Caractères irréductibles du groupe symétrique et grands diagrammes de Young

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Young symmetrizer

Let T be a filling of $\lambda = (3, 2, 2)$.

2	3	6
4	1	
7	5	

Consider :

row-stabilizer $RS(T) = S_{\{2,3,6\}} \times S_{\{1,4\}} \times S_{\{5,7\}}$. Let $a_T = \sum_{\sigma \in RS(T)} \sigma$.

column-stabilizer $CS(T) = S_{\{2,4,7\}} \times S_{\{1,3,5\}}$. Let $b_T = \sum_{\sigma \in CS(T)} (-1)^{\sigma} \sigma$.

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$$\text{Then } (a_T \cdot b_T)^2 = \alpha_T (a_T \cdot b_T) \text{ for some } \alpha_T.$$

$$\Rightarrow \text{If} : p_T = \frac{(a_T \cdot b_T)}{\alpha_T}, \text{ then } r_{p_T} : x \mapsto x p_T \text{ is a projector.}$$

Fact : its image $\mathbb{C}[\mathfrak{S}_n] p_T$ is the irreducible representation of $\mathfrak{S}_n$ associated to λ ($\mathfrak{S}_n$ acts by left multiplication).

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Then $(a_T \cdot b_T)^2 = \alpha_T (a_T \cdot b_T)$ for some α_T.

$\Rightarrow$ If : $p_T = \frac{(a_T \cdot b_T)}{\alpha_T}$, then $r_{p_T} : x \mapsto x p_T$ is a projector.

Rk : the proof does not give the value of α_T. Note that

$$\mathrm{tr}(p_T) = n!/\alpha_T = \dim \lambda.$$

Computing the trace (1)

- we would like to compute the trace of :

$$\rho_\pi : \begin{array}{rcl} \mathbb{C}[\mathfrak{S}_n]p_T & \longrightarrow & \mathbb{C}[\mathfrak{S}_n]p_T \\ x & \longmapsto & \pi \cdot x \end{array}$$

it seems hard : there is no explicit basis.

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it seems hard : there is no explicit basis.

- we use the following trick : consider

$$\varphi_\pi : \begin{array}{ccc} \mathbb{C}[\mathfrak{S}_n] & \longrightarrow & \mathbb{C}[\mathfrak{S}_n] \\ x & \longmapsto & \pi \cdot x \cdot p_T \end{array}$$

Then, if we write $\mathbb{C}[\mathfrak{S}_n] = \mathbb{C}[\mathfrak{S}_n]p_T \oplus Ker(p_T)$,

$$\varphi_\pi = \rho_\pi \oplus 0.$$

In particular $\mathrm{tr}(\rho_\pi) = \mathrm{tr}(\varphi_\pi)$.

Computing the trace (2)

$$\mathrm{tr}(\varphi_\pi) = \sum_{\tau \in \mathfrak{S}_n} \langle \pi \tau p_T, \tau \rangle$$

$$\chi^\lambda(\pi) = \frac{\dim \lambda}{n!} \sum_\tau \langle \pi \tau a_T b_T, \tau \rangle$$

$$\frac{n! \chi^\lambda(\pi)}{\dim \lambda} = \sum_{\tau \in \mathfrak{S}_n} \sum_{\substack{\sigma_1 \in RS(T) \\ \sigma_2 \in CS(T)}} (-1)^{\sigma_2} \langle \pi \tau \sigma_1 \sigma_2, \tau \rangle$$

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Equivalently, in terms of Schur functions :

$$\frac{n! s_\lambda}{\dim \lambda} = \sum_{\substack{\sigma_1 \in RS(T) \\ \sigma_2 \in CS(T)}} (-1)^{\sigma_2} p_{\mathrm{type}(\sigma_2 \sigma_1)}$$

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Work in progress with P. Śniady : analog for zonal polynomials

An equivalent formulation

$$\frac{n!\chi^\lambda(\pi)}{\dim \lambda} = \sum_{\sigma_2\sigma_1=\pi} (-1)^{\sigma_2} \sum_T [\sigma_1 \in RS(T)] \cdot [\sigma_2 \in CS(T)]$$

An equivalent formulation

Definition

$N'_{\sigma_1,\sigma_2}(\lambda)$ is the number of bijections $f : \{1,\dots,n\} \simeq \lambda$ such that

$$[\sigma_1 \in RS(T)] \cdot [\sigma_2 \in CS(T)] = 1$$

i.e. for all i, $f(i)$ and $f(\sigma_1(i))$ (resp. $f(\sigma_2(i))$) are in the same row (resp. column).

$$\frac{n!\chi^\lambda(\pi)}{\dim \lambda} = \sum_{\sigma_2\sigma_1=\pi} (-1)^{\sigma_2} N'_{\sigma_1,\sigma_2}(\lambda)$$

Nice behaviour on short permutations

If $\pi \in S_k \overset{\iota}{\hookrightarrow} S_n$, $(\pi(i) = i \ \forall\ i > k)$,
then $N'_{\sigma_1,\sigma_2}(\lambda) = 0$ unless $\sigma_1(i) = \sigma_2(i) = \pi(i) \ \forall\ i > k$.

In this case the formula becomes :

$$\frac{n!\chi^\lambda(\pi)}{\dim\lambda} = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_2\sigma_1 = \pi}} (-1)^{\sigma_2} N'_{\iota(\sigma_1),\iota(\sigma_2)}(\lambda)$$

But $N'_{\iota(\sigma_1),\iota(\sigma_2)} = \#\{f : \{1,\dots,k\} \hookrightarrow \lambda \text{ with usual conditions}\} \cdot \underbrace{(n-k)!}_{\text{choices of the places of } k+1,\dots,n}$

Nice behaviour on short permutations

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$N'_{\sigma_1,\sigma_2}(\lambda)$ is the number of injections $f : \{1,\dots,k\} \hookrightarrow \lambda$ such that

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$$\Sigma_\pi(\lambda) := \frac{n\cdot(n-1)\dots(n-k+1)\chi^\lambda(\pi)}{\dim\lambda} = \sum_{\substack{\sigma_1,\sigma_2\in S_k \\ \sigma_2\sigma_1=\pi}} (-1)^{\sigma_2} N'_{\sigma_1,\sigma_2}(\lambda).$$

Forgetting injectivity

Definition

$N_{\sigma_1,\sigma_2}(T)$ is the number of functions $f : \{1, \dots, k\} \to \lambda$ such that

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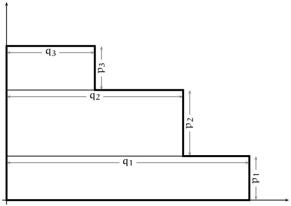
Idea of proof : the total contribution of a non-injective function in rhs is easily seen to be 0. □

Stanley's coordinates



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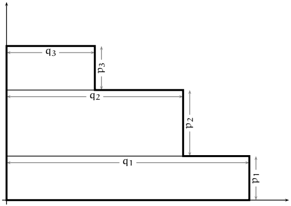
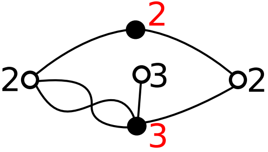




weight : $p_2^2 p_3 q_2 q_3$

Theorem

$$\Sigma_\pi(\lambda) = (-1)^k \sum_{(M,\phi)} \prod_{w \in V_\circ(M)} p_{\phi(w)} \prod_{b \in V_\bullet(M)} (-q_{\phi(b)}),$$

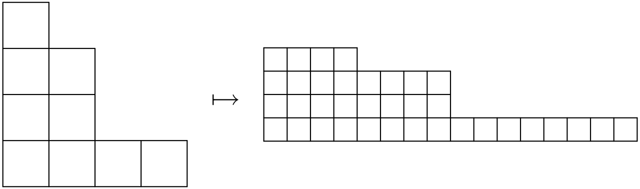
where (M,ϕ) runs over unions of vertex-labeled marked bipartite maps st :

- *the lengths of the faces correspond to the lengths of the cycles of π ;*
- *the label of $b \in V_\bullet$ is the maximum of the labels of its neighbours.*

Analog for zonal polynomials : same with maps on locally oriented surfaces !

Asymptotics is easy to read on this formula

Model : fix a permutation π and multiply the diagram λ by a constant c (*i.e.* multiply **q**'s by a constant c).



Question : asymptotics of $\frac{\chi^\lambda(\pi)}{\dim \lambda}$?

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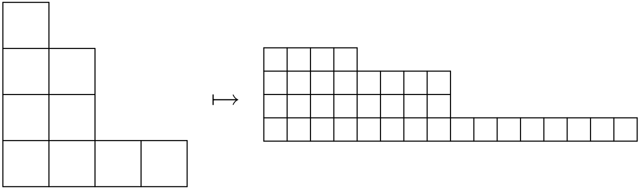

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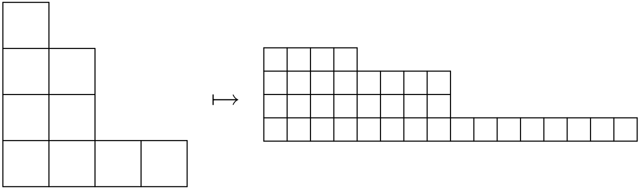

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Dominant term of $\Sigma_\pi(\lambda)$:

$$N_{\pi,\mathrm{Id}_k}(\lambda) = \prod_{\mu_i \in \mathrm{type}(\pi)} \left(\sum_j (\lambda_j \qquad)^{\mu_i} \right)$$

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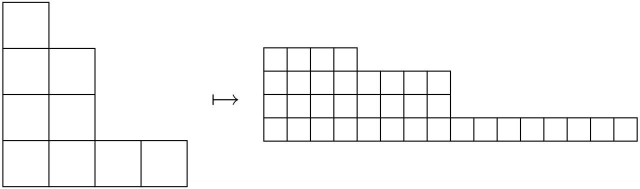

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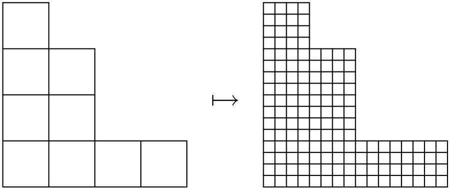
Model : fix a permutation π and dilate the diagram λ by a constant c (*i.e.* multiply the **p**'s and **q**'s by a constant c).



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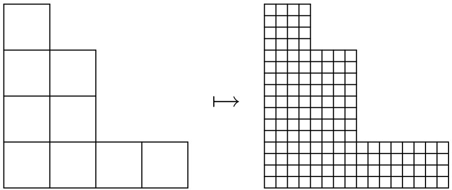

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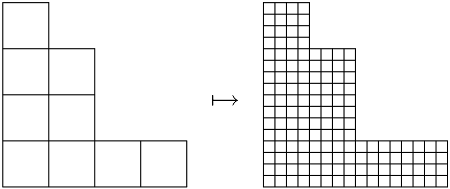

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Dominant term of $\Sigma_\pi(\lambda)$:

$$\sum_{\substack{(M,\phi)\dots \\ M \text{ forest}}} \pm \prod_{w \in V_\circ(M)} p_{\phi(w)} \prod_{b \in V_\bullet(M)} q_{\phi(b)} =: \prod_{\mu_i \in \text{type}(\pi)} R_{\mu_i+1}(\lambda)$$

Remarks

These results were already known, but :

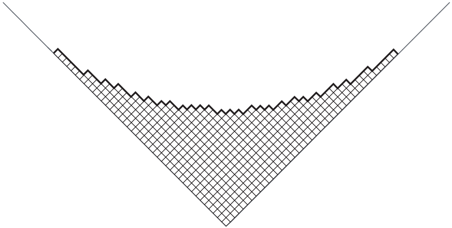
- we provide unified approach of both cases ;
- our bound for error terms are better.

Description of the problem

Consider Plancherel's probability measure on Young diagrams of size n

$$P(\lambda) = \frac{(\dim \lambda)^2}{n!}$$

Question : is there a limit shape for (renormalized rotated) Young diagram taken randomly with Plancherel's measure when $n \to \infty$?



Normalized character values have simple expectations !

Fix $\pi \in \mathfrak{S}_n$. Let us consider the random variable :

$$X_\pi(\lambda) = \chi^\lambda(\pi) = \frac{\mathrm{tr}\left(\rho_\lambda(\pi)\right)}{\dim V_\lambda}.$$

Let us compute its expectation :

$$\mathbb{E}(X_\pi) = \frac{1}{n!} \sum_{\lambda \vdash n} (\dim V_\lambda) \cdot \mathrm{tr}\left(\rho_\lambda(\pi)\right)$$

$$= \frac{1}{n!} \mathrm{tr}_{\left(\bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda}\right)}(\pi) = \frac{1}{n!} \mathrm{tr}_{\mathbb{C}[\mathfrak{S}_n]}(\pi)$$

Last expression is easy to evaluate :

$$\mathbb{E}(X_\pi) = \delta_{\pi, \mathrm{Id}_n}$$

Convergence of cumulants

Recall : we proved that $\prod_i R_{k_i+1} \approx \Sigma_{k_1,\dots,k_r}$.
Thus

$$\begin{aligned} \mathbb{E}(R_2) &\approx n \\ \mathbb{E}(R_i) &\approx 0 \text{ if } i > 2 \\ \mathrm{Var}(R_i) &\approx 0 \text{ if } i \geq 2 \end{aligned}$$

$\Rightarrow$ after a suitable renormalization, the random variables R_i tends in probability towards the sequence $(0, 1, 0, 0, \dots)$

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Easy to make it formal because $R_k \in \mathrm{Vect}(\Sigma_\pi)$.

Harder part : this implies convergence of Young diagrams (general result of Kerov)

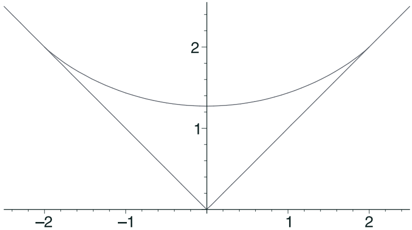
Existence of a limiting curve

Theorem (Logan and Shepp 77, Kerov and Vershik 77)

Let us take randomly (with Plancherel measure) a sequence of Young diagram λ_n of size n. Then, in probability, for the uniform convergence topology on continuous functions, one has :

$$r_{45^\circ}(h_{1/\sqrt{n}}(\lambda_n)) \to \delta_\Omega,$$

where Ω is an explicit function drawn here :



Convergence of q-Plancherel measure

Case where expectation of character values are big :

- there can not be a limit shape after dilatation.
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Example : $q-$Plancherel measure $(q < 1)$

- defined using representation of Hecke algebras
- one can prove

$$\mathbb{E}_q(\Sigma_\pi) = \frac{(1-q)^{|\mu|}}{\prod_i 1 - q^{\mu_i}} n(n-1)\dots(n-|\mu|+1)$$

Thus

$$\mathbb{E}_q(p_k) \approx \frac{(1-q)^k}{\prod_i 1 - q^k} n^k \operatorname{Var}_q(p_k) \approx 0$$

Convergence of q-Plancherel measure

Theorem (F., Méliot, 2010)

Let $q < 1$. In probability, under q-Plancherel measure,

$$\forall k \geq 1, \; \frac{p_k(\lambda)}{|\lambda|^k} \longrightarrow_{M_{n,q}} \frac{(1-q)^k}{1-q^k}.$$

Moreover,

$$\forall i \geq 1, \; \frac{\lambda_i}{n} \longrightarrow_{M_{n,q}} (1-q)\, q^{i-1};$$

We also obtained the second-order asymptotics.

Questions

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Qui va au bleu ?