Shifted symmetric functions III: Jack and Macdonald analogues

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Third lecture

In the first lectures, we have seen

- Two bases of the shifted symmetric function ring: $s^\star_\mu$ and Ch_μ with nice vanishing characterizations, multiplication tables and multirectangular expansions.

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Today:

- Can we define analogue in the Jack/Macdonald setting?
- Do they still have nice vanishing characterizations, multiplication tables and multirectangular expansions? What is the combinatorics involved?
- Application to random Young diagrams.

(No pre-requisite on Jack/Macdonald symmetric functions.)

Transition

Shifted Jack/Macdonald polynomials
through vanishing conditions

α shifted symmetric functions

Definition

A polynomial $f(x_1, \dots, x_N)$ is α-shifted symmetric if it is symmetric in $x_1 - \frac{1}{\alpha}$, $x_2 - \frac{2}{\alpha}$, $\dots$, $x_N - \frac{N}{\alpha}$.

Examples: $p_k^\star(x_1, \dots, x_N) = \sum_{i=1}^N \left(x_i - \frac{i}{\alpha}\right)^k$.

$\alpha = 1$ gives previous case.

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$\alpha=1$ gives previous case.

α-shifted symmetric function: sequence $f_N(x_1,\dots,x_N)$ of shifted symmetric polynomials with

$$f_{N+1}(x_1,\dots,x_N,0)=f_N(x_1,\dots,x_N).$$

Examples: $p^\star_k=\sum_{i\geq 1}\left[(x_i-\frac{i}{\alpha})^k-(\frac{-i}{\alpha})^k\right]$.

Shifted Jack polynomials

Proposition (Sahi, '94)

Let μ be a partition. There exists a unique α-shifted symmetric function $P_\mu^{(\alpha),\star}$ of degree at most $|\mu|$ such that $P_\mu^{(\alpha),\star}(\lambda) = \delta_{\lambda,\mu}\alpha^{-|\mu|}H_\alpha(\lambda)$ for $|\lambda| \leq |\mu|$.

$H_\alpha(\lambda)$: deformation of the hook product.

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$H_\alpha(\lambda)$: deformation of the hook product.

Note on the proof: looking for $P_\mu^{(\alpha),\star}$ under the form $\sum_{|\nu|\leq|\mu|} c_\nu p_\nu^\star$ the conditions $P_\mu^{(\alpha),\star}(\lambda) = \delta_{\lambda,\mu}H_\alpha(\lambda)$ defines a square system of linear equations in indeterminates c_ν. We need to prove that it is non-degenerate...

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$H_\alpha(\lambda)$: deformation of the hook product.

Theorem (Knop-Sahi '96, Okounkov '98)

1. $P_\mu^{(\alpha),\star}(\lambda) = 0$ *if* $\lambda \not\supset \mu$ *(extra-vanishing property);*
2. *in general,* $P_\mu^{(\alpha),\star}(\lambda)$ *counts* α*-weighted skew SYT.*
3. *the top degree component of* $P_\mu^{(\alpha),\star}$ *is the usual Jack polynomial* $J_\mu^{(\alpha)}$.

$P_\mu^{(\alpha),\star}$ is called shifted Jack polynomials (because of 3.)

No determinantal formula as for shifted Schur functions!...

Jack deformation of normalized characters (Lassalle '08)

Set $\tilde{p}_\mu = \alpha^{\frac{|\mu|-\ell(\mu)}{2}} p_\mu$, consider the expansion $\tilde{p}_\mu = \sum_{|\nu|=|\mu|} \theta_\mu^\nu \, P_\nu^{(\alpha)}$ (in usual symmetric function ring) and define

$$\mathrm{Ch}_\mu^{(\alpha)} = \sum_{|\nu|=|\mu|} \theta_\mu^\nu \, P_\nu^{(\alpha),\star}.$$

Vanishing characterization

(F., Śniady, 2015) $\mathrm{Ch}_\mu^{(\alpha)}$ is the unique α-shifted sym. function F of degree at most $|\mu|$ such that

1. $F(\lambda) = 0$ if $|\lambda| < |\mu|$;
2. The top-degree component of F is $\tilde{p}_\mu$.

Deformation of characters

Note that θ_μ^λ is a deformation of the character χ_μ^λ. We can prove

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = (|\lambda| \downharpoonleft |\mu|) \frac{\theta_\mu^\lambda}{\theta_{(1^k)}^\lambda}.$$

(Easy from the SYT interpretation of $P_\nu^{(\alpha),\star}(\lambda)$.)

t shifted symmetric functions

Definition

A polynomial $f(y_1, \dots, y_N)$ is t-shifted symmetric if it is symmetric in $y_1\, t^{-1},\ y_2\, t^{-2},\ \dots,\ y_N\, t^{-N}$.

Examples: $p_k^\star(y_1, \dots, y_N) = \sum_{i=1}^N \left(y_i\, t^{-i}\right)^k$.

Set $y_i = q^{x_i},\ q = t^\alpha \to 1$
and divide by $(q-1)^*$
to recover Jack case.

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t-shifted symmetric function: sequence $f_N(y_1, \dots, y_N)$ of shifted symmetric polynomials with

$$f_{N+1}(y_1, \dots, y_N, 1) = f_N(y_1, \dots, y_N).$$

Examples: $p^\star_k = \sum_{i \geq 1} \left[(y_i^k - 1)\, t^{-ki}\right]$.

Shifted Macdonald polynomials

Proposition (Sahi '96, Knop '97)

Let μ be a partition. There exists a unique t-shifted symmetric function $P_\mu^{(q,t),\star}$ of degree at most $|\mu|$ such that, for $|\lambda| \leq |\mu|$,

$$P_\mu^{(q,t),\star}(q^{\lambda_1}, q^{\lambda_2}, \dots) = \delta_{\lambda,\mu} H_{(q,t)}(\lambda).$$

$H_{(q,t)}(\lambda)$: deformation of the hook product.

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Theorem (Sahi' 96, Knop '97, Okounkov '98)

1. $P_\mu^{(q,t),\star}(\lambda) = 0$ *if* $\lambda \not\supset \mu$ *(extra-vanishing property);*
2. *the top degree component of* $P_\mu^{(q,t),\star}$ *is the usual Macdonald polynomial* $P_\mu^{(q,t)}$ *evaluated in* $y_1, y_2 t^{-1}, \dots, y_n t^{-n}$.

$P_\mu^{(q,t),\star}$ is called shifted Macdonald polynomial.

Note: no interpretation of $P_\mu^{(q,t),\star}(\lambda)$ as counting weighted SYTs!

Macdonald deformation of normalized characters?

Consider the expansion $p_\mu = \sum_{|\nu|=|\mu|} \theta_\mu^\nu \, P_\nu^{(q,t)}$ (in usual symmetric function ring) and define

$$\mathrm{Ch}_\mu^{(q,t)} = \sum_{|\nu|=|\mu|} \theta_\mu^\nu \, P_\nu^{(q,t),\star}.$$

Vanishing characterization

$\mathrm{Ch}_\mu^{(q,t)}$ is the unique t-shifted sym. function F of degree at most $|\mu|$ such that

1. $F(\lambda) = 0$ if $|\lambda| < |\mu|$;
2. The top-degree component of F is p_μ.

Deformation of characters

☹ We cannot relate $\mathrm{Ch}_\mu^{(q,t)}(\lambda)$ and θ_μ^λ.

In fact, I could not find a normalization of θ_μ^λ, that is t-shifted symmetric.

Macdonald deformation of normalized characters?

Consider the expansion $p_\mu = \sum_{|\nu|=|\mu|} \theta^\nu_\mu \, P^{(q,t)}_\nu$ (in usual symmetric function ring) and define

$$\mathrm{Ch}^{(q,t)}_\mu = \sum_{|\nu|=|\mu|} \theta^\nu_\mu \, P^{(q,t),\star}_\nu .$$

Vanishing characterization

$\mathrm{Ch}^{(q,t)}_\mu$ is the unique t-shifted sym. function F of degree at most $|\mu|$ such that

1. $F(\lambda) = 0$ if $|\lambda| < |\mu|$;
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Deformation of characters

☹ We cannot relate $\mathrm{Ch}^{(q,t)}_\mu(\lambda)$ and θ^λ_μ.

In fact, I could not find a normalization of θ^λ_μ, that is t-shifted symmetric.

$\Rightarrow$ Are the functions $\mathrm{Ch}^{(q,t)}_\mu$ nevertheless interesting? I don't know...
Here, we'll focus on the Jack case...

Combinatorial formula for $P_\mu^{(q,t),\star}$ and $P_\mu^{(\alpha),\star}$

Theorem (Okounkov '98)

$$P_\mu^{(q,t),\star}(x_1,\dots,x_N) = \sum_T \Psi_T^{(q,t)} \prod_{(i,j)\in T} t^{1-T(i,j)}(x_{T(i,j)} - q^{j-1}t^{1-i}).$$

where the sum runs over reverse[a] semi-std Young tableaux T, and $\Psi_T^{(q,t)}$ is the same weight as for usual Macdonald polynomials (rational function in q and t).

[a] filling with decreasing columns and weakly decreasing rows

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Corollary

$$P_\mu^{(\alpha),\star}(x_1,\dots,x_N) = \sum_T \Psi_T^{(\alpha)} \prod_{\square\in T} (x_{T(\square)} - c_\alpha(\square)),$$

where $c(i,j) = \alpha(j-1) - (i-1)$ is the α- content of the box and $\Psi_T^{(\alpha)}$ a rational function in α.

Transition

Multiplications tables

Multiplication tables

Question

Can we understand the multiplication tables of our favorite bases?

$$P_\mu^{(\alpha),\star}\, P_\nu^{(\alpha),\star} = \sum_{\rho : |\rho| \leq |\mu|+|\nu|} c_{\mu,\nu}^{\rho,(\alpha)}\, P_\rho^{(\alpha),\star}$$

$$\mathrm{Ch}_\mu^{(\alpha)}\ \mathrm{Ch}_\nu^{(\alpha)} = \sum_{\rho : |\rho| \leq |\mu|+|\nu|} g_{\mu,\nu}^{\rho,(\alpha)}\, \mathrm{Ch}_\rho^{(\alpha)}$$

Are $c_{\mu,\nu}^{\rho,(\alpha)}$ and $g_{\mu,\nu}^{\rho,(\alpha)}$ polynomials in α? with nonnegative coefficients? Do they have a combinatorial interpretation?

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$$P^{(\alpha),\star}_{\mu} P^{(\alpha),\star}_{\nu} = \sum_{\rho : |\rho| \le |\mu|+|\nu|} c^{\rho,(\alpha)}_{\mu,\nu} P^{(\alpha),\star}_{\rho}$$

$$\mathrm{Ch}^{(\alpha)}_{\mu} \mathrm{Ch}^{(\alpha)}_{\nu} = \sum_{\rho : |\rho| \le |\mu|+|\nu|} g^{\rho,(\alpha)}_{\mu,\nu} \mathrm{Ch}^{(\alpha)}_{\rho}$$

Are $c^{\rho,(\alpha)}_{\mu,\nu}$ and $g^{\rho,(\alpha)}_{\mu,\nu}$ polynomials in α? with nonnegative coefficients? Do they have a combinatorial interpretation?

Note: when $|\rho| = |\mu| + |\nu|$, then $c^{\rho,(\alpha)}_{\mu,\nu}$ is the Jack analogue of LR coefficients (multiplication table of usual Jack symmetric functions). When suitably renormalized, they are conjectured to be polynomials with nonnegative coefficients in α (Stanley '89, still open).

Jack shifted LR coefficients

Conjecture (Alexandersson, F.)

$\alpha^{|\mu|+|\nu|-|\rho|-2} H_\alpha(\mu)\, H_\alpha(\nu)\, H'_\alpha(\rho) c_{\mu,\nu}^{\rho,(\alpha)}$ is a polynomial in α with nonnegative integer coefficients.

- It implies Stanley's conjecture;
- A weaker form had been formulated earlier by Sahi, '11: namely, $c_{\mu,\nu}^{\rho,(\alpha)}$ is a quotient of two polynomials in α with nonnegative integer coefficients.
- We can prove polynomiality in α with rational coefficients.

Computing Jack shifted LR coefficients by induction

As in the Schur case, we have:

1. $c_{\mu,\nu}^{\rho,(\alpha)} = 0$ if $\rho \not\supseteq \mu$ or $\rho \not\supseteq \nu$;
2. $c_{\mu,\nu}^{\nu,(\alpha)} = P_\mu^{(\alpha),\star}(\nu)$.
3. $c_{\mu,\nu}^{\rho,(\alpha)} = \frac{1}{|\rho|-|\nu|} \left(\sum_{\nu \nwarrow \nu^+} \psi'_{\nu^+/\nu} c_{\mu,\nu^+}^{\rho,(\alpha)} - \sum_{\rho^- \nearrow \rho} \psi'_{\rho/\rho^-} c_{\mu,\nu}^{\rho^-} \right)$.

($\psi'_{\nu^+/\nu}$ is the weight appearing in Pieri's formula for Jack polynomials)

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$\rightarrow$ we can prove the weaker conjecture when $|\rho| \leq |\nu| + 1$.

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$\rightarrow$ we can prove the weaker conjecture when $|\rho| \leq |\nu| + 1$.

Strong version completely open (even for $|\rho| = |\nu|$)

Multiplication table of deformed characters

Reminder: $\mathrm{Ch}^{(\alpha)}_\mu \, \mathrm{Ch}^{(\alpha)}_\nu = \sum_{\rho : |\rho| \le |\mu| + |\nu|} g^{\rho,(\alpha)}_{\mu,\nu} \, \mathrm{Ch}^{(\alpha)}_\rho$

Conjecture (Śniady, '16)

$g^{\rho,(\alpha)}_{\mu,\nu}$ is a polynomial with nonnegative integer coefficients in

$$\delta := \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}.$$

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Nonnegativity for $\alpha = 1, 2$ (Ivanov, Kerov, '99, Tout, '14).
Polynomiality with rational coefficients is known (Dołęga, F., '16).

Implies the matching-Jack conjecture of Goulden and Jackson ('96).

Transition

Multirectangular expansions

Multirectangular expansion of normalized characters

Conjecture (Lassalle, '08)

$(-1)^k \alpha^{\frac{|\mu|-\ell(\mu)}{2}} \mathrm{Ch}_\mu^{(\alpha)}$ is a polynomial with nonnegative integer coefficients in $\boldsymbol{p}$, $-\boldsymbol{q}$ and $\alpha - 1$.

Polynomiality with rational coefficients also follows from Dołęga, F. '16.

Nonnegativity has been proved by Lassalle for rectangular partitions (i.e. one single p, resp. q).

Question: is there again a nice formula in terms of N_G functions??
Such formulas exist for $\alpha = 2$ (F., Śniady, '11) and for rectangular partitions (F. Dołęga, Śniady, '14).

Multirectangular expansion of shifted Jack polynomials

Conjecture (Alexandersson, F., '17)

$\alpha^{|\mu|-\mu_1} H'_\alpha(\mu)\, P_\mu^{(\alpha),\star}$ is a polynomial with nonnegative integer coefficients in the falling factorial basis

$$\alpha^c\, (p_1 \downharpoonleft a_1) \dots (p_m \downharpoonleft a_m)(r_1 \downharpoonleft b_1) \dots (r_m \downharpoonleft b_m).$$

Polynomiality with rational coefficients also follows from Dołęga, F. '16.

We could prove it for $\mu = (k)$, by finding a new combinatorial formula for this case.

A new formula for $P_{(k)}^{(\alpha),\star}$

Theorem (Alexandersson, F., '17)

For any integer $k \geq 1$ and Young diagram λ, one has:

$$\frac{1}{k!} H'_\alpha((k))\, P_{(k)}^{(\alpha),\star}(\lambda) = \sum_{\substack{A \subseteq \lambda,\ |A|=k \\ \text{column-dinstinct}}} \left(\prod_{\substack{R \text{ row} \\ \text{of } \lambda}} P_{|R\cap A|}(\alpha) \right),$$

where, for $i \geq 0$, we set $P_i(\alpha) = \prod_{j=0}^{i-1}(1 + j\,\alpha)$.

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Main steps of proof:

1. observe that $\frac{1}{k!} H'_\alpha(\mu) P^{(\alpha),\star}_{(k)}(\lambda) = \text{const} \cdot \left[m_{(k,1^{|\lambda|-k})} \right] P^{(\alpha)}_\lambda$.

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2. Knop-Sahi combinatorial formula ('97) gives a combinatorial formula for the right-hand side.
3. We transform it into the one in the theorem through a nontrivial bijection.

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where, for $i \geq 0$, we set $P_i(\alpha) = \prod_{j=0}^{i-1}(1 + j\,\alpha)$.

Question 1: proof through vanishing characterization? (only the shifted symmetry is hard.)

Question 2: is there such a formula for $P_\mu^{(\alpha),\star}(\lambda)$? (no direct relation with monomial coefficients anymore.)

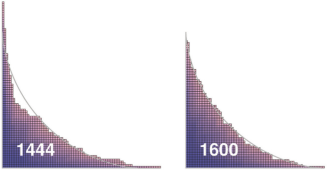
Transition

A motivation:
global fluctuations of
Jack-Plancherel random diagrams

A motivation: random Young diagram

Plancherel measure

$$\mathbb{P}(\lambda) = \frac{\dim(\rho^\lambda)^2}{n!}$$



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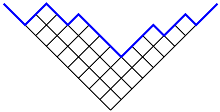

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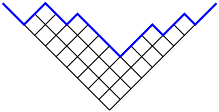
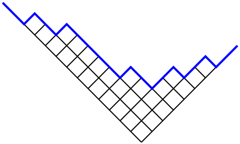


1. limit shape (Vershik-Kerov/Logan-Shepp '77): $\widetilde{\omega_\lambda}$ tends almost surely towards a deterministic shape Ω;
2. global fluctuations (Kerov-Ivanov-Olshanski '93-'03): $\widetilde{\omega_\lambda}(x) \approx \Omega(x) + \frac{2}{\sqrt{n}}\Delta_\infty(x)$ for some Gaussian process λ;
3. edge fluctuations (Borodin-Okounkov-Olshanski '00/Johansson '01): first few rows fluctuations are similar to first few eigenvalue fluctuations in GUE random matrices;

Jack-Plancherel measure

Jack-Plancherel measure

$$\mathbb{P}(\lambda) = \frac{\alpha^n n!}{H_\alpha(\lambda) H'_\alpha(\lambda)}$$



Simulation for $\alpha = 3$, $n = 30$.

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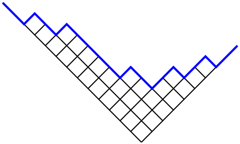

Simulation for $\alpha = 3$, $n = 30$.

We rescale rows by $\sqrt{\frac{\alpha}{n}}$ and columns by $\frac{1}{\sqrt{\alpha n}}$ and consider the function $\widetilde{\omega_\lambda}$ defined by the blue zigzag line.

Theorem (Dołęga, F., '16, informal version)

Let λ be a random Jack-Plancherel distributed Young diagram. Then

$$\widetilde{\omega_\lambda}(x) \approx \Omega(x) + \frac{2}{\sqrt{n}} \Delta_\infty^{(\alpha)}(x),$$

where $\Omega(x)$ is the limit shape independent on α and

$$\Delta_\infty^{(\alpha)}(2\cos(\theta)) = \frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\Xi_k}{\sqrt{k}} \sin(k\theta) - \gamma/4 + \gamma\theta/2\pi.$$

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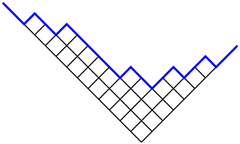

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Recent result (Guionnet, Huang): fluctuation of the first row lengths

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How is it related to multiplication tables?

Key point is to prove (idea due to Kerov):

Proposition

$$(\mathrm{Ch}^{(\alpha)}_{(k)})_{k=2,3,\dots} \xrightarrow{d} (\Xi_k)_{k=2,3,\dots},$$

where Ξ_k are independent Gaussian variables with appropriate variances.

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Proposition

$$(\mathrm{Ch}^{(\alpha)}_{(k)})_{k=2,3,\dots} \xrightarrow{d} (\Xi_k)_{k=2,3,\dots},$$

where Ξ_k are independent Gaussian variables with appropriate variances.

for any $k \geq 1$, $\int_{-\infty}^{\infty} x^k \widetilde{\omega_\lambda}(x)dx$ is α-shifted symmetric and thus can be expressed as a polynomials in $\mathrm{Ch}^{(\alpha)}_{(k)}$.

$\rightarrow$ we can describe the fluctuations of $\int_{-\infty}^{\infty} x^k \widetilde{\omega_\lambda}(x)dx$ for all k.

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Multiplication table of $\mathrm{Ch}^{(\alpha)}_{\mu}$ is little understood but the polynomiality with bound on the degree, together with special values $\alpha = 1/2, 1, 2$ are enough here.

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- The multirectangular expansion conjectures suggest new combinatorial formulae related to Jack polynomials;
- In most problems, if we could conjecture a combinatorial formula, we have tools to try to prove it (induction relation, vanishing characterization).