Asymptotic of characters of symmetric groups and limit shape of Young diagrams

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Outline of the talk

1. Character values of symmetric groups
   - An exact formula
   - Asymptotic behaviours

2. Application : limit shape of Young diagrams

Young symmetrizer

Let T be a filling of $\lambda = (3, 2, 2)$.

2	3	6
4	1	
7	5	

Consider :

row-stabilizer $RS(T) = S_{\{2,3,6\}} \times S_{\{1,4\}} \times S_{\{5,7\}}$.

column-stabilizer $CS(T) = S_{\{2,4,7\}} \times S_{\{1,3,5\}}$.

Young symmetrizer

Let T be a filling of $\lambda = (3, 2, 2)$.

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$$\frac{n! s_\lambda}{\dim \lambda} = \sum_{\substack{\sigma_1 \in RS(T) \\ \sigma_2 \in CS(T)}} (-1)^{\sigma_2} p_{\text{type}(\sigma_2 \sigma_1)}$$

Work in progress with P. Śniady : analog for zonal polynomials

An equivalent formulation

Recall : character value $\chi^\lambda(\mu)$ fulfills $s_\lambda = \sum_\mu \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu$.

$$\frac{n!\chi^\lambda(\pi)}{\dim \lambda} = \sum_{\sigma_2\sigma_1=\pi} (-1)^{\sigma_2} N'_{\sigma_1,\sigma_2}(\lambda),$$

where

Definition

$N'_{\sigma_1,\sigma_2}(\lambda)$ is the number of bijections $f : \{1,\dots,n\} \simeq \lambda$ such that for all i, $f(i)$ and $f(\sigma_1(i))$ (resp. $f(\sigma_2(i))$) are in the same row (resp. column).

Nice behaviour on short permutations

If $\pi \in S_k \overset{\iota}{\hookrightarrow} S_n$, $(\pi(i) = i \ \forall \ i > k)$,
then $N'_{\sigma_1,\sigma_2}(\lambda) = 0$ unless $\sigma_1(i) = \sigma_2(i) = \pi(i) \ \forall \ i > k$.

In this case the formula becomes :

$$\frac{n! \chi^\lambda(\iota(\pi))}{\dim \lambda} = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_2\sigma_1=\pi}} (-1)^{\sigma_2} N'_{\iota(\sigma_1),\iota(\sigma_2)}(\lambda)$$

But $N'_{\iota(\sigma_1),\iota(\sigma_2)} = \#\{f : \{1,\dots,k\} \hookrightarrow \lambda \text{ with usual conditions}\} \cdot \underbrace{(n-k)!}_{\text{choices of the places of } k+1,\dots,n}$

Nice behaviour on short permutations

Definition

$N'_{\sigma_1,\sigma_2}(\lambda)$ is the number of injections $f : \{1, \dots, k\} \hookrightarrow \lambda$ such that, for all i, $f(i)$ and $f(\sigma_1(i))$ (resp. $f(\sigma_2(i))$) are in the same row (resp. column).

$$\Sigma_\pi(\lambda) := \frac{n \cdot (n-1) \dots (n-k+1) \chi^\lambda(\iota(\pi))}{\dim \lambda} = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_2\sigma_1=\pi}} (-1)^{\sigma_2} N'_{\sigma_1,\sigma_2}(\lambda).$$

Forgetting injectivity

Definition

$N_{\sigma_1,\sigma_2}(T)$ is the number of functions $f : \{1,\dots,k\} \to \lambda$ such that, for all i, $f(i)$ and $f(\sigma_1(i))$ (resp. $f(\sigma_2(i))$) are in the same row (resp. column).

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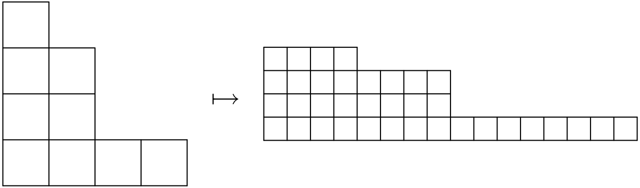
Idea of proof : the total contribution of a non-injective function in rhs is easily seen to be 0. □

Asymptotics is easy to read on this formula

Model : fix a permutation $\pi_0 \in S_k$ and a partition $\lambda_0 \vdash k$
Consider $\pi = \iota(\pi_0)$ (*i.e.* we just add fixpoints) and $\lambda = c \cdot \lambda_0 = \lambda$ multiplied by c (*i.e.* horizontal lengths are multiplied by c)



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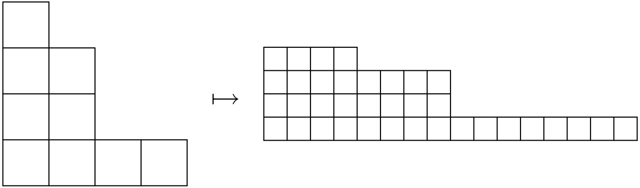

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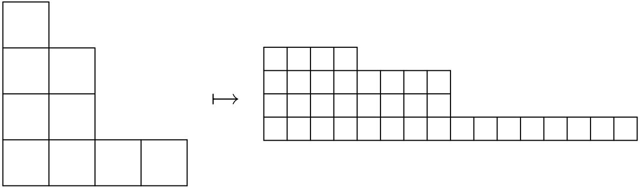

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$$N_{\pi,\mathrm{Id}_k}(\lambda) = \prod_{\mu_i \in \mathrm{type}(\pi)} \left(\sum_j \lambda_j^{\mu_i} \right)$$

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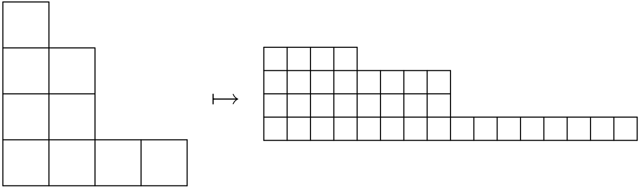

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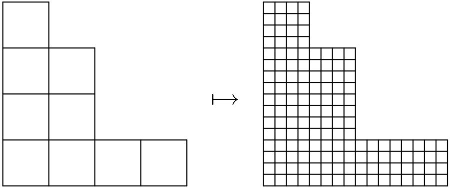

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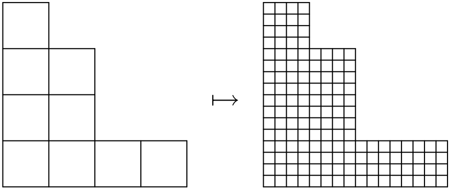

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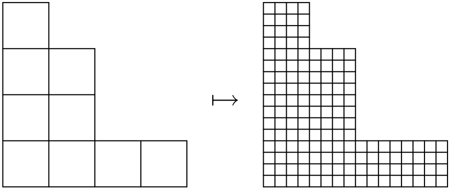

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$$\sum_{\substack{\sigma_1\sigma_2=\pi \\ |C(\sigma_1)|+|C(\sigma_2)| \text{ maximal}}} \pm N_{\sigma_1,\sigma_2}(\lambda)$$

Free cumulants

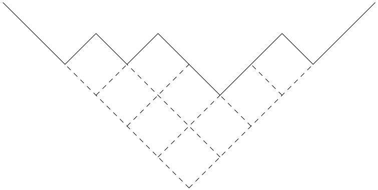
$$\Sigma_\pi(c \bullet \lambda) = \sum_{\substack{\sigma_1\sigma_2=\pi \\ |C(\sigma_1)|+|C(\sigma_2)| \text{ maximal}}} \pm N_{\sigma_1,\sigma_2}(\lambda)$$

But, $\left\{ \begin{array}{c} \sigma_1\sigma_2 = \pi \\ |C(\sigma_1)| + |C(\sigma_2)| \text{ maximal} \end{array} \right\} \simeq \prod NC_{\mu_i} \simeq \prod \text{Trees}(\mu_i)$.

With generating series, one can prove (Rattan, 2006)

$$rhs = \prod R_{\mu_i+1}(\lambda)$$

R_k : free cumulants defined from the shape ω_λ by Biane (1998).



Remarks

Works in more general context than sequences $c \cdot \lambda_0$ and $c \bullet \lambda_0$ (in fact, works as soon as a sequence of Young diagram has a *limit*)

These results were already known (Vershik & Kerov 81, Biane 98), but :

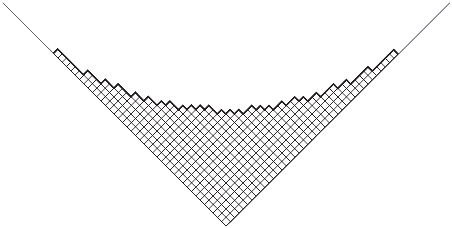
- we provide unified approach of both cases ;
- our bound for error terms are better.

Description of the problem

Consider Plancherel's probability measure on Young diagrams of size n

$$P(\lambda) = \frac{(\dim \lambda)^2}{n!}$$

Question : is there a limit shape for (renormalized rotated) Young diagram taken randomly with Plancherel's measure when $n \to \infty$?



Normalized character values have simple expectations !

Fix $\pi \in \mathfrak{S}_n$. Let us consider the random variable :

$$X_\pi(\lambda) = \chi^\lambda(\pi) = \frac{\mathrm{tr}\left(\rho_\lambda(\pi)\right)}{\dim V_\lambda}.$$

Let us compute its expectation :

$$\mathbb{E}(X_\pi) = \frac{1}{n!} \sum_{\lambda \vdash n} (\dim V_\lambda) \cdot \mathrm{tr}\left(\rho_\lambda(\pi)\right)$$

$$= \frac{1}{n!} \mathrm{tr}_{\left(\bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda}\right)}(\pi) = \frac{1}{n!} \mathrm{tr}_{\mathbb{C}[\mathfrak{S}_n]}(\pi)$$

Last expression is easy to evaluate :

$$\mathbb{E}(X_\pi) = \delta_{\pi, \mathrm{Id}_n}$$

Convergence of cumulants

Recall : we proved that $\prod_i R_{k_i+1} \approx \Sigma_{k_1,\dots,k_r}$.
Thus

$$\mathbb{E}(R_2) \approx n$$
$$\mathbb{E}(R_i) \approx 0 \text{ if } i > 2$$
$$\mathrm{Var}(R_i) \approx 0 \text{ if } i \geq 2$$

Convergence of cumulants

Recall : we proved that $\prod_i R_{k_i+1} \approx \Sigma_{k_1,\dots,k_r}$.
Thus

$$\lim \mathbb{E}\left(R_2/n\right) = 1$$
$$\lim \mathbb{E}\left(R_i/\sqrt{n}^i\right) = 0 \text{ if } i > 2$$
$$\lim \mathrm{Var}\left(R_i/\sqrt{n}^i\right) = 0 \text{ if } i \geq 2$$

Easy to make it formal because $R_k \in \mathrm{Vect}(\Sigma_\pi)$.

$\Rightarrow$ Random variables $R_i/\sqrt{n}^i$ converge in probability towards the sequence $(0, 1, 0, 0, \dots)$

General lemma from Kerov :

$$\text{convergence of cumulants} \Rightarrow \text{convergence of Young diagrams}$$

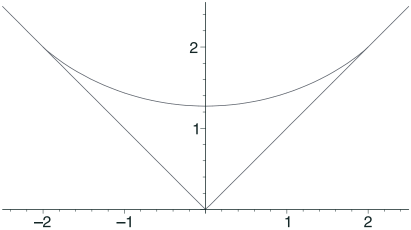
Existence of a limiting curve

Theorem (Logan and Shepp 77, Kerov and Vershik 77)

Let us take randomly (with Plancherel measure) a sequence of Young diagram λ_n of size n. Then, in probability, for the uniform convergence topology on continuous functions, one has :

$$r_{45^\circ}(h_{1/\sqrt{n}}(\lambda_n)) \to \delta_\Omega,$$

where Ω is an explicit function drawn here :



Convergence of q-Plancherel measure

Case where expectation of character values are big :

- there can not be a limit shape after dilatation.
- we use the first approximation for characters $\Sigma_\pi(\lambda) \approx \prod p_{\mu_i}(\lambda)$.

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Example : $q-$Plancherel measure ($q < 1$)

- defined using representation of Hecke algebras
- one can prove

$$\mathbb{E}_q(\Sigma_\pi) = \frac{(1-q)^{|\mu|}}{\prod_i 1 - q^{\mu_i}} n(n-1)\dots(n-|\mu|+1)$$

Thus

$$\mathbb{E}_q(p_k) \approx \frac{(1-q)^k}{\prod_i 1 - q^k} n^k \qquad \mathrm{Var}_q(p_k) \approx 0$$

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Thus

$$\lim \mathbb{E}_q(p_k/n^k) = \frac{(1-q)^k}{\prod_i 1 - q^k} \qquad \lim \mathrm{Var}_q(p_k/n^k) = 0$$

Convergence of q-Plancherel measure

Theorem (F., Méliot, 2010)

Let $q < 1$. In probability, under q-Plancherel measure,

$$\forall k \geq 1, \; \frac{p_k(\lambda)}{|\lambda|^k} \longrightarrow_{M_{n,q}} \frac{(1-q)^k}{1-q^k}.$$

Moreover,

$$\forall i \geq 1, \; \frac{\lambda_i}{n} \longrightarrow_{M_{n,q}} (1-q)\, q^{i-1};$$

We also obtained the second-order asymptotics.

End of the talk

Thank you for listening

Questions ?