1}^{n} -\partial_i(a_{ij} \partial_j u) + a_0 u = h \quad \text{in } \omega_0 \\ u = 0 \quad \text{on } \partial\omega_0\end{gathered} \tag{1.4}$$

where θ is a positive parameter. Since we are interested in θ close to 0, we can take $\theta < 1$. Assume that

$$f, h \in L^2(\omega_0). \tag{1.5}$$

Consider then

$$a_{ij}^\theta \in L^\infty(P_{X_1}\Omega_\theta \times \omega_0), \tag{1.6}$$

for all $i, j = 1, \dots, n$.

Remark 1.1. We can only suppose that

$$a_{ij}^{\theta} \in L^{\infty}(\Omega_\theta), \quad \text{for } j = 1, \dots, n$$

and we extend the coefficients on $P_{X_1}\Omega_\theta \times \omega_0$, keeping the assumptions below.

Assume that the coefficients a_{ij}^{θ} are independent of X_1 for $j \geq p+1$, and independent of θ for $i \geq p+1$ and $j \geq p+1$, i.e.

$$a_{ij}^{\theta}(x) = a_{ij}^{\theta}(X_2) \quad \text{for } j \geq p+1 \tag{1.7}$$

$$a_{ij}^{\theta}(x) = a_{ij}(X_2) \quad \text{for } i \geq p+1,\ j \geq p+1. \tag{1.8}$$

Furthermore, we assume the ellipticity condition; i.e., there exist a constant $\lambda > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}^{\theta}(x)\xi_i\xi_j \geq \lambda\theta|\boldsymbol{\xi}^1|^2 + \lambda|\boldsymbol{\xi}^2|^2, \quad \text{a.e. } x \in P_{X_1}\Omega_\theta \times \omega_0,\ \forall \xi \in \mathbb{R}^n, \tag{1.9}$$

where $\boldsymbol{\xi}^1 = (\xi_1, \dots, \xi_p)$ and $\boldsymbol{\xi}^2 = (\xi_{p+1}, \dots, \xi_n)$. Consequently,

$$\sum_{i,j=p+1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \text{a.e. } x \in \omega_0,\ \forall \xi \in \mathbb{R}^{n-p}. \tag{1.10}$$

In addition, we suppose that there exist constants α $(0 < \alpha \leq 1/2)$ and $C > 0$, such that

$$|a_{ij}^{\theta}(x)| \leq C\theta^{\frac{1}{2}+\alpha} \quad \text{for } i \leq p,\ j \leq p \tag{1.11}$$

$$|a_{ij}^{\theta}(x)| \leq C\theta^{\alpha} \quad \text{for } i \geq p+1,\ j \leq p \text{ or } i \leq p,\ j \geq p+1 \tag{1.12}$$

a.e. $x \in \Delta \times \omega$. The existence of the term a_0 does not have any influence on the final result, then we put $a_0 = 0$.

Remark 1.2. As a model example, we consider the singularly perturbed Laplacian problem, defined on a cylindrical domain $\Omega_\theta = \Delta \times \omega$,

$$\begin{gathered} -\theta\Delta_{X_1}u - \Delta_{X_2}u = f \quad \text{in } \Omega_\theta \\ u = 0 \quad \text{on } \partial\Omega_\theta. \end{gathered}$$

The variational problems corresponding to (1.2), (1.3) and (1.4) are

$$\begin{gathered} a(u,v) = \int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^{\theta}(x)\partial_j u_\theta \partial_i v dx = \int_{\Omega_\theta} f v dx, \\ u, v \in H_0^1(\Omega_\theta), \end{gathered} \tag{1.13}$$

$$\begin{gathered} a_\omega(u,v) = \int_{\omega} \sum_{i,j=p+1}^{n} a_{ij}(X_2)\partial_j u_\infty \partial_i v dX_2 = \int_{\omega} f v dX_2, \\ u, v \in H_0^1(\omega), \end{gathered} \tag{1.14}$$

and

$$\begin{gathered} a_{\omega_0}(u,v) = \int_{\omega_0} \sum_{i,j=p+1}^{n} a_{ij}(X_2)\partial_j u_h \partial_i v dX_2 = \int_{\omega_0} h v dX_2, \\ u, v \in H_0^1(\omega_0). \end{gathered} \tag{1.15}$$

According to the Lax-Milgram theorem, the existence and the uniqueness of the solution u_θ in $H_0^1(\Omega_\theta)$ of the problem (1.13), the solution u_∞ in $H_0^1(\omega)$ of the problem (1.14) and the solution u_h in $H_0^1(\omega_0)$ of the problem (1.14) are assured. First of all, we need to introduce some preliminary results.

2. Some estimates

We start with the following Lemmas which will be used frequently in this paper.

Lemma 2.1. *Let v be an element of $H_0^m(\Omega_\theta)$. Then*

$$v(X_1,.) \in H_0^m(\omega_0) \quad a.e.\ X_1 \in P_{X_1}\Omega_\theta, \tag{2.1}$$

$$v(X_1,.) \in H_0^m(\omega) \quad a.e.\ X_1 \in \Delta. \tag{2.2}$$

Proof. By the density of $\mathcal{D}(\Omega_\theta)$ in $H_0^m(\Omega_\theta)$, there exists a sequence ϕ_n of $\mathcal{D}(\Omega_\theta)$, such that

$$\int_{\Omega_\theta} \nabla(v-\phi_n)dx \to 0 \quad \text{as } n \to \infty.$$

We extend v and ϕ_n by 0 on $P_{X_1}\Omega_\theta \times \omega_0$ ($\Omega_\theta \subset P_{X_1}\Omega_\theta \times \omega_0$), then we have $\phi_n \in \mathcal{D}(P_{X_1}\Omega_\theta \times \omega_0)$, $v \in H_0^m(P_{X_1}\Omega_\theta \times \omega_0)$ and

$$\int_{P_{X_1}\Omega_\theta}\int_{\omega_0} |\nabla(v-\phi_n)|^2 dx \to 0 \quad \text{as } n \to \infty.$$

We can extract a subsequence ϕ_{n_k}, such that as $k \to \infty$:

$$\int_{\omega_0} |\nabla_{X_2}(v-\phi_{n_k})|^2 dx \to 0 \quad \text{a.e. } X_1 \in P_{X_1}\Omega_\theta,$$

$$\int_{\omega} |\nabla_{X_2}(v-\phi_{n_k})|^2 dx \to 0 \quad \text{a.e. } X_1 \in \Delta,$$

which give (2.1) and (2.2). □

Lemma 2.2. *Under the preceding hypotheses, we assume that $h = f \geq 0$ (resp. $h = f \leq 0$). Then we have*

$$0 \leq u_\theta \leq u_h, \quad (\textit{resp. } u_h \leq u_\theta \leq 0).$$

Proof. We apply the weak maximal principle for elliptic problems (see [6]) to obtain the inequalities $u_\theta \geq 0$ and $u_h \geq 0$. For the second inequality, if we use (2.1) we can take $v \in H_0^1(\Omega_\theta)$ in (1.15) and integrate on $P_{X_1}\Omega_\theta$, to get

$$\int_{\Omega_\theta} \sum_{i,j=p+1}^{n} a_{ij}(x)\partial_j u_h \partial_i v dx = \int_{\Omega_\theta} f v dx,$$

because v vanishes in the exterior of Ω_θ. By comparison with (1.13), we deduce

$$\int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^\theta(x)\partial_j u_\theta \partial_i v dx = \int_{\Omega_\theta} \sum_{i,j=p+1}^{n} a_{ij}(x)\partial_j u_h \partial_i v dx.$$

Taking into account the independence of u_∞ on X_1, we deduce

$$\begin{aligned}\int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^\theta(x)\partial_j (u_\theta - u_h) \partial_i v dx &= \int_{\Omega_\theta} \sum_{\substack{1\leq i\leq p\\ p+1\leq j\leq n}}^{n} a_{ij}^\theta(x)\partial_j u_h \partial_i v dx \\ &= \int_{\Omega_\theta} \sum_{\substack{1\leq i\leq p\\ p+1\leq j\leq n}}^{n} \partial_i(a_{ij}^\theta(x)\partial_j u_h v) dx\end{aligned}$$

$$= \int_{\partial\Omega_\theta} \sum_{\substack{1\le i\le p\\ p+1\le j\le n}}^{n} a_{ij}^\theta(x)\partial_j u_h v\nu_i dx,$$

because a_{ij}^θ is independent of X_1 for $1 \le i \le p$ and $p+1 \le j \le n$. Then, since v vanishes on the boundary, we deduce that

$$\int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^\theta(x)\partial_j(u_\theta - u_h)\partial_i v dx = 0, \tag{2.3}$$

for all $v \in H_0^1(\Omega_\theta)$. On the other hand, Theorem 2.8 in [4] shows that

$$\gamma[(u_\theta - u_h)^+] = [\gamma(u_\theta - u_h)]^+.$$

Then since $u_\theta \in H_0^1(\Omega_\theta)$ and $u_h \ge 0$, we have

$$\gamma[(u_\theta - u_h)^+] = 0,$$

which allows us to take $v = (u_\theta - u_h)^+ \in H_0^1(\Omega_\theta)$ in (2.3), then we get

$$\begin{aligned}
&\int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^\theta(x)\partial_j(u_\theta - u_h)\partial_i(u_\theta - u_\infty)^+ v dx\\
&= \int_{u_\theta - u_h \ge 0} \sum_{i,j=1}^{n} a_{ij}^\theta(x)\partial_j(u_\theta - u_h)\partial_i(u_\theta - u_h)^+ v dx = 0.
\end{aligned}$$

By the ellipticity assumption (1.9), it follows that

$$|\nabla(u_\theta - u_h)^+|_{L^2(\Omega_\theta)}^2 \le 0.$$

Therefore, $(u_\theta - u_h)^+ =$const and $(u_\theta - u_h)^+ \in H_0^1(\Omega_\theta)$, then we have $(u_\theta - u_h)^+ = 0$, which gives the second inequality $u_\theta \le u_\infty$. For the second case when $f \le 0$, it is enough to take $-f$ in place of f above. ☐

Let u_+ (resp. u_-) be the solution of (1.15) replacing h by f^+ (resp. $-f^-$).

Lemma 2.3. *Under the preceding assumptions, we have*

$$u_- \le u_\theta \le u_+.$$

Proof. Let $u_{\theta,+}$ (resp. $u_{\theta,-}$) be the solution of (1.13) replacing f by f^+ (resp. $-f^-$). Let us notice that

$$-f^- \le f \le f^+, \quad f^+ \ge 0, \quad -f^- \le 0$$

a.e. $x \in \omega_0$, then applying the weak maximal principle for elliptic problems, we get

$$u_{\theta,-} \le u_\theta \le u_{\theta,+}.$$

If we use lemma 2.2, we obtain $u_- \le u_{\theta,-}$, $u_{\theta,+} \le u_+$. This completes the proof. ☐

Next, we show the convergence of u_θ to u_∞ and we estimate the rate of this convergence.

3. Asymptotic behavior

According to Lemma 2.1, testing (1.14) with $v \in H_0^1(\Delta \times \omega)$ and integrating on Δ yields
$$\int_{\Omega_\theta} \sum_{i,j=p+1}^{n} a_{ij}^{\theta}(x)\partial_j u_\infty \partial_i v dx = \int_{\Omega_\theta} f v dx,$$
because v vanishes in the exterior of $\Delta \times \omega$. By (1.13), we remark that
$$\int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^{\theta}(x)\partial_j u_\theta \partial_i v \, dx = \int_{\Omega_\theta} \sum_{i,j=p+1}^{n} a_{ij}(x)\partial_j u_\infty \partial_i v dx.$$
Using the independence of u_∞ on X_1, it comes
$$\int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^{\theta}(x)\partial_j (u_\theta - u_\infty) \partial_i v dx = \int_{\Omega_\theta} \sum_{\substack{1\le i\le p\\ p+1\le j\le n}}^{n} a_{ij}^{\theta}(x)\partial_j u_\infty \partial_i v dx. \tag{3.1}$$
On the other hand, the independence of u_∞ and of the coefficients a_{ij}^{θ} on X_1 for $1 \le i \le p$ and $p+1 \le j \le n$, gives
$$\begin{aligned}\int_{\Omega_\theta} \sum_{\substack{1\le i\le p\\ p+1\le j\le n}}^{n} a_{ij}^{\theta}(x)\partial_j u_\infty \partial_i v dx &= \sum_{\substack{1\le i\le p\\ p+1\le j\le n}}^{n} \int_{\Omega_\theta} \partial_i (a_{ij}^{\theta}(x)\partial_j u_\infty v) dx \\ &= \sum_{\substack{1\le i\le p\\ p+1\le j\le n}}^{n} \int_{\partial\Omega_\theta} a_{ij}^{\theta}(x)\partial_j u_\infty v \nu_i dx = 0,\end{aligned}$$
because v vanishes on the boundary. Consequently, (3.1) becomes
$$\int_{\Omega_\theta} \sum_{i,j=1}^{n} a_{ij}^{\theta}(x)\partial_j (u_\theta - u_\infty) \partial_i v dx = 0 \quad \text{for all } v \in H_0^1(\Delta \times \omega)). \tag{3.2}$$
For $\epsilon > 0$, we set
$$\Delta_\epsilon = \{x \in \Delta : d(\partial\Delta, x) > \epsilon\}.$$
Let $(\rho_\epsilon)_{\epsilon>0}$ be a family of smooth functions on $\mathbb{R}^p$, such that
$$\operatorname{supp} \rho_\epsilon \subset \Delta_{\frac{\epsilon}{2}}, \quad (\Delta_\epsilon \subset \Delta_{\frac{\epsilon}{2}}),$$
$\rho_\epsilon(x) = 1$ for all x in Δ_ϵ and for all x in Δ, ρ_ϵ satisfies
$$0 \le \rho_\epsilon(x) \le 1.$$
If we take $v = \rho_\epsilon^2 (u_\theta - u_\infty) \in H_0^m(\Delta \times \omega))$ in (3.2), we deduce that
$$\int_{\Delta\times\omega} \sum_{i,j=1}^{n} a_{ij}^{\theta}(x)\partial_j (u_\theta - u_\infty) \partial_i (\rho_\epsilon^2 (u_\theta - u_\infty)) dx = 0,$$
whence
$$\begin{aligned}&\int_{\Delta\times\omega} \sum_{i,j=1}^{n} \rho_\epsilon^2 a_{ij}^{\theta}(x)\partial_j (u_\theta - u_\infty) \partial_i (u_\theta - u_\infty) dx \\ &= -2 \int_{\Delta\times\omega} \sum_{\substack{1\le i\le p\\ 1\le j\le n}} a_{ij}^{\theta}(x)\rho_\epsilon \partial_j (u_\theta - u_\infty)(u_\theta - u_\infty)\partial_i \rho_\epsilon dx.\end{aligned}$$

Using (1.9) and noting that ρ_ϵ vanishes in the exterior of $\Delta_{\frac{\epsilon}{2}}$ and depends only on X_1, it follows that

$$\begin{aligned}
&\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \lambda\theta \sum_{i=1}^{p} \rho_\epsilon^2(\partial_i(u_\theta - u_\infty))^2 dx + \int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \lambda' \sum_{i=p+1}^{n} \rho_\epsilon^2(\partial_i(u_\theta - u_\infty))^2 dx \\
&\le -2\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \sum_{\substack{1\le i\le p\\ 1\le j\le p}} a_{ij}^\theta(x)\rho_\epsilon\partial_j(u_\theta - u_\infty)(u_\theta - u_\infty)\partial_i\rho_\epsilon dx \\
&\quad - 2\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \sum_{\substack{1\le i\le p\\ p+1\le j\le n}} a_{ij}^\theta(x)\rho_\epsilon\partial_j(u_\theta - u_\infty)(u_\theta - u_\infty)\partial_i\rho_\epsilon dx.
\end{aligned}$$

We estimate the second member using (1.11), (1.12) and the fact that the derivative of ρ_ϵ is bounded, we get

$$\begin{aligned}
&\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \theta \sum_{i=1}^{p} (\rho_\epsilon\partial_i(u_\theta - u_\infty))^2 dx + \int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \sum_{i=p+1}^{n} (\rho_\epsilon\partial_i(u_\theta - u_\infty))^2 dx \\
&\le C\theta^{\frac{1}{2}+\alpha}\Big[\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \sum_{1\le j\le p} (\rho_\epsilon\partial_j(u_\theta - u_\infty))^2 dx\Big]^{1/2}\Big[\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} (u_\theta - u_\infty)^2 dx\Big]^{1/2} \\
&\quad + C\theta^{\alpha}\Big[\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \sum_{1\le j\le p} (\rho_\epsilon\partial_j(u_\theta - u_\infty))^2 dx\Big]^{1/2}\Big[\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} (u_\theta - u_\infty)^2 dx\Big]^{1/2}.
\end{aligned}$$

According to the Young inequality $ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$ with $\varepsilon = \frac{1}{2C}\theta^{\frac{1}{2}-\alpha}$ in the first term of the right hand side, and $\varepsilon = \frac{1}{2C}\theta^{-\alpha}$ in the second term of the right hind side, we deduce

$$\begin{aligned}
&\frac{1}{2}\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \theta \sum_{i=1}^{p} (\rho_\epsilon\partial_i(u_\theta - u_\infty))^2 dx + \frac{1}{2}\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \sum_{i=p+1}^{n} (\rho_\epsilon\partial_i(u_\theta - u_\infty))^2 dx \\
&\le C\theta^{2\alpha}\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} (u_\theta - u_\infty)^2 dx.
\end{aligned} \tag{3.3}$$

Using Poincaré's inequality and since $u_\theta - u_\infty$ vanishes on $\partial\omega$ for a.e. X_1,

$$\frac{1}{|\omega|^2}\int_\omega (u_\theta - u_\infty)^2 dX_2 \le \frac{1}{2}\int_\omega \sum_{p+1\le i\le n} (\partial_i(u_\theta - u_\infty))^2 dX_2 \text{ a.e. } X_1 \text{ in } \Delta_{\frac{\epsilon}{2}},$$

where $|\omega|$ is the diameter of ω, then (3.3) becomes

$$\begin{aligned}
&\frac{1}{|\omega|^2}\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} (\rho_\epsilon(u_\theta - u_\infty))^2 dx + \int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \theta\sum_{i=1}^{p} (\rho_\epsilon\partial_i(u_\theta - u_\infty))^2 dx \\
&+ \int_{\Delta_{\frac{\epsilon}{2}}\times\omega} \sum_{i=p+1}^{n} (\rho_\epsilon\partial_i(u_\theta - u_\infty))^2 dx \\
&\le C\theta^{2\alpha}\int_{\Delta_{\frac{\epsilon}{2}}\times\omega} (u_\theta - u_\infty)^2 dx.
\end{aligned}$$

According to the definition of ρ_ϵ, we obtain

$$\begin{aligned}
&\frac{1}{|\omega|^2}\int_{\Delta_\epsilon\times\omega}(u_\theta-u_\infty)^2dx+\int_{\Delta_\epsilon\times\omega}\theta\sum_{i=1}^{p}(\partial_i(u_\theta-u_\infty))^2dx\\
&\quad+\int_{\Delta_\epsilon\times\omega}\sum_{i=p+1}^{n}(\partial_i(u_\theta-u_\infty))^2dx\\
&\le C\theta^{2\alpha}\int_{\Delta_{\frac{\epsilon}{2}}\times\omega}(u_\theta-u_\infty)^2dx,
\end{aligned}\tag{3.4}$$

in particular

$$\int_{\Delta_\epsilon\times\omega}(u_\theta-u_\infty)^2dx\le C(\theta^\alpha|\omega|)^2\int_{\Delta_{\frac{\epsilon}{2}}\times\omega}(u_\theta-u_\infty)^2dx. \tag{3.5}$$

Choosing $\epsilon=\frac{\varepsilon}{2^k}$ for $k=0,\dots,\tau-1$ and $\varepsilon>0$, we get

$$\int_{\Delta_{\frac{\varepsilon}{2^k}}\times\omega}(u_\theta-u_\infty)^2dx\le C(\theta^\alpha|\omega|)^2\int_{\Delta_{\frac{\varepsilon}{2^{k+1}}}\times\omega}(u_\theta-u_\infty)^2dx.$$

Iterating the above formula, leads to

$$\int_{\Delta_{\frac{\varepsilon}{2}}\times\omega}(u_\theta-u_\infty)^2dx\le C(\theta^\alpha|\omega|)^{2(\tau-1)}\int_{\Delta_{\frac{\varepsilon}{2^\tau}}\times\omega}(u_\theta-u_\infty)^2dx.$$

Applying Lemma 2.3, we obtain

$$\int_{\Delta_{\frac{\varepsilon}{2}}\times\omega}(u_\theta-u_\infty)^2dx\le C(\theta^\alpha|\omega|)^{2(\tau-1)}\int_\omega(|u_+|+|u_-|+|u_\infty|)^2dx,$$

whence

$$\int_{\Delta_{\frac{\varepsilon}{2}}\times\omega}(u_\theta-u_\infty)^2dx\le C_\omega\theta^{2\alpha(\tau-1)},$$

with

$$C_\omega=C|\omega|^{2(\tau-1)}\int_\omega(|u_+|+|u_-|+|u_\infty|)^2dx. \tag{3.6}$$

Using (3.4) with $\epsilon=\varepsilon$, we get the estimates

$$\int_{\Delta_\varepsilon\times\omega}\sum_{i=1}^{p}(\partial_i(u_\theta-u_\infty))^2dx\le C_\omega\theta^{2\alpha\tau-1}, \tag{3.7}$$

$$\int_{\Delta_\varepsilon\times\omega}\sum_{i=p+1}^{n}(\partial_i(u\theta-u_\infty))^2dx\le C_\omega\theta^{2\alpha\tau}. \tag{3.8}$$

Finally, for any constant $r>0$, choosing τ such that $\tau\alpha>r$. Hence, we can state the following theorem.

Theorem 3.1. *Under conditions* (1.5)–(1.9), (1.11) *and* (1.12)*, for any open subset Φ of $\Delta\times\omega$ with boundary disjoint of $\partial\Delta\times\omega$, it holds that*

$$u_\theta\to u_\infty \quad in\ H^1(\Phi),$$

and for any $r>0$,

$$\int_\Phi|\nabla_{X_1}u_\theta|^2dx\le C_\omega\theta^{2r-1}, \tag{3.9}$$

$$\int_{\Phi} |\nabla_{X_2}(u_\theta - u_\infty)|^2 dx \le C_\omega \theta^{2r}, \tag{3.10}$$

where C_ω is a constant given above and independent of θ.

Proof. It is sufficient to take $\varepsilon = d(\partial\Delta, P_{X_1}\Phi)$. □

Remark 3.2. We can take $f \in H^{-1}(\omega_0)$ to show the same results. In this case we can consider f as an element of $H^{-1}(\Omega_\theta)$ by

$$< \widetilde{f}(t), v >_{H^{-1}(\Omega_\theta)} = \int_{P_{X_1}\Omega_\theta} < f(t), \widetilde{v}(X_1, .) >_{H^{-1}(\omega_0)} dX_1, \quad v \in H_0^1(\Omega_\theta),$$

where $\widetilde{v}$ is the extension of v by 0 on $P_{X_1}\Omega_\theta \times \omega_0$.

4. Application to the case of large size domains

We will see in this paragraph that the asymptotic behavior of the solution of linear elliptic problems of order two on domain $\overline{\Omega}_\ell$ satisfied for $\ell' \ge \ell$

$$\overline{\Omega}_\ell = (-\ell, \ell)^p \times \omega \text{ or } (-\ell, \ell)^p \times \omega \subset \overline{\Omega}_\ell \subset (-\ell', \ell')^p \times \omega \tag{4.1}$$

which is studied in the book of Chipot [3, Chapter 2 and 3], can be casted in the preceding study without supposing any assumption on ℓ' (considering domains more general than (4.1)), by giving a particular form to the coefficients a_{ij}^θ. Indeed, Let $(\overline{\Omega}_\ell)_{\ell>0}$ be a family of bounded Lipschitz domains of $\mathbb{R}^p \times \omega_0$ (see Figures 2 and 3), such that for any $\ell > 0$, $\overline{\Omega}_\ell$ contains the cylinder $(-\ell, \ell)^p \times \omega$ and $(-\ell, \ell)^p \times \partial\omega$ is a part of the boundary of $\overline{\Omega}_\ell$, where ω_0 and ω are defined in the first section.

We consider the two boundary-value problems defined by

$$\begin{gathered} \sum_{i,j=1}^{n} -\partial_i(a_{ij}\partial_j u) + a_0 u = f \quad \text{in } \overline{\Omega}_\ell \\ u = 0 \quad \text{on } \partial\overline{\Omega}_\ell, \end{gathered} \tag{4.2}$$

and

$$\begin{gathered} \sum_{i,j=p+1}^{n} -\partial_i(a_{ij}\partial_j u) + a_0 u = f \quad \text{in } \omega \\ u = 0 \quad \text{on } \partial\omega. \end{gathered} \tag{4.3}$$

We suppose that $f \in L^2(\omega)$,

$$a_0,\ a_{ij} \in L^\infty(\mathbb{R}^p \times \omega_0), \tag{4.4}$$

and

$$a_0(x) = a_0(X_2) \ge 0, \quad a_{ij}(x) = a_{ij}(X_2) \quad \text{for } j \ge p+1. \tag{4.5}$$

Moreover, we assume that there exists a constant $\lambda > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \lambda|\boldsymbol{\xi}|^2, \quad \text{a.e. } x \in \mathbb{R}^p \times \omega_0,\ \forall \xi \in \mathbb{R}^n. \tag{4.6}$$

Then the solutions $\overline{u}_\ell$ and u_∞ of (4.2) and (4.3) respectively satisfy

$$\int_{\overline{\Omega}_\ell} \sum_{i,j=1}^{n} a_{ij}(x)\partial_j\overline{u}_\ell\partial_i v + a_0(x)\overline{u}_\ell v dx = \int_{\overline{\Omega}_\ell} f v dx, \quad \text{a.e. } v \in H_0^1(\overline{\Omega}_\ell), \tag{4.7}$$

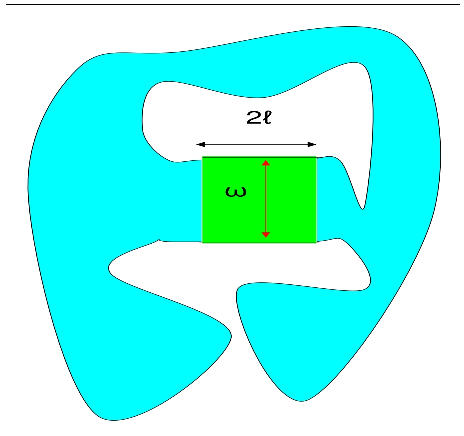

FIGURE 2. The domain $\overline{\Omega}_\ell$.

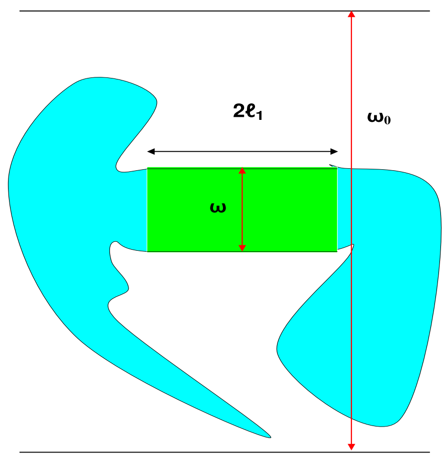

FIGURE 3. The domain $\overline{\Omega}_{\ell_1}$ has another form for $\ell_1 > \ell$.

and

$$\int_\omega \sum_{i,j=p+1}^{n} a_{ij}(X_2)\partial_j u_\infty \partial_i v + a_0(X_2) u_\infty v dX_2 = \int_\omega f v dX_2, \quad \text{a.e. } v \in H_0^1(\omega). \tag{4.8}$$

We take $\theta = \frac{1}{\ell^2}$ and use the change of variable given by

$$\psi : (X_1, X_2) \mapsto y = \Big(Y_1 = \frac{X_1}{\ell}, Y_2 = X_2\Big), \tag{4.9}$$

in (4.7), and we set $\psi(\overline{\Omega}_\ell) = \Omega_\theta$, thus we obtain

$$\int_{\Omega_\theta} \sum_{i,j=1}^{p} \frac{1}{\ell^2} a_{ij}(\ell Y_1, Y_2)\partial_j \overline{u}_\ell(\ell Y_1, Y_2)\partial_i v(\ell Y_1, Y_2)\ell^p dy$$

$$
\begin{aligned}
&+\int_{\Omega_\theta}\sum_{\substack{1\le i\le p,\ p+1\le j\le n\\ or\\ 1\le j\le p,\ p+1\le i\le n}}\frac{1}{\ell}a_{ij}(\ell Y_1,Y_2)\partial_j\overline{u}_\ell(\ell Y_1,Y_2)\partial_i v(\ell Y_1,Y_2)\ell^p dy\\
&+\int_{\Omega_\theta}\sum_{i,j=p+1}^{p}a_{ij}(Y_2)\partial_j\overline{u}_\ell(\ell Y_1,Y_2)\partial_i v(\ell Y_1,Y_2)\ell^p dy\\
&=\int_{\Omega_\theta}f(Y_2)v(\ell Y_1,Y_2)\ell^p dy.
\end{aligned}
$$

Setting

$$
\begin{gathered}
u_\theta(Y_1,Y_2)=\overline{u}_\ell(\ell Y_1,Y_2),\\
a^\theta_{ij}(Y_1,Y_2)=\frac{1}{\ell^2}a_{ij}(\ell Y_1,Y_2)\quad\text{for } i,j=1,\dots,p,\\
a^\theta_{ij}(Y_1,Y_2)=\frac{1}{\ell}a_{ij}(\ell Y_1,Y_2)\quad\text{for } 1\le i\le p<j\le n \text{ or } 1\le j\le p<i\le n,\\
a^\theta_{ij}(Y_2)=a_{ij}(Y_2)\quad\text{for}\quad i,j=p+1,\dots,n.
\end{gathered}
$$

In addition, it is clear that $(Y_1,Y_2)\mapsto v(\ell Y_1,Y_2)\in H^1_0(\Omega_\theta)$ if and only if $(X_1,X_2)\mapsto v(X_1,X_2)\in H^1_0(\overline{\Omega}_\ell)$. Consequently, the problem (4.7) is equivalent to

$$
\int_{\Omega_\theta}\sum_{i,j=1}^{n}a^\theta_{ij}(x)\partial_j u_\theta\partial_i v dx=\int_{\Omega_\theta}f(X_2)v dx,\quad\text{for all } v\in H^1_0(\Omega_\theta). \tag{4.10}
$$

Therefore, $\overline{u}_\ell$ is a solution of (4.7) if and only if u_θ is a solution of (4.10). Moreover we can examine the conditions of the first paragraph on the problem (4.10). According to the definition of $\overline{\Omega}_\ell$, the domain Ω_θ satisfies the condition (1.1) with $\Delta=(-1,1)^p$. The conditions (1.6)–(1.8) are satisfied by definition, for the condition (1.9), we have

$$
\begin{aligned}
\sum_{i,j=1}^{n}a^\theta_{ij}(y)\xi_i\xi_j&=\sum_{i,j=1}^{p}a_{ij}(\ell Y_1,Y_2)(\frac{1}{\ell}\xi_i)(\frac{1}{\ell}\xi_j)+\sum_{1\le i\le p,\,p+1\le j\le n}a_{ij}(Y_2)(\frac{1}{\ell}\xi_i)(\xi_j)\\
&\quad+\sum_{1\le j\le p,\ p+1\le i\le n}a_{ij}(\ell Y_1,Y_2)(\xi_i)(\frac{1}{\ell}\xi_j)+\sum_{i,j=p+1}^{n}a_{ij}(\ell Y_1,Y_2)\xi_i\xi_j,
\end{aligned}
$$

then using (4.6), we obtain

$$
\sum_{i,j=1}^{n}a^\theta_{ij}(y)\xi_i\xi_j\ge\lambda\theta|\boldsymbol{\xi}^1|^2+\lambda|\boldsymbol{\xi}^2|^2,
$$

a.e. $y\in\Omega_\theta$ and $\forall\xi\in\mathbb{R}^n$, therefore we have (1.9). Finally, if we use (4.4), we get the conditions (1.11) and (1.12) with $\alpha=\frac{1}{2}$. Then, if we apply Theorem 3.1, we deduce for $r>0$ and for $\Phi=(-\sigma,\sigma)^p\times\omega$ with $0<\sigma<1$, that there exists $C>0$ independent of ℓ, such that

$$
\begin{gathered}
\int_{(-\sigma,\sigma)^p\times\omega}|\nabla_{X_1}u_\theta|^2dy\le C\theta^{r+p-2},\\
\int_{(-\sigma,\sigma)^p\times\omega}|\nabla_{X_2}(u_\theta-u_\infty)|^2dy\le C\theta^{r+p}.
\end{gathered}
$$

Again, we use the change of variable (4.9) to obtain

$$\|\overline{u}_\ell - u_\infty\|_{H^1((-\sigma\ell,\sigma\ell)^p\times\omega)} \le \frac{C}{\ell^r}.$$

5. Estimate according to all directions

In the applications, we say that the size of the domain is large specifically in some directions if we take into account the size ratio between all the directions, for instance in the domain $(0,1)\times(0,\varepsilon)$, the size of $(0,1)$ is considered large when ε become negligible. However all the estimates of $\overline{u}_\ell - u_\infty$ given in [2], [3] and [5], only show an estimate of the error of convergence with respect to ℓ. In the following, we investigate this estimate with respect to the size ratio between ℓ and $|\omega|$. Then, we suppose in this section that $\omega = \omega_0$ and a bounded domain $\overline{\Omega}_\ell$ satisfies

$$(-\ell,\ell)^p \times \omega \subset \overline{\Omega}_\ell \subset \mathbb{R}^p \times \omega; \tag{5.1}$$

in addition, we assume that

$$f \in L^\infty(\omega). \tag{5.2}$$

First, we show the following estimate.

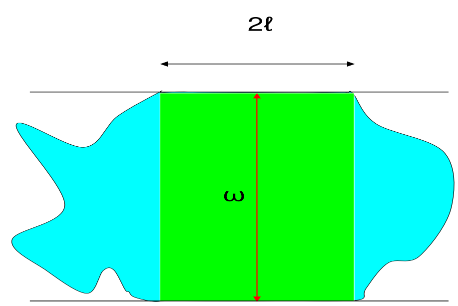

Figure 4. The domain $\overline{\Omega}_\ell$.

Lemma 5.1. *Let u_+ (resp. u_-) be the solution of* (1.14) *replacing h by f^+ (resp. $-f^-$). It holds that*

$$|u_+|_{L^2(\omega)},\ |u_-|_{L^2(\omega)},\ |u_\infty|_{L^2(\omega)} \le C\,[\operatorname{meas}\omega)]^{1/2}\,|\omega|^2 \tag{5.3}$$

where C is a constant independent of ω and $\operatorname{meas}\omega)$ *denotes the measure of ω.*

Proof. We give the proof for u_+, the proof for u_- and u_∞ are similar. Taking $v = u_+$ in (1.14) and using the ellipticity condition (1.10), we obtain

$$\lambda' \int_\omega |\nabla u_+|^2 dX_2 \le |f^+|_{L^2(\omega)} |u_+|_{L^2(\omega)}.$$

Using (5.2) and applying Poincaré's inequality, then there exists a constant C independent of ω, such that

$$\frac{1}{|\omega|^2} |u_+|^2_{L^2(\omega)} \le C[\operatorname{meas}\omega)]^{1/2} |u_+|_{L^2(\omega)},$$

which gives (5.3). ∎

This enables us to state the following corollary.

Corollary 5.2. *Let $\overline{u}_\ell$ be the solution of* (4.7) *where* $\overline{\Omega}_\ell$ *is given by* (5.1)*. If we suppose that* (4.4)*,* (4.5)*,* (4.6) *and* (5.2) *hold, then for any* $\tau > 0$ *and any* $0 < \sigma < 1$ *there exists a constant* $C_\sigma > 0$ *independent of* ℓ *and* ω*, such that*

$$|\nabla(\overline{u}_\ell - u_\infty)|_{L^2((-\sigma\ell,\sigma\ell)^p\times\omega)} \leq C_\sigma \ell^p \operatorname{meas}(\omega)|\omega|^2\Big(\frac{|\omega|}{\ell}\Big)^{2\tau}. \tag{5.4}$$

Proof. If we use the change of variable (4.9) in (3.7) and (3.8), and we apply the lemma above to estimate the constant C_ω defined in (3.6), then we deduce (5.4). □

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