Weighted dependency graphs

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Stochastic Processes and their Applications
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Central limit theorems

Theorem (De Moivre, Laplace, Lyapunov)

If $Y_1, Y_2, \dots$ are independent identically distributed variables with finite variance, and $X_n = \sum_{i=1}^n Y_i$, then

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\operatorname{Var} X_n}} \xrightarrow{d} \mathcal{N}(0,1). \qquad \text{(CLT)}$$

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Goal of the talk: give an extension of dependency graphs and applications to statistical mechanics models.

Weighted dependency graphs

Dependency graphs

Definition (Petrovskaya and Leontovich, 1982, Janson, 1988)

A graph L with vertex set A is a dependency graph for the family $\{Y_\alpha, \alpha \in A\}$ if

- if A_1 and A_2 are disconnected subsets in L, then $\{Y_\alpha, \alpha \in A_1\}$ and $\{Y_\alpha, \alpha \in A_2\}$ are independent.

Roughly: there is an edge between pairs of dependent random variables.

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Example 1

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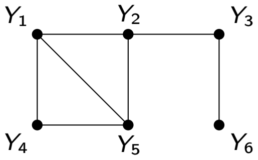

Y_2 and Y_4 are independent;
$\{Y_1, Y_4, Y_5\}$ independent from $\{Y_3, Y_6\}$.

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Example 2 (triangles in $G(n,p)$)

Consider $G = G(n,p)$. Let $A = \{\Delta \in \binom{[n]}{3}\}$ (set of potential triangles) and

$$\{\Delta_1,\, \Delta_2\} \in E_L \text{ iff } \Delta_1 \text{ and } \Delta_2 \text{ share an edge in } G.$$

Then L is a dependency graph for the family $\{\mathbf{1}_{\Delta \subset G}, \Delta \in \binom{[n]}{3}\}$.

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Note: L has degree $\mathcal{O}(n)$

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Janson's normality criterion

Setting: for each n,

- $\{Y_{n,i}, 1 \leq i \leq N_n\}$ is a family of bounded random variables; $|Y_{n,i}| < M$ a.s.
- we have a dependency graph L_n with maximal degree $\Delta_n - 1$.
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Theorem (Janson, 1988)

Assume that $\left(\frac{N_n}{\Delta_n}\right)^{1/s} \frac{\Delta_n}{\sigma_n} \to 0$ *for some integer* s. *Then* X_n *satisfies a CLT.*

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For triangles, $N_n = \binom{n}{3}$, $\Delta_n = \mathcal{O}(n)$, while $\sigma_n \asymp n^2$. (for fixed p)

Corollary

Fix p *in* $(0, 1)$*. Then the number* T_n *of triangles in* $G(n, p)$ *satisfies a CLT.*

(also true for $p_n \to 0$ with $np_n \to \infty$; originally proved by Rucinski, 1988).

Applications of dependency graphs to CLT results

- mathematical modelization of cell populations (Petrovskaya, Leontovich, 82);
- subgraph counts in random graphs (Janson, Baldi, Rinott, Penrose, 88, 89, 95, 03);
- Geometric probability (Avram, Bertsimas, Penrose, Yukich, Bárány, Vu, 93, 05, 07);
- pattern occurrences in random permutations (Bóna, Janson, Hitchenko, Nakamura, Zeilberger, 07, 09, 14).
- m-dependence (Hoeffding, Robbins, 53, . . . ; now widely used in statistics) is a special case.

(Some of these applications use variants of Janson's normality criterion, which are more technical to state and omitted here. . .)

A weighted variant

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- We need to quantify the dependence somehow: we'll use cumulants.

What are (mixed) cumulants?

- The r-th mixed cumulant κ_r of r random variables is a specific r-linear symmetric polynomial in joint moments. Examples:

$$\kappa_1(X) := \mathbb{E}(X), \quad \kappa_2(X,Y) := \mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\kappa_3(X,Y,Z) := \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).$$

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  Notation: $\kappa_r(X) := \kappa_r(X, \dots, X)$.

- If a set of variables can be split in two mutually independent sets, then its mixed cumulant vanishes.

- If, for each r big enough, we have $\kappa_r(X_n) = o(\mathrm{Var}(X_n)^{r/2})$, then X_n satisfies a CLT. (Janson, 1988)

Weighted dependency graphs

Definition (F., 2016)

A weighted graph $\widetilde{L}$ with vertex set A is a weighted dependency graph for the family $\{Y_\alpha, \alpha \in A\}$ if, for any $\alpha_1, \dots, \alpha_r$ in A,

$$\left|\kappa(Y_{\alpha_1}, \cdots, Y_{\alpha_r})\right| \leq C_r \, \mathcal{M}\left(\widetilde{L}[\alpha_1, \cdots, \alpha_r]\right).$$

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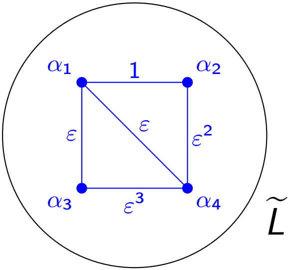

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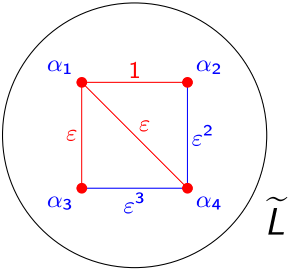
$\mathcal{M}(K)$: Maximum weight of a spanning tree of K.

In the example,
$\mathcal{M}\left(\widetilde{L}[\alpha_1, \cdots, \alpha_4]\right) = \varepsilon^2$.



Weighted dependency graphs: an example

Example (triangles in $G(n, M)$)

Consider $G = G(n, M_n)$, where $M_n = p\binom{n}{2}$. Let $A = \{\Delta \in \binom{[n]}{3}\}$ and

$$\mathrm{wt}_{\widetilde{L}}(\{\Delta_1, \Delta_2\}) = \begin{cases} 1 & \text{if } \Delta_1 \text{ and } \Delta_2 \text{ share an edge in } G. \\ 1/n^2 & \text{otherwise.} \end{cases}$$

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Then $\widetilde{L}$ is a weighted dependency graph for the family $\{\mathbf{1}_{\Delta \subset G}, \Delta \in \binom{[n]}{3}\}$.

Note: $\widetilde{L}$ has degree $\mathcal{O}(n^3)$, but weighted degree $\mathcal{O}(n)$.

A normality criterion for weighted dependency graphs

Setting: for each n,

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For triangles in $G(n, M_n)$, $N_n = \binom{n}{3}$, $\Delta_n = \mathcal{O}(n)$, while $\sigma_n \asymp n^{3/2}$.

Corollary

Fix p *in* $(0,1)$ *and set* $M_n = p\binom{n}{2}$*. Then* T_n *satisfies a CLT.*

(also true for $n \ll M_n \ll n^2$; originally proved by Janson 1994).

Stability by powers

Setting:

- Let $\{Y_\alpha, \alpha \in A\}$ be r.v. with weighted dependency graph $\widetilde{L}$;
- fix an integer $m \geq 2$;
- for a multiset $B = \{\alpha_1, \cdots, \alpha_m\}$ of elements of A, denote

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Proposition

The set of r.v. $\{\boldsymbol{Y}_B\}$ has a weighted dependency graph $\widetilde{L}^m$, where

$$\mathrm{wt}_{\widetilde{L}^m}(\boldsymbol{Y}_B, \boldsymbol{Y}_{B'}) = \max_{\alpha \in B, \alpha' \in B'} \mathrm{wt}_{\widetilde{L}}(Y_\alpha, Y_{\alpha'}).$$

In short: if we have a dependency graph for some variables Y_α, we have also one for monomials in the Y_α.
(And potentially CLT for polynomials in the Y_α).

Applications of weighted dependency graphs

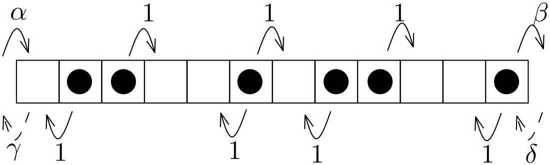
- crossings in random pair-partitions;
- subgraph counts in $G(n, M)$;
- random permutations;
- particles in symmetric simple exclusion process (SSEP);
- subword counts in Markov chains;
- patterns in multiset permutations*, in set-partitions*;
- spins in Ising model (with Jehanne Dousse);
- determinantal point process**.

*in progress with Marko Thiel. **project

(⚠ Some of these applications use a variant of the above definition and normality criterion, which is more technical to state...)

Applications to ASEP and Ising model

Symmetric simple exclusion process (SSEP)



$\tau = (\tau_1, \cdots, \tau_N)$ particle configuration with stationary distribution.

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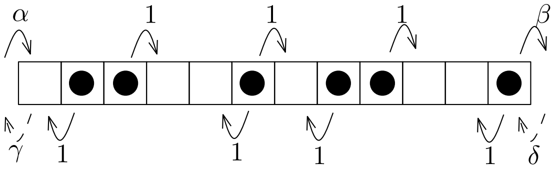

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Theorem

The complete graph on $[N]$ with weight $1/N$ on each edge is a weighted dependency graph for the family $\{\tau_i, 1 \le i \le N\}$.

In particular, for disjoint $i_1, \cdots, i_r$,

$$\kappa(\tau_{i_1}, \dots, \tau_{i_r}) = \mathcal{O}_r(N^{-r+1}).$$

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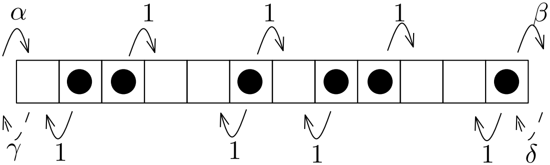

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Ingredients of the proof:

- joint moments of the τ_i given by matrix ansatz;
- in case of SSEP, this gives an induction formula for cumulants (Derrida, Lebowitz, Speer, 2006).

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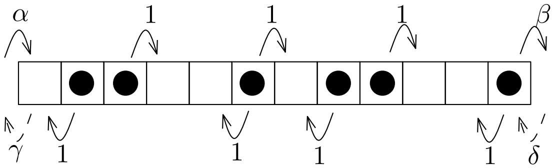

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Consequences:

- functional CLT for the particle distribution function;
- also, e.g., for the number $\sum_i \tau_i(1 - \tau_{i+1})$ of particles that can jump to their right (using stability by powers).

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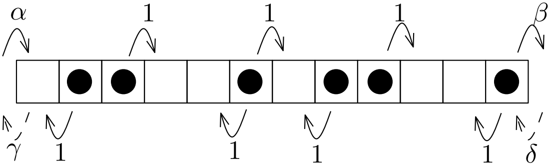

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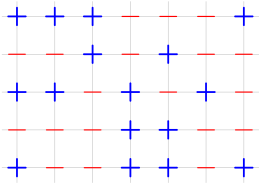
The same is conjectured for ASEP in general.

Ising model



$$\mathbb{P}(\omega) \propto \exp\big[-H(\omega)\big];$$
$$H(\omega) = -\beta \textstyle\sum_{x\sim y} \omega_x \omega_y - h \sum_x \omega_x.$$

Theorem

In presence of a magnetic field or at very low or very large temperature, there exists $\varepsilon = \varepsilon(d, h, \beta) > 0$ such that the complete graph on $\mathbb{Z}^d$ with weight $\varepsilon^{\|x-y\|_1}$ on the edge $\{x, y\}$ is a weighted dependency graph for $\{\sigma_x, x \in \mathbb{Z}^d\}$

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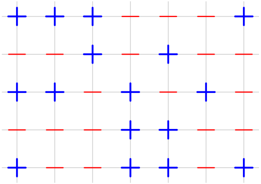

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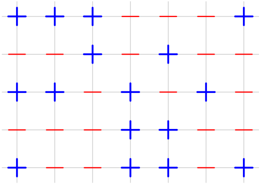
In particular, for disjoint $x_1, \cdots, x_r$,

$$\kappa(\sigma_{x_1}, \dots, \sigma_{x_r}) = \mathcal{O}_r(\varepsilon^{\ell_T(x_1,\dots,x_r)}),$$

where $\ell_T(x_1,\dots,x_r)$ is the smallest length of a tree connecting $x_1,\dots,x_r$.

Ising model



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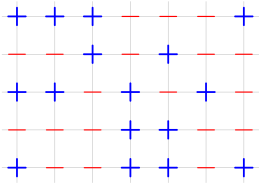
The bound on cumulants was proved by Duneau, Iagolnitzer and Souillard (with magnetic field or in very high temperature) and Malyshev and Minlos in very low temperature.
Proofs based on cluster expansion. . .

Ising model



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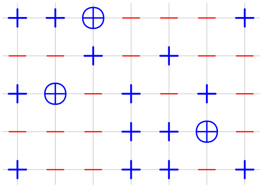
Question: does it hold near the critical point?
(At the critical point, the answer is NO, since already covariances do not decay exponentially)

Ising model: CLT for global patterns



Circled spins:
occurrence of the + pattern 2 3 1

(notion inspired from patterns in permutations.)

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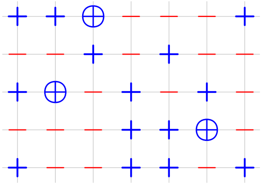

Circled spins:
occurrence of the $+$ pattern $2\,3\,1$

$S_n^{\mathcal{P}} :=$ number of occurrences of $\mathcal{P}$ within $\Lambda_n = [-n, n]^d$.

Theorem (Dousse, F., 2016)

Assume $\mathrm{Var}(S_n^{\mathcal{P}}) \geq cst|\Lambda_n|^{2|\mathcal{P}|-2+\eta}$ *for* $\eta > 0$. *Then we have*

$$\frac{S_n^{\mathcal{P}} - \mathbb{E}(S_n^{\mathcal{P}})}{\sqrt{\mathrm{Var}(S_n^{\mathcal{P}})}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1).$$

The lower bound of the variance is always fulfilled for patterns of only positive spins (as in the example).

Discrete determinantal point processes

Setting: S discrete state space; X random subset of S.

Definition

X is a discrete determinantal point process (DPP) with kernel K if for any distinct $s_1, \dots, s_r$ in S,

$$\mathbb{P}(\{s_1, \dots, s_r\} \subseteq X) = \mathbb{E}\left(\prod_{i=1}^{r} \mathbf{1}_{s_i \in X}\right) = \det\Big(K(s_i, s_j)\Big)_{1 \le i,j \le r}.$$

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Strange definition (not even clear *a priori* if such a process exists at all), but there are lots of example:

- random Young diagrams, taken with Poissonized Plancherel measure;
- spanning trees in graphs;
- eigenvalues of random matrices (continuous DPP).

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Lemma (Soshnikov, 2000)

If X is a discrete determinantal point process with kernel K, then, for any distinct $s_1, \dots, s_r$ in S,

$$\kappa(\mathbf{1}_{s_1 \in X}, \dots, \mathbf{1}_{s_r \in X}) = \sum_\sigma \varepsilon(\sigma) \prod_i K(s_i, s_{\sigma(i)}),$$

where the sum runs over cyclic permutation in S_r.

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Soshnikov cumulant formula $\Rightarrow$ for each DPP, we have a weighted dependency graph for $\{1_{s \in X},\ s \in S\}$ with weights $K(s,t)_{s,t \in S}$.

CLT for linear statistics is known;
Project: investigate CLT for "multilinear" statistics.

Большое Спасибо

Thanks for your attention !