Cumulants mixtes et arbres couvrants

Valentin Féray
travail en commun avec Pierre-Loïc Méliot (Orsay)
and Ashkan Nighekbali (Zürich)

Institut für Mathematik, Universität Zürich

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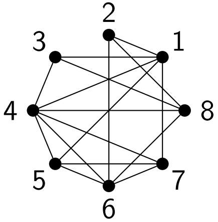
A problem in random graphs

Erdös-Rényi model of random graphs $G(n, p)$:

- G has n vertices labelled $1, \ldots, n$;
- each edge (i, j) is taken independently with probability p;



Example : $n = 8, p = 1/2$

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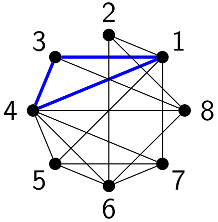

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Fix $p \in]0; 1[$.
Describe asymptotically the fluctuations of the number T_n of triangles.

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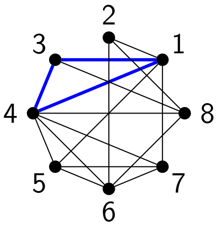

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Describe asymptotically the fluctuations of the number T_n of triangles.

Answer (Ruciński, 1988)

The fluctuations are asymptotically Gaussian.

A good tool for that: mixed cumulants

- the r-th mixed cumulant k_r of r random variables is a r-linear symmetric polynomial in joint moments. Examples:

$$\kappa_1(X) = \mathbb{E}(X), \quad \kappa_2(X,Y) = \mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\kappa_3(X,Y,Z) = \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).$$

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- if the variables can be split in two mutually independent sets, then the cumulant vanishes.

- if, for each $r \neq 2$, the sequence $\kappa_r(X_n, \dots, X_n)$ converges towards 0 and if $\mathrm{Var}(X_n)$ has a limit, then X_n converges in distribution towards a Gaussian law.

Application to the number of triangles

$$T_n = \sum_{\Delta=\{i,j,k\}\subset[n]} B_\Delta, \text{ where } B_\Delta(G) = \begin{cases} 1 & \text{if } G \text{ contains the triangle } \Delta; \\ 0 & \text{otherwise.} \end{cases}$$

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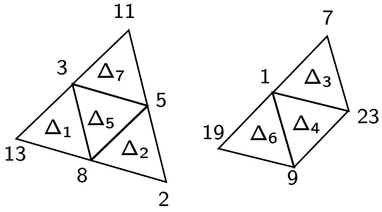
Example: 

$\{\Delta_1, \Delta_2, \Delta_5, \Delta_7\}$ is independent from $\{\Delta_3, \Delta_4, \Delta_6\}$. Reminder: presence of different edges are independent events.

$$\kappa_\ell(B_{\Delta_1},\dots,B_{\Delta_7}) = 0.$$

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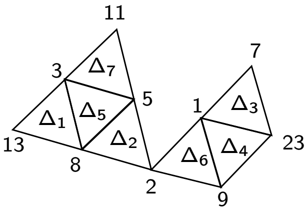

$\{\Delta_1,\Delta_2,\Delta_5,\Delta_7\}$ is independent from $\{\Delta_3,\Delta_4,\Delta_6\}$. Triangles need to share an edge to be dependent!

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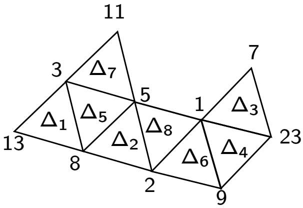

This configuration contributes to the sum. Call it configuration of dependent triangles. Note that it has only $\ell + 2$ vertices (here $\ell = 8$).

$$\kappa_\ell(B_{\Delta_1},\dots,B_{\Delta_8}) \neq 0.$$

Bound on the cumulant

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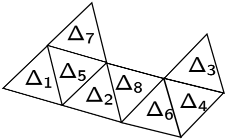

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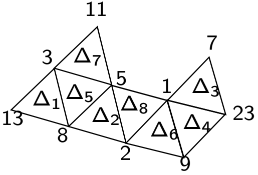

- each configuration can be labelled in at most $n^{\ell+2}$ ways.

Conclusion

$$|\kappa_\ell(T_n)| = O_\ell(n^{\ell+2})$$

The central limit theorem for triangles

Proposition (Leonov, Shirryaev, 1955)

If $X_1, \dots, X_\ell$ can be split into two sets of mutually independent variables, then

$$\kappa_\ell(X_1, \cdots, X_\ell) = 0$$

Corollary (Janson, 1988)

For each ℓ, there exists a constant C_ℓ such that

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Corollary (Ruciński, 1988)

$$\frac{T_n - \mathbb{E}(T_n)}{\sqrt{\mathrm{Var}(T_n)}} \to \mathcal{N}(0,1)$$

Proof: $\mathrm{Var}(T_n) \approx n^4$ and $\kappa_\ell(T_n/n^2) = n^{2-\ell} = o_\ell(1)$ for $\ell > 2$.

Our work

Theorem (F., Méliot, Nighekbali, 2014)

Let $X_1, \ldots, X_\ell$ be random variables with finite moments of order ℓ,

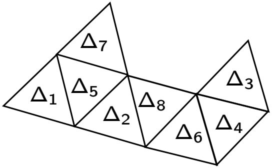
$$|\kappa_\ell(X_1, \cdots, X_\ell)| \leq 2^{\ell-1} ||X_1||_\ell \cdots ||X_\ell||_\ell \cdot \mathrm{ST}\left(G_{\mathrm{dep}}(X_1, \cdots, X_\ell)\right),$$

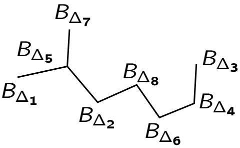
where $\mathrm{ST}\left(G_{\mathrm{dep}}(X_1, \cdots, X_\ell)\right)$ is the number of spanning trees of the dependency graph of $X_1, \cdots, X_\ell$.

Dependency graphs for a list $(B_{\Delta_1}, \cdots, B_{\Delta_\ell})$:

$$B_{\Delta_i} \sim B_{\Delta_j} \Leftrightarrow \Delta_i \text{ and } \Delta_j \text{ share an edge}$$

Example:





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Corollary (FMN, 2014)

There exists an absolute constant C such that

$$|\kappa_\ell(T_n)| = (C\ell)^\ell n^{\ell+2}$$

Naive bound: $(C\ell)^{3\ell} n^{\ell+2}$

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Corollary (FMN, 2014)

Very precise extension of the central limit theorem: if $1 << v << n^{1/2}$,

$$\mathbb{P}\left[T_n \geq \binom{n}{3} p^3 + v \cdot n^2\right] \sim \frac{1}{\sqrt{\pi\, p^5 (1-p) v^2}} \exp\left(-\frac{v^2}{p^5\,(1-p)} + \frac{(7-8p)\, v^3}{n\sqrt{p(1-p)/2}}\right)$$

Moment-cumulant relation

Mixed cumulants can be expressed in terms of mixed moments:

$$\kappa(X_1, \dots, X_r) = \sum_{\pi} \mu(\pi) M_\pi,$$

where

- π runs over set-partitions of $[\ell]$,
- $\mu(\pi) = \mu(\pi, \{[\ell]\})$ is the Möbius function of the set-partition poset (it is explicit!),
- $M_\pi = \prod_{B \in \pi} \mathbb{E}\left[\prod_{i \in B} X_i\right]$.

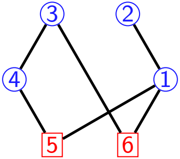
Example:

$$\kappa_3(X, Y, Z) = \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) \\ - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).$$

Using independence to simplify M_π

Example: $\pi = \big\{\{1,2,3,4\},\{5,6\}\big\}$ and

$$H := G_{\text{dep}}(X_1,\dots,X_6) =$$



$$\text{Then } M_\pi := \mathbb{E}(X_1X_2X_3X_4)\mathbb{E}(X_5X_6) = \mathbb{E}(X_1X_2)\mathbb{E}(X_3X_4)\mathbb{E}(X_5)\mathbb{E}(X_6).$$

In general, $M_\pi = M_{\phi_H(\pi)}$, with obvious definition of $\phi_H(\pi)$.

Rewriting the summation

$$\kappa(X_1, \ldots, X_r) = \sum_{\pi} \mu(\pi) M_\pi = \sum_{\pi} \mu(\pi) M_{\phi_H(\pi)}$$
$$= \sum_{\pi'} M_{\pi'} \left(\sum_{\substack{\pi \text{ s.t.} \\ \phi_H(\pi) = \pi'}} \mu(\pi) \right)$$

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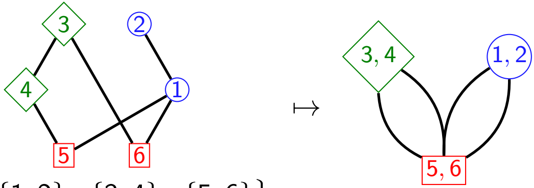
- $\phi_H(\pi) = \pi' \Rightarrow$ for all part π'_i of π', the induced graph $H[\pi'_i]$ is connected.
- if so, we have to compute

$$\alpha_H^{\pi'} := \sum_{\substack{\pi \text{ s.t.} \\ \phi_H(\pi)=\pi'}} \mu(\pi).$$

$\alpha_H^{\pi'}$ and Tutte polynomial

Consider the contracted graph H/π. Example:



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It is a multigraph.

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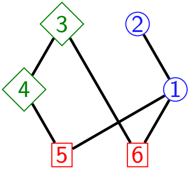
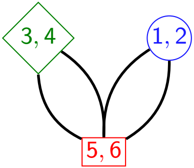


$\mapsto$



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Lemma

$$\alpha_H^{\pi'} = \sum_{E \subset E(H/\pi')} (-1)^{|E|},$$

where the sum runs over spanning connected subgraphs of H/π'.

If H/π' is connected, $\left|\alpha_H^{\pi'}\right|$ is Tutte polynomial evaluated at $(1,0)$.

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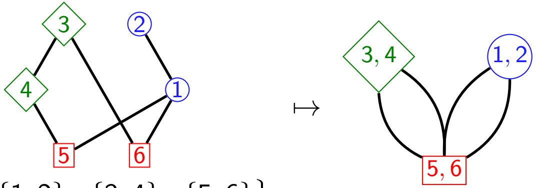

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Corollary: $\left|\alpha_H^{\pi'}\right| \leq \mathrm{ST}(H/\pi')$.

Bounding everything

Reminder:

$$\kappa(X_1, \dots, X_\ell) = \sum_{\pi'} M_{\pi'} \alpha_H^{\pi'}$$

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We have the following inequalities

$$\begin{aligned} |M_\pi| &\leq ||X_1||_\ell \cdots ||X_\ell||_\ell \quad \text{(Hölder inequality)}; \\ \left|\alpha_H^{\pi'}\right| &\leq \mathrm{ST}(H/\pi'); \\ \mathbf{1}_{H[\pi'_i] \text{ connected}} &\leq \mathrm{ST}(H[\pi'_i]) \end{aligned}$$

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Thus

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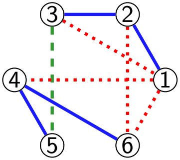
$T =$ 

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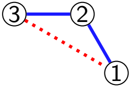
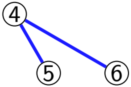
$T_1 =$ 

, $T_2 =$ 

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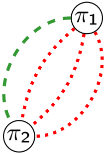
$\overline{T} =$ 

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A precise bound on cumulants of T_n

Recall that $\kappa_\ell(T_n) = \sum_{\Delta_1,\dots,\Delta_\ell} \kappa_\ell(B_{\Delta_1},\dots,B_{\Delta_\ell})$.

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$$|\kappa_\ell(T_n)| \leq \sum_{\Delta_1,\dots,\Delta_\ell} 2^{\ell-1} \left| \mathrm{ST}\left(G_{\mathrm{dep}}(B_{\Delta_1,\dots,\Delta_\ell})\right)\right|.$$

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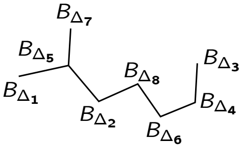

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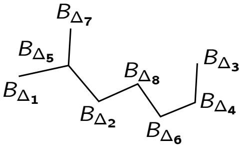

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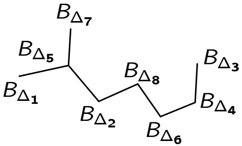

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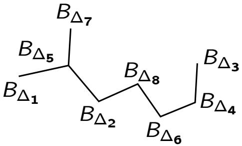

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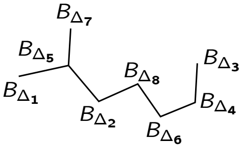

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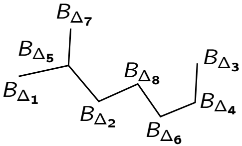

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Moderate deviations

Let $X_n = (T_n - \mathbb{E}(T_n))/n^{5/3}$, then

$$\log \mathbb{E}(\exp(zX_n)) = \sum_{\ell \geq 2} \kappa^{(\ell)}(X_n) z^\ell/\ell!$$
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$\rightarrow$ looks like, but is stronger than the hypotheses in Hwang's quasi-power theorem (convergence on $\mathbb{C}$!) $\Rightarrow$ stronger results.

Conclusion

- very general bound on mixed cumulants, with a strong combinatorial flavor ;
- implies a good uniform bound on cumulants of sums of partially dependent random variables (number of copies of subgraphs, character of a random irreducible representation, ...) ;
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Questions:

- Large deviations $\mathbb{P}(T_n \geq \mathbb{E}(T_n) + v\, n^3) \sim$? ;
- other models: $p_n \to 0$, $G(n, M)$ (fixed number of edges $\Rightarrow$ almost-independence).