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ASYMPTOTIC BEHAVIOR OF SOLUTION OF HYPERBOLIC PROBLEMS ON A CYLINDRICAL DOMAIN

B. Brighi and S. Guesmia

Université de Haute Alsace
Laboratoire Mathématiques, Informatique et Applications
4, rue des Frères Lumière
68093 MULHOUSE CEDEX France

Abstract. The asymptotic behavior of the hyperbolic evolution problems of order two, on a cylindrical domain $\Omega = \Delta \times \omega$, with coefficients dependent on a parameter is examined. The convergence of the solution of such problems towards a solution of a problem of the same type defined in ω is proved, and the rate of convergence estimates is given. One can see this work as a singular perturbation of the hyperbolic problems in some directions.

1. **Introduction.** We expose in this paper some results which were inspired to us, on the one hand by the theory of "singular Perturbation" of boundary problem, which is the framework of this work, and on the other hand by the ideas and the tools given in some works of Chipot and Rougirel [1], [2], [3]. We study the asymptotic behavior of the solution of hyperbolic problems, on a cylindrical domain $\Omega = \Delta \times \omega$, with coefficients dependent on a parameter θ, and we proof the convergence of the solution of such problems towards a solution of an other problems of the same type defined in ω. In this work, for hyperbolic problems with some small parameter, we show that dimension of the space can be reduced, and so numerical calculation can be made easier.

Let $\Omega = \Delta \times \omega$ be a cylindrical domain of $\mathbb{R}^n$, where Δ and ω are respectively bounded Lipschitz domains of $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$, n and p integers with $n > p \geq 1$. For $x \in \mathbb{R}^n$, we set $X_1 = (x_1, ..., x_p)$, $X_2 = (x_{p+1}, ..., x_n)$ and we denote $Q = [0,T] \times \Omega$, $Q_\infty = [0,T] \times \omega$, where T is a positive constant. For a positive parameter θ, we consider the two evolution problems defined by

$$\begin{cases} u'' - \sum\limits_{0 \leq i,j \leq n} \partial_{x_i}(a_{ij}^\theta(t,x)\partial_{x_j}u) + c(t,x)u = f & \text{in } Q \\ u = 0 & \text{on } [0,T] \times \partial\Omega \\ u(0,.) = u_{\theta,0}, \quad u'(0,.) = u_{\theta,1} & \text{in } \Omega \end{cases} \tag{1}$$

$$\begin{cases} u'' - \sum\limits_{p+1 \leq i,j \leq n} \partial_{x_i}(a_{ij}^\theta(t,x)\partial_{x_j}u) + c(t,x)u = f & \text{in } Q_\infty \\ u = 0 & \text{on } [0,T] \times \partial\omega \\ u(0,.) = u_0, \quad u'(0,.) = u_1 & \text{in } \omega \end{cases} \tag{2}$$

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such that $f : Q \longrightarrow \mathbb{R}$, u_0, $u_1 : \omega \longrightarrow \mathbb{R}$, $u_{\theta,0}$, $u_{\theta,1} : \Omega \longrightarrow \mathbb{R}$, a_{ij}^{θ}, $a : Q \longrightarrow \mathbb{R}$. For the initial conditions and the source term f, we suppose

$$f \in L^2(Q_\infty), \tag{3}$$

$$u_0 \in H_0^1(\omega), \quad u_1 \in L^2(\omega), \quad u_{\theta,0} \in H_0^1(\Omega), \quad u_{\theta,1} \in L^2(\Omega). \tag{4}$$

We put the conditions on the coefficients a_{ij}^{θ} , c

$$a_{ij}^{\theta} \in C^1(\overline{Q}),\ c \in C(\overline{Q}), \quad a_{ij}^{\theta} = a_{ji}^{\theta}, \tag{5}$$

for all $i, j = \widehat{1, n}$. In addition, we assume that the coefficients a_{ij}^{θ} are independent of X_1 for $i \geq p+1$, and independent of θ for $i \geq p+1$ and $j \geq p+1$, i.e.

$$a_{ij}^{\theta}(t,x) \quad = \quad a_{ij}^{\theta}(t, X_2) \quad \text{for} \quad i \geq p+1, \tag{6}$$

$$a_{ij}^{\theta}(t,x) \quad = \quad a_{ij}(t, X_2) \quad \text{for} \quad i, j \geq p+1. \tag{7}$$

For a positive number α with $0 < \alpha \leq \frac{1}{2}$, we suppose that there exists a constant $C > 0$, such as for $a.e.$ $(t,x) \in Q$

$$\left|a_{ij}^{\theta}(t,x)\right| \leq C\theta^{1/2+\alpha}, \quad \left|\partial_t a_{ij}^{\theta}(t,x)\right| \leq C\theta \quad \text{for} \quad i \leq p, \quad j \leq p, \tag{8}$$

$$\left|a_{ij}^{\theta}(t,x)\right| \leq C\theta^{\alpha}, \quad \left|\partial_t a_{ij}^{\theta}(t,x)\right| \leq C\theta^{1/2} \quad \text{for} \quad i \geq p+1,\ j \leq p. \tag{9}$$

The hyperbolicity of the problem is assumed, so that there are two constants $\lambda > 0$ and $\lambda' > 0$, such that for $\xi^1 = (\xi_1, ..., \xi_p)$ and $\xi^2 = (\xi_{p+1}, ..., \xi_n)$

$$\sum_{i,j=1}^{n} a_{ij}^{\theta}(t,x)\xi_i\xi_j \geq \lambda\theta \left|\xi^1\right|^2 + \lambda' \left|\xi^2\right|^2, \text{ for } a.e.\ (t,x) \in Q \text{ and } \forall \xi \in \mathbb{R}^n. \tag{10}$$

Under the conditions (3)-(5) and (10), the problems (1) and (2) admit weak solutions u_θ and u_∞ respectively, which means that

$$\frac{d^2}{dt^2}\int_\Omega u_\theta v dx + \int_\Omega \left(\sum_{1\leq i,j\leq n} a_{ij}^{\theta}(t,x)\partial_{x_j}u_\theta \partial_{x_i} v + c(t,x)u_\theta v \right) dx = \int_\Omega f v dx \tag{11}$$

$$\forall v \in H_0^1(\Omega),\ a.e.\ t \in [0,T] \text{ and } u_\theta(0,.) = u_{\theta,0}, \quad u_\theta'(0,.) = u_{\theta,1},$$

$$\frac{d^2}{dt^2}\int_\omega u_\infty v dx + \int_\omega \left(\sum_{p+1\leq i,j\leq n} a_{ij}(t,x)\partial_{x_j}u_\infty \partial_{x_i} v + c(t,x)u_\infty v \right) dx = \int_\omega f v dx \tag{12}$$

$$\forall v \in H_0^1(\omega),\ a.e.\ t \in [0,T] \text{ and } u_\infty(0,.) = u_0, \quad u_\infty'(0,.) = u_1$$

and

$$u_\theta \in C^0(0,T; H_0^1(\Omega)),\ u_\theta' \in C^0(0,T; L^2(\Omega)), \tag{13}$$

$$u_\infty \in C^0(0,T; H_0^1(\omega)),\ u_\infty' \in C^0(0,T; L^2(\omega)). \tag{14}$$

The injection $H_0^1(\omega) \hookrightarrow H^1(\Omega)$ is continuous, then we obtain

$$u_\infty \in C^0(0,T; H^1(\Omega)),\ u_\infty' \in C^0(0,T; L^2(\Omega)). \tag{15}$$

We will give in this work, the asymptotic behavior for the solution u_θ when $\theta \to 0$, and more precisely, we justify the convergence $u_\theta \to u_\infty$ when $\theta \to 0$, and we give the rate of convergence (the estimate of $u_\theta - u_\infty$). We will set $w_\theta = u_\theta - u_\infty$, $w_{\theta,0} = u_{\theta,0} - u_0$ and $w_{\theta,1} = u_{\theta,1} - u_1$.

2. **Equality of the energy type.** If we take $v \in H_0^1(\Omega)$, then for a.e. $X_1 \in \Delta$ we can take $v(X_1, .)$ as a test function in (12). It is possible because $v(X_1, .) \in H_0^1(\omega)$, see [1, Prop 3.1.]. Next, if we integrate on Δ, then the right hand side is the same as in (11). By comparison, we find

$$\frac{d^2}{dt^2}\int_\Omega u_\theta v dx + \int_\Omega \left(\sum_{1\leq i,j\leq n} a_{ij}^\theta(t,x)\partial_{x_j} u_\theta \partial_{x_i} v + c(t,x) u_\theta v \right) dx$$
$$= \frac{d^2}{dt^2}\int_\Omega u_\infty v dx + \int_\Omega \left(\sum_{p+1\leq i,j\leq n} a_{ij}(t,x)\partial_{x_j} u_\infty \partial_{x_i} v + c(t,x) u_\infty v \right) dx.$$

The coefficients $a_{ij}^\theta(x)$ for $p+1 \leq j \leq n$ and the solution u_∞ depend only on X_2, hence we deduce

$$\frac{d^2}{dt^2}\int_\Omega w_\theta v dx + \int_\Omega \sum_{1\leq i,j\leq n} a_{ij}^\theta(t,x)\partial_{x_j} w_\theta \partial_{x_i} v + c(t,x) w_\theta v dx$$
$$= -\int_\Omega \sum_{\substack{p+1\leq j\leq n,\\ 1\leq i\leq p}} \partial_{x_i}\left(a_{ij}^\theta(t,x)\partial_{x_j} u_\infty v\right) dx = -\int_\omega \int_{\partial\Delta} \sum_{\substack{p+1\leq j\leq n,\\ 1\leq i\leq p}} a_{ij}(t,x)\partial_{x_j} u_\infty v \nu_i dx.$$

Since v vanishes on the boundary of Ω, then we obtain

$$\frac{d^2}{dt^2}\int_\Omega w_\theta v dx + \int_\Omega \sum_{1\leq i,j\leq n} a_{ij}^\theta(t,x)\partial_{x_j} w_\theta \partial_{x_i} v + c(t,x) w_\theta v dx = 0 \quad \forall v \in H_0^1(\Omega). \tag{16}$$

Generally, for hyperbolic problems, the idea to have an estimate of w_θ, is to use an energy type equality, by replacing v by w_θ' in (16), when the regularity of the solutions allows to do it. But in our case we only have $w_\theta'(t) \in L^2(\Omega)$ and v in (16) has to vanish on the boundary of Ω. To avoid these obstacles, the method is based on the idea of approaching $u_\theta - u_\infty$ by functions with value in $H_0^1(\Omega)$, which is done by triple regularization. For $\epsilon > 0$, we set $\Delta_\epsilon = \{x \in \Delta \; ; \; d(\partial\Delta, x) > \epsilon\}$ and we consider a family of infinitely differentiable functions ρ_ϵ, such that $\operatorname{supp}\rho_\epsilon \subset \Delta_{\frac{\epsilon}{2}}$, for all $x \in \Delta_\epsilon$ we have $\rho_\epsilon(x) = 1$, and for all $x \in \Delta$ we have $0 \leq \rho_\epsilon(x) \leq 1$. For any $\delta > 0$, we introduce the continuous function ϑ_δ defined on $\mathbb{R}$, by $\vartheta_\delta(t) = \frac{t}{\delta}$ if $t \in [0, \delta]$, $\vartheta_\delta(t) = 1$ if $t \in [\delta, t_0 - \delta]$, $\vartheta_\delta(t) = \frac{t_0 - t}{\delta}$ if $t \in [t_0 - \delta, t_0]$ and $\vartheta_\delta(t) = 0$ if $t \notin [0, t_0]$, and the function ϑ_0 equal to 1 on $[0, t_0]$ and to 0 otherwise. Let $\overline{\rho}_n$ be a regularization sequence of even functions defined on $\mathbb{R}$, verifying $\int_{\mathbb{R}} \overline{\rho}_n(t)dt = 1$. We start with the energy type equality.

Theorem 1. *For any t we have*

$$\begin{aligned}
&\int_{\Delta_{\epsilon/2}\times\omega} (\rho_\epsilon w_\theta')^2 dx + b_\epsilon(t, w_\theta, w_\theta) = \int_{\Delta_{\epsilon/2}\times\omega} |\rho_\epsilon w_{\theta,1}|^2 dx \\
&- 2\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \sum_{\substack{1\leq j\leq n\\ 1\leq i\leq p}} a_{ij}^\theta(\sigma,x)\partial_{x_j} w_\theta(\sigma)\partial_{x_i}\rho_\epsilon \rho_\epsilon w_\theta'(\sigma) dx d\sigma \\
&+ b_\epsilon(0, w_{\theta,0}, w_{\theta,0}) + \int_0^t b_\epsilon'(\sigma, w_\theta, w_\theta) d\sigma + 2\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 c(t,x) w_\theta w_\theta' dx dt
\end{aligned} \tag{17}$$

where

$$\forall u,v\in H^1(\Omega),\ b_\epsilon(t,u,v):=\int_{\Delta_{\epsilon/2}\times\omega}\sum_{1\le i,j\le n}\rho_\epsilon^2 a_{ij}^\theta\partial_{x_j}u\partial_{x_i}vdx,$$
$$\forall u,v\in H^1(\Omega),\ b'_\epsilon(t,u,v):=\int_{\Delta_\epsilon\times\omega}\sum_{1\le i,j\le n}\rho_\epsilon^2\left(a_{ij}^\theta\right)'\partial_{x_j}u\partial_{x_i}vdx.$$

Proof. Supposing that the functions a_{ij}^θ and c are defined for any $t\in\mathbb{R}$, with same properties of regularity on $\mathbb{R}$ as on $[0,T]$ (extension by continuity), and the same for w_θ (for example extension by reflection). It is clear that $\rho_\epsilon^2\overline{\rho}_n*(\vartheta_\delta w'_\theta)=\rho_\epsilon^2\overline{\rho}'_n*(\vartheta_\delta w_\theta)-\rho_\epsilon^2\overline{\rho}_n*(\vartheta'_\delta w_\theta)\in L^2\left(-\infty,+\infty;H_0^1(\Omega)\right)$. The proof uses the equality (16), replacing v by $\rho_\epsilon^2\overline{\rho}_n*(\vartheta_\delta w'_\theta)$, and leaving $\delta\to 0$ in the first step, next $n\to+\infty$. For the first step, we have the following lemma.

Lemma 1. *Under the assumptions above, for any $t_0\in[0,T]$, we have*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} b'_\epsilon\left(t,\overline{\rho}_n*(\vartheta_0 w_\theta),\overline{\rho}_n*(\vartheta_0 w_\theta)\right)dt+2\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\overline{\rho}_n*\overline{\rho}_n*(\vartheta_0 w_\theta)(0)w'_\theta(0)dx\\
&+2\int_{\Delta_{\epsilon/2}\times\omega}\overline{\rho}_n*\overline{\rho}_n*\left(\sum_{1\le i,j\le n}a_{ij}^\theta(.,x)\partial_{x_j}(\vartheta_0 w_\theta)\right)(0)\rho_\epsilon^2\partial_{x_i}w'_\theta(0)dx\\
&-2\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\overline{\rho}_n*\overline{\rho}_n*(\vartheta_0 w'_\theta)(t_0)w'_\theta(t_0)dx\\
&-2\int_{\Delta_{\epsilon/2}\times\omega}\overline{\rho}_n*\overline{\rho}_n*\left(\sum_{1\le i,j\le n}a_{ij}^\theta(.,x)\partial_{x_j}(\vartheta_0 w_\theta)\right)(t_0)\,\rho_\epsilon^2\partial_{x_i}w_\theta(t_0)\,dx+\Upsilon_n\\
=&2\int_{-\infty}^{+\infty}\int_{\Delta_{\epsilon/2}\times\omega}\overline{\rho}_n*\left(c(t,x)\vartheta_0 w_\theta\right)\rho_\epsilon^2\left(\overline{\rho}_n*(\vartheta_0 w'_\theta)\right)dxdt\\
&+2\int_{-\infty}^{+\infty}\int_{\Delta_{\epsilon/2}\times\omega}\sum_{1\le j\le n,\ 1\le i\le p}\left(\overline{\rho}_n*\left(a_{ij}^\theta(t,x)\partial_{x_j}(\vartheta_0 w_\theta)\right)\right)\partial_{x_i}\rho_\epsilon\rho_\epsilon\left(\overline{\rho}_n*(\vartheta_0 w'_\theta)\right)dxdt.
\end{aligned}\tag{18}$$

with $\Upsilon_n=\int_{-\infty}^{+\infty}\int_{\Delta_{\epsilon/2}\times\omega}\sum_{1\le i,j\le n}\left(a_{ij}^\theta(t,x)\partial_{x_j}\overline{\rho}_n*(\vartheta_0 w_\theta)-\overline{\rho}_n*\left(a_{ij}^\theta(t,x)\partial_{x_j}(\vartheta_0 w_\theta)\right)\right)$ $\rho_\epsilon^2\partial_{x_i}\overline{\rho}'_n*(\vartheta_0 w_\theta)\,dxdt.$

The proof is quite technical and is based on the convergence $\vartheta_\delta\to\vartheta_0$ in $L^2(\mathbb{R})$, as $\delta\to 0$, the fact that $\int_{\mathbb{R}}|\vartheta'_\delta(t)|\,dt=2$ and [7, Théorème p. 120].

In the second step, we take $n\to+\infty$ in formula (18). We start by Υ_n.

$$\begin{aligned}
\Upsilon_n=&\int_{-\infty}^{+\infty}\int_{\Delta_{\epsilon/2}\times\omega}\sum_{1\le i,j\le n}a_{ij}^\theta(t,x)\partial_{x_j}\overline{\rho}_n*(\vartheta_0 w_\theta)\,\rho_\epsilon^2\left(\partial_{x_i}\overline{\rho}_n*(\vartheta_0 w_\theta)\right)'dxdt\\
&-\int_{-\infty}^{+\infty}\int_{\Delta_{\epsilon/2}\times\omega}\sum_{1\le i,j\le n}a_{ij}^\theta(t,x)\partial_{x_j}(\vartheta_0 w_\theta)\,\rho_\epsilon^2\partial_{x_i}\overline{\rho}_n*\left(\overline{\rho}_n*(\vartheta_0 w_\theta)\right)'dxdt.
\end{aligned}\tag{19}$$

The bilinear forms $b_\epsilon(t,.,.)$ given in Theorem 1 are continuous on $H^1(\Omega)$, therefore there exists a family of operators $\mathcal{B}_\epsilon(t)\in\mathcal{L}\left(H^1(\Omega),H^1(\Omega)\right)$, such that $b_\epsilon(t,u,v)=((\mathcal{B}_\epsilon(t)u,v))$, for $u,v\in H^1(\Omega)$. The mapping $t\longrightarrow((\mathcal{B}_\epsilon(t)u,v))$ $\forall u,v\in H^1(\Omega)$ is once continuously differentiable according to (5). The notation $((,))$ represents the scalar product in $H^1(\Omega)$. Replacing in (19), an elementary calculation gives

$$\Upsilon_n=-\int_{-\infty}^{+\infty}((\frac{d}{dt}\left[\mathcal{B}_\epsilon(t)\overline{\rho}_n*(\vartheta_0 w_\theta)-\overline{\rho}_n*\mathcal{B}_\epsilon(t)\left(\vartheta_0 w_\theta\right)\right],\overline{\rho}_n*(\vartheta_0 w_\theta)))dt.$$

Using the fact that $\int_{\mathbb{R}}\|\overline{\rho}_n*(\vartheta_0 w_\theta)\|^2dt\le C\int_{\mathbb{R}}|\vartheta_0|^2dt$, (where C does not depend on n) and the Friedrichs lemma [5, lemme 7.2. p. 72], we deduced that $\Upsilon_n\longrightarrow 0$. Moreover, $\overline{\rho}_n*(\vartheta_0 w_\theta)\to(\vartheta_0 w_\theta)$ in $L^2\left(-\infty,+\infty;H^1(\Omega)\right)$ and $\overline{\rho}_n*(\vartheta_0 w'_\theta)\to(\vartheta_0 w'_\theta)$ in $L^2\left(-\infty,+\infty;L^2(\Omega)\right)$, as $n\to+\infty$. Then we get

$$\int_{-\infty}^{+\infty}b'_\epsilon\left(t,\overline{\rho}_n*(\vartheta_0 w_\theta),\overline{\rho}_n*(\vartheta_0 w_\theta)\right)dt\underset{n\to+\infty}{\longrightarrow}\int_0^{t_0}b'_\epsilon\left(t,w_\theta,w_\theta\right)dt,$$

$$\int_{-\infty}^{+\infty}\int_\Omega\overline{\rho}_n*\left(c(t,x)\vartheta_0 w_\theta\right)\rho_\epsilon^2\left(\overline{\rho}_n*(\vartheta_0 w'_\theta)\right)dxdt\underset{n\to+\infty}{\longrightarrow}\int_0^{t_0}\int_\Omega\rho_\epsilon^2 c(t,x)w_\theta w'_\theta dxdt$$

$$\int_{-\infty}^{+\infty}\int_\Omega\overline{\rho}_n*\left(a_{ij}^\theta(t,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)\partial_{x_i}\rho_\epsilon\rho_\epsilon\overline{\rho}_n*\left(\vartheta_0 w'_\theta\right)dxdt$$
$$\underset{n\to+\infty}{\longrightarrow}\int_0^{t_0}\int_\Omega a_{ij}^\theta(t,x)\partial_{x_j}w_\theta\partial_{x_i}\rho_\epsilon\rho_\epsilon w'_\theta dxdt$$

for $1\le i\le p$ and $1\le j\le n$. For the remainder of the terms in (18), a same way gives the convergence. We explain it in detail for one of them. First, we set $\sigma_n=\overline{\rho}_n*\overline{\rho}_n$, and hence $\int_{-t_0}^{0}\sigma_n(t)dt=\int_0^{t_0}\sigma_n(t)dt=\frac{1}{2}$. Therefore, we obtain

$$\begin{aligned}
\Lambda_n:=&2\int_\Omega\overline{\rho}_n*\overline{\rho}_n*\left(\sum_{1\le i,j\le n}a_{ij}^\theta(.,x)\partial_{x_j}\left(\vartheta_0 w_\theta\right)\right)(t_0)\,\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)dx\\
&-\int_\Omega\sum_{1\le i,j\le n}a_{ij}^\theta(t_0,x)\partial_{x_j}w_\theta\left(t_0\right)\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)dx\\
=&2\int_0^{t_0}\sigma_n(t)\int_\Omega\sum_{1\le i,j\le n}\left(a_{ij}^\theta(.,x)\partial_{x_j}w_\theta\right)\left(t_0-t\right)\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)dx\\
&-2\int_0^{t_0}\sigma_n(t)\int_\Omega\sum_{1\le i,j\le n}a_{ij}^\theta(t_0,x)\partial_{x_j}w_\theta\left(t_0\right)\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)dx\\
=&2\int_{\mathrm{supp}\sigma_n\cap[0,t_0]}\sigma_n(t)\int_\Omega\sum_{1\le i,j\le n}\left(a_{ij}^\theta(t_0-t,x)\partial_{x_j}w_\theta\left(t_0-t\right)\right.\\
&\left.-\,a_{ij}^\theta(t_0,x)\partial_{x_j}w_\theta\left(t_0\right)\right)\rho_\epsilon^2\partial_{x_i}w_\theta\left(t_0\right)dx),
\end{aligned}$$

from the continuity of a_{ij}^{θ} and w_θ, we deduce that $\Lambda_n \to 0$, i.e.

$$2\int_{\Omega} \overline{\rho}_n * \overline{\rho}_n * \left(\sum_{1\le i,j\le n} a_{ij}^{\theta}(.,x)\partial_{x_j}(\vartheta_0 w_\theta) \right)(t_0)\, \rho_\epsilon^2 \partial_{x_i} w_\theta(t_0)\, dx \underset{n\to+\infty}{\longrightarrow} \int_{\Omega} \sum_{1\le i,j\le n} a_{ij}^{\theta}(t_0,x)\partial_{x_j} w_\theta(t_0)\, \rho_\epsilon^2 \partial_{x_i} w_\theta(t_0)\, dx.$$

The same way for the other terms allows to achieve the proof of the theorem. □

3. **Estimation.** We estimate the terms of the second member of (17), using the properties of the family of the functions ρ_ϵ and (5), (8) and (9). For the first term we obtain

$$\begin{aligned}
&\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \sum_{\substack{1\le j\le n\\ 1\le i\le p}} a_{ij}^{\theta}(\sigma,x)\partial_{x_j} w_\theta(\sigma)\partial_{x_i}\rho_\epsilon \rho_\epsilon w_\theta'(\sigma) dx d\sigma \\
\le & C\theta^{\frac{1}{2}+\alpha} \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} w_\theta|\, |\rho_\epsilon w_\theta'(\sigma)|\, dx d\sigma \\
&+ C\theta^{\alpha} \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_\theta|\, |\rho_\epsilon w_\theta'(\sigma)|\, dx d\sigma \\
\le & C\theta^{\alpha} \left(\theta \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} w_\theta|^2\, dx d\sigma + \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_\theta|^2\, dx d\sigma \right) \\
&+ C \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\rho_\epsilon w_\theta'(\sigma)|^2\, dx d\sigma,
\end{aligned}$$

where we applied Young inequality $ab \le \mu a^2 + \frac{b^2}{\mu}$ for the two last terms of the first inequality, with $\mu = \theta^{1/2}$ and $\mu = \theta^{\alpha}$ (C independent of θ). For the other terms of (17), we apply the same techniques. We deduce

$$b_\epsilon(0, w_{\theta,0}, w_{\theta,0}) \le C\theta^{2\alpha} \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} u_{\theta,0}|^2\, dx + C \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_{\theta,0}|^2\, dx,$$

$$\begin{aligned}
\int_0^t b_\epsilon'(\sigma, w_\theta, w_\theta)\, d\sigma \le & C\theta \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_1} w_\theta|^2\, dx d\sigma \\
&+ C \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_2} w_\theta|^2\, dx d\sigma,
\end{aligned}$$

$$\begin{aligned}
\int_0^t \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 c(t,x) w_\theta w_\theta' dx dt \le & C \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\rho_\epsilon w_\theta'(\sigma)|^2\, dx d\sigma \\
&+ C \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\rho_\epsilon w_\theta(\sigma)|^2\, dx d\sigma.
\end{aligned}$$

If we use the hyperbolicity condition and the Poincaré inequality on $\int_\omega |\rho_\epsilon w_\theta|^2 \, dX_2$ and since $\rho_\epsilon w_\theta (t, X_1, .)$ vanishes on the boundary of ω for a.e. (t, X_1), then (17) gives

$$\begin{aligned}
&\int_{\Delta_{\epsilon/2}\times\omega} (\rho_\epsilon w_\theta')^2 \, dx + \theta \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_1} w_\theta|^2 \, dx + \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_2} w_\theta|^2 \, dx \\
&\quad \le C\theta^\alpha \left(\theta \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} w_\theta|^2 \, dx d\sigma + \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_\theta|^2 \, dx d\sigma \right) \\
&\quad + C \int_0^t \left(\int_{\Delta_{\epsilon/2}\times\omega} |\rho_\epsilon w_\theta'(\sigma)|^2 \, dx + \theta \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_1} w_\theta|^2 \, dx \right. \\
&\quad \left. + \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_2} w_\theta|^2 \, dx \right) d\sigma + C \int_{\Delta_{\epsilon/2}\times\omega} |w_{\theta,1}|^2 \, dx \\
&\quad + C\theta^{2\alpha} \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} u_{\theta,0}|^2 \, dx + C \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_{\theta,0}|^2 \, dx.
\end{aligned}$$

Using the Gronwall inequality, we deduce that

$$\begin{aligned}
&\int_{\Delta_{\epsilon/2}\times\omega} (\rho_\epsilon w_\theta'(t))^2 \, dx + \theta \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_1} w_\theta(t)|^2 \, dx \\
&+ \int_{\Delta_{\epsilon/2}\times\omega} \rho_\epsilon^2 |\nabla_{X_2} w_\theta(t)|^2 \, dx \le C\theta^\alpha \left(\theta \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} w_\theta|^2 \, dx d\sigma \right. \\
&\left. + \int_0^t \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_\theta|^2 \, dx d\sigma \right) + C \int_{\Delta_{\epsilon/2}\times\omega} |w_{\theta,1}|^2 \, dx \\
&+ C\theta^{2\alpha} \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} u_{\theta,0}|^2 \, dx + C \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_{\theta,0}|^2 \, dx.
\end{aligned}$$

This inequality is verified for any t of $(0, T)$, then by (14) and (15), we obtain

$$\begin{aligned}
&\sup_{0\le t\le T} \int_{\Delta_\epsilon\times\omega} (w_\theta'(t))^2 \, dx + \theta \sup_{0\le t\le T} \int_{\Delta_\epsilon\times\omega} |\nabla_{X_1} w_\theta(t)|^2 \, dx \\
&+ \sup_{0\le t\le T} \int_{\Delta_\epsilon\times\omega} |\nabla_{X_2} w_\theta(t)|^2 \, dx \le C\theta^\alpha \left(\theta \sup_{0\le t\le T} \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} w_\theta(t)|^2 \, dx \right. \\
&\left. + \sup_{0\le t\le T} \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_\theta(t)|^2 \, dx \right) + C \int_{\Delta_{\epsilon/2}\times\omega} |w_{\theta,1}|^2 \, dx \\
&+ C\theta^{2\alpha} \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} u_{\theta,0}|^2 \, dx + C \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_{\theta,0}|^2 \, dx, \qquad (20)
\end{aligned}$$

since $\rho_\epsilon = 1$ on Δ_ϵ. We take $\epsilon = \frac{\varepsilon}{2^k}$ for $k = \widehat{0, \tau - 1}$ in (20) and we set

$$\chi_\epsilon^\theta = C \int_{\Delta_{\epsilon/2}\times\omega} |w_{\theta,1}|^2 \, dx + C\theta^{2\alpha} \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_1} u_{\theta,0}|^2 \, dx + C \int_{\Delta_{\epsilon/2}\times\omega} |\nabla_{X_2} w_{\theta,0}|^2 \, dx.$$

If we vary k from 0 to $\tau-1$, we obtain

$$\begin{aligned}
&\sup_{0\le t\le T}\int_{\Delta_\epsilon\times\omega}\left(w_\theta'(t)\right)^2dx+\theta\sup_{0\le t\le T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx\\
&+\sup_{0\le t\le T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\le\ C\theta^\alpha\left(\theta\sup_{0\le t\le T}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx\right.\\
&\left.+\sup_{0\le t\le T}\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\right)+\chi_\varepsilon^\theta\le\cdots\le\\
&C\theta^{\tau\alpha}\left(\theta\sup_{0\le t\le T}\int_{\Delta_{\frac{\varepsilon}{2^\tau}}\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx+\sup_{0\le t\le T}\int_{\Delta_{\frac{\varepsilon}{2^\tau}}\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\right)\\
&+C(\chi_\varepsilon^\theta+\theta^\alpha\chi_{\frac{\varepsilon}{2}}^\theta+\cdots+\theta^{(\tau-1)\alpha}\chi_{\frac{\varepsilon}{2^{\tau-1}}}^\theta).
\end{aligned}\tag{21}$$

Thus, the estimate of $u_\theta-u_\infty$ depends on the proximity of the initial conditions, and the estimate of the quantity $\sup_{0\le t\le T}\int_{\Delta_{\frac{\varepsilon}{2^\tau}}\times\omega}\left|\nabla\left(u_\theta-u_\infty\right)(t)\right|^2dx$. The following lemma gives an estimate of u_θ.

Lemma 2. *Under the preceding conditions, we have*

$$\begin{aligned}
&\sup_{0\le t\le T}\int_{\Delta\times\omega}\left|\nabla_{X_2}u_\theta(t)\right|^2dx\le C\left(\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx+1\right),\\
&\sup_{0\le t\le T}\int_{\Delta\times\omega}\left|\nabla_{X_1}u_\theta(t)\right|^2dx\le\frac{C}{\theta}\left(\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx+1\right),\\
&\sup_{0\le t\le T}\int_{\Delta\times\omega}\left|u_\theta'(t)\right|^2dx\le C\left(\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx+1\right)
\end{aligned}\tag{22}$$

where C is independent of θ.

Proof. Using the energy equality [6, lemme 8.3. p. 298], (8), (9) and (10), we obtain

$$\begin{aligned}
&\int_{\Delta\times\omega}\left|u_\theta'(t)\right|^2dx+\lambda\theta\int_{\Delta\times\omega}\left|\nabla_{X_1}u_\theta(t)\right|^2dx+\lambda'\int_{\Delta\times\omega}\left|\nabla_{X_2}u_\theta(t)\right|^2dx\\
&\le\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2dx+C\int_{\Delta\times\omega}\left|\nabla u_{\theta,0}\right|^2dx+C\theta\int_0^t\int_{\Delta\times\omega}\left|\nabla_{X_1}u_\theta(\sigma)\right|^2dxd\sigma\\
&+C\int_0^t\int_{\Delta\times\omega}\left|\nabla_{X_2}u_\theta(\sigma)\right|^2dxd\sigma+C\theta^{1/2}\int_0^t\int_{\Delta\times\omega}\left|\nabla_{X_1}u_\theta(\sigma)\right|\left|\nabla_{X_2}u_\theta(\sigma)\right|dxd\sigma\\
&+C\int_0^t\int_{\Delta\times\omega}\left|u_\theta(\sigma)\right|\left|u_\theta'(\sigma)\right|dxd\sigma+C\int_0^t\int_{\Delta\times\omega}\left|f(\sigma)\right|\left|u_\theta'(\sigma)\right|dxd\sigma,
\end{aligned}$$

where C is independent of θ. Using Young inequality with $\mu=\theta^{1/2}$ and $\mu=1$, and applying the Poincaré inequality to the term $\int_\omega\left|u_\theta(\sigma,X_1,X_2)\right|^2dX_2$ for a.e. (σ,X_1), implies that

$$\begin{aligned}
&\int_{\Delta\times\omega}\left|u_\theta'(t)\right|^2dx+\theta\left|\nabla_{X_1}u_\theta(t)\right|^2dx+\left|\nabla_{X_2}u_\theta(t)\right|^2dx\le C\int_{\Delta\times\omega}\left|u_{\theta,1}\right|^2+\left|\nabla u_{\theta,0}\right|^2dx\\
&+C\int_0^t\int_{\Delta\times\omega}\left|u_\theta'(\sigma)\right|^2dx+\theta\left|\nabla_{X_1}u_\theta(\sigma)\right|^2+\left|\nabla_{X_2}u_\theta(\sigma)\right|^2dxd\sigma+C\left\|f\right\|_{L^2(Q_\infty)}^2.
\end{aligned}$$

Then by Gronwall inequality, we obtain the estimates of Lemma 2. □

Using Lemma 2, inequality (21) implies

$$\sup_{0\le t\le T}\int_{\Delta_\epsilon\times\omega}\left(w_\theta'(t)\right)^2dx+\theta\sup_{0\le t\le T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx+$$
$$\sup_{0\le t\le T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\le C\left(\theta^{\tau\alpha}+\Xi_{\theta,\tau}\right),$$

where C is a constant independent of θ, and $\Xi_{\theta,\tau}$ is given by

$$\Xi_{\theta,\tau}=\theta^{\tau\alpha}\left(\int_{\Delta\times\omega}|u_{\theta,1}|^2dx+\int_{\Delta\times\omega}|\nabla u_{\theta,0}|^2dx\right)+\chi^\theta_\varepsilon+\theta^\alpha\chi^\theta_{\frac{\varepsilon}{2}}+\cdots+\theta^{(\tau-1)\alpha}\chi^\theta_{\frac{\varepsilon}{2^{\tau-1}}}. \tag{23}$$

Then, for $r>0$, we take τ such that $\tau\alpha>r$ to give a basic theorem in the study of the asymptotic behavior.

Theorem 2. *Under the conditions (3)-(9), for all $r>0$ and $\varepsilon>0$, there exists a constant $C>0$ independent of θ, such that*

$$\sup_{0\le t\le T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_1}w_\theta(t)\right|^2dx\le\frac{C}{\theta}\left(\theta^r+\Xi_{\theta,r}\right), \tag{24}$$

$$\sup_{0\le t\le T}\int_{\Delta_{2\varepsilon}\times\omega}\left(w_\theta'(t)\right)^2dx+\sup_{0\le t\le T}\int_{\Delta_\varepsilon\times\omega}\left|\nabla_{X_2}w_\theta(t)\right|^2dx\le C\left(\theta^r+\Xi_{\theta,r}\right), \tag{25}$$

where u_θ and u_∞ are solutions of (1) and (2) respectively, and $\Xi_{\theta,r}$ is given by (23).

4. **The asymptotic behavior.** Let Φ be an open subset of Ω with boundary disjoint of $\partial\Delta\times\omega$. Let $\varepsilon=\operatorname{dist}(\Phi,\partial\Delta\times\omega)$. We have $\Phi\subset\Delta_\varepsilon\times\omega$. We introduce the functional spaces

$$W(0,T;H^1(\Phi),L^2(\Phi))=\left\{u\in L^2(0,T;H^1(\Phi)),\ u'\in L^2(0,T;L^2(\Phi))\right\},$$
$$W^\infty(0,T;H^1(\Phi),L^2(\Phi))=\left\{u\in L^\infty(0,T;H^1(\Phi)),\ u'\in L^\infty(0,T;L^2(\Phi))\right\}.$$

According to (24) and (25), we see that the initial conditions play an important role to estimate w_θ. Therefore, let us assume that

$$\chi^\theta_{\frac{\varepsilon}{2^k}}=O(\theta^{r-k\alpha})\quad k=\widehat{0,\tau-1} \tag{26}$$

$$\int_{\Delta\times\omega}|u_{\theta,1}|^2dx+\int_{\Delta\times\omega}|\nabla u_{\theta,0}|^2dx=O(1) \tag{27}$$

with $\tau=\left[\frac{r}{\alpha}\right]+1$. The result of convergence is the following.

Corollary 1. *Let Φ be an open subset of Ω with boundary disjoint to $\partial\Delta\times\omega$. We suppose that (26), (27) and the conditions in Theorem 2 are satisfied. Then $u_\theta\to u_\infty$ in $W^\infty(0,T;H^1(\Phi),L^2(\Phi))$ and, for any $r'=r-\alpha>0$, there exists a constant $C>0$ independent of θ, such that the estimates*

$$\sup_{0\le t\le T}\left|(u_\theta-u_\infty)'(t)\right|_{L^2(\Phi)}\le C\theta^{r'},\quad\sup_{0\le t\le T}\left\|(u_\theta-u_\infty)(t)\right\|_{H^1(\Phi)}\le C\theta^{r'} \tag{28}$$

hold for any $\theta>0$.

We now ask if the conditions of the type (26) and (27), are necessary for convergence. It is the case for the space $W^\infty(0,T;H^1(\Delta_\varepsilon\times\omega),L^2(\Delta_\varepsilon\times\omega))$. For the space $W(0,T;H^1(\Delta_\varepsilon\times\omega),L^2(\Delta_\varepsilon\times\omega))$ we start by (17), and using (8), (9) and (10), we deduce

$$\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|w_{\theta,1}\right|^2dx+\theta\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_1}u_{\theta,0}\right|^2dx$$
$$+\int_{\Delta_{\epsilon/2}\times\omega}\rho_\epsilon^2\left|\nabla_{X_2}w_{\theta,0}\right|^2dx\le C\left(H'(t)+H(t)\right)\le C\left(e^tH(t)\right)'$$

where

$$H(t)=\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left(w_\theta'\right)^2dxd\sigma+C\int_0^t\int_{\Delta_{\epsilon/2}\times\omega}\left|\nabla w_\theta\right|^2dxd\sigma.$$

We take $\epsilon=2\varepsilon$ and $\varepsilon'<\varepsilon$, we choose $\rho_{2\varepsilon}$ such that $\rho_{2\varepsilon}>0$ on $\overline{\Delta}_{\varepsilon+\varepsilon'}$ then, if we integrate from 0 to T, and for $c=\min_{\overline{\Delta}_{\varepsilon+\varepsilon'}}\rho_{2\varepsilon}$, we obtain

$$\int_{\Delta_{\varepsilon+\varepsilon'}\times\omega}\left|w_{\theta,1}\right|^2dx+\theta\int_{\Delta_{\varepsilon+\varepsilon'}\times\omega}\left|\nabla_{X_1}u_{\theta,0}\right|^2dx+\int_{\Delta_{\varepsilon+\varepsilon'}\times\omega}\left|\nabla_{X_2}w_{\theta,0}\right|^2dx\le CH(T).$$

Thus, we can state the theorem.

Theorem 3. *Necessary conditions to have the convergence $u_\theta\longrightarrow u_\infty$ in the space $W^\infty(0,T;H^1(\Delta_\varepsilon\times\omega),L^2(\Delta_\varepsilon\times\omega))$ are*

$$u_{\theta,0}\longrightarrow u_0\ \text{in}\ H^1(\Delta_\varepsilon\times\omega)\ \text{and}\ \ u_{\theta,1}\longrightarrow u_1\ \text{in}\ L^2(\Delta_\varepsilon\times\omega).$$

In the space $W(0,T;H^1(\Delta_\varepsilon\times\omega),L^2(\Delta_\varepsilon\times\omega))$, they are

$$\nabla_{X_2}u_{\theta,0}\longrightarrow\nabla_{X_2}u_0,\ u_{\theta,1}\longrightarrow u_1\ \text{in}\ L^2(\Delta'\times\omega)\ \text{and}\ \int_{\Delta'\times\omega}\left|\nabla_{X_1}u_{\theta,0}\right|^2dx=o(1/\theta),$$

for any compact Δ' of Δ_ε.

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E-mail address: bernard.brighi@uha.fr
E-mail address: senoussi.guesmia@uha.fr