Two first-order logics of permutations

Valentin Féray
(joint work with Michael Albert and Mathilde Bouvel)

Institut für Mathematik, Universität Zürich

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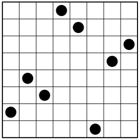
$\longrightarrow$ natural point of view to study the group structure, the cycle decomposition, ...

Answer 2: a collection of dots in a square grid, with exactly one dot per row or per column.

$\longrightarrow$ natural point of view to study patterns.



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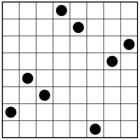

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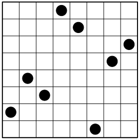

General belief: these two points of view are orthogonal.

We will give a mathematical statement supporting this belief using formal logic.

Informal presentation of the main result

We will define two first-order logical theories:

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- one representing *permutations-as-bijections*;
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Expressibility: For each theory, some properties of permutations are *expressible* (i.e. there exist a first-order formula describing the property), some are not.

Theorem (Albert, Bouvel & F. '18)

Let P be a property expressible in both logical theories. Then P is in some sense trivial.

TOOB: Theory Of One Bijection (models)

Models: pair (A, R), where A is a (finite) set and R is a binary relation on A.

Any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is a model of the theory.

$$A_\sigma := \{1, \dots, n\}$$
$$x \, R_\sigma \, y \overset{\text{def}}{\Leftrightarrow} y = \sigma(x)$$

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Remark: Models corresponding to conjugate permutations are isomorphic!

TOOB: Theory Of One Bijection (formulas)

First-order formulas: every formula that you can write with quantifiers $\exists$, $\forall$, conjunctions $\wedge$, disjunctions $\vee$, negation $\neg$ and the relation R.

Examples:

- σ has a fixed point:
$$\exists x,\ xRx.$$
- σ is an involution:
$$\forall x,\ \exists y,\ xRy \ \wedge\ yRx.$$

Important: variables represent elements of the permutation, not sets!

TOTO: Theory Of Two Orders (models)

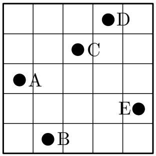
Models: pair $(A, <_P, <_V)$, where A is a (finite) set and $<_P$, $<_V$ are binary relations on A.

Any permutation (seen as a diagram) is a model of the theory.

$A_\sigma :=$ {dots in the diagram};

$(x <_P y) \overset{\text{def}}{\Leftrightarrow} x$ is on the left of y;

$(x <_V y) \overset{\text{def}}{\Leftrightarrow} x$ is below y;



$A <_P B <_P C <_P D <_P E$;

$B <_V E <_V A <_V C <_V D$.

TOTO: Theory Of Two Orders (models)

Models: pair $(A, <_P, <_V)$, where A is a (finite) set and $<_P$, $<_V$ are binary relations on A.

Axioms: $<_P$ and $<_V$ are total orders.

Any permutation (seen as a diagram) is a model of the theory.

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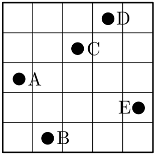

$A <_P B <_P C <_P D <_P E$;

$B <_V E <_V A <_V C <_V D$.

Conversely, any model is isomorphic to a permutation (and to exactly one).

TOTO: Theory Of Two Orders (formulas)

First-order formulas: every formula that you can write with quantifiers $\exists$, $\forall$, conjunctions $\wedge$, disjunctions $\vee$, negation $\neg$ and the relations $<_P$ and $<_V$.

Examples:

- σ contains the pattern 213:

$$\exists x, y, z : (x <_P y <_P z) \wedge (y <_V x <_V z).$$

- σ contains the vincular pattern $\underline{21}3$:

$$\exists x, y, z : (x <_P y <_P z) \wedge (y <_V x <_V z) \\ \wedge \left[\forall t, \neg(x <_P t <_P y)\right]$$

- containment of all notions of generalized patterns, being a simple permutation.

Transition

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- Can we give examples of properties that they cannot describe?
- Immediate: "σ contains the pattern 213" is not expressible in TOOB, since this property disguishes some conjugate permutations.
- To go further, we need Ehrenfeucht-Fraïssé games, which we explain in the context of TOTO in the next few slides.

Ehrenfeucht-Fraïssé game EF(π, π', ℓ)

Data: two permutations π, π' (the board of the game), an integer $\ell \geq 1$ (number of rounds).

Two players: Duplicator and Spoiler.

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Each round: Spoiler chooses an element in either π or π'. Duplicator then chooses an element in the other permutation.

$\longrightarrow$ denote a_i and a'_i the chosen elements in π and π'.

Note: a_i can have been chosen either by Spoiler or Duplicator

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Who wins? Duplicator wins if the $(a_1, \dots, a_\ell)$ and $(a'_1, \dots, a'_\ell)$ are in the same P and V orders.

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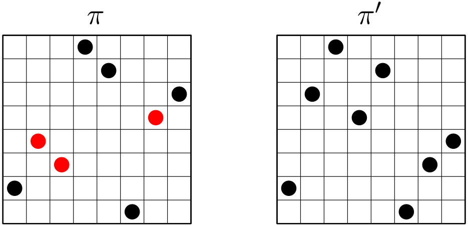
Theorem (Ehrenfeucht-Fraïssé)

Duplicator has a winning strategy in the game EF(π, π', ℓ) *if and only if* π *and* π' *satisfies the same* TOTO *formulas with quantifier depth at most* ℓ.

Notation: $\pi \sim_\ell \pi'$.

Ehrenfeucht-Fraïssé game (an example)



Note: π contains a $\underline{21}3$ pattern (in red), but not π'.

Reminder: there is a FO-formula of quantifier-depth 4, expressing "σ contains a $\underline{21}3$ pattern".

$\Rightarrow$ Spoiler should win the game $\text{EF}(\pi, \pi', 4)$.

Let us see her winning strategy.

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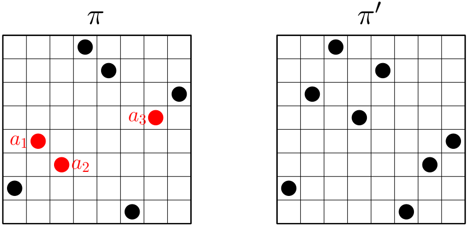

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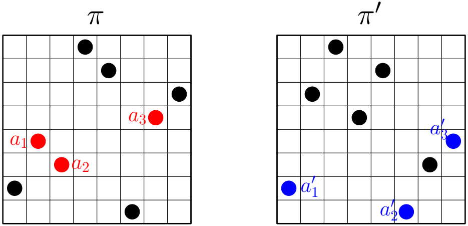

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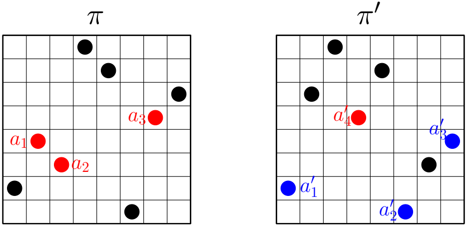

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- Spoiler chooses a point a'_4 between a'_1 and a'_2. Duplicator should choose a point a_4 between a_1 and a_2, but there is none.

  $\longrightarrow$ Spoiler wins.

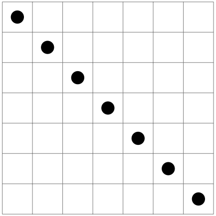
Ehrenfeucht-Fraïssé game (second example)

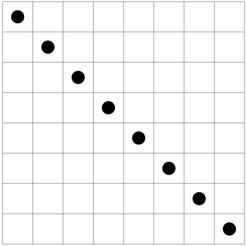
Let $\pi = \delta_{2^\ell - 1}$ and $\pi' = \delta_{2^\ell}$ are decreasing permutations of sizes $2^\ell - 1$ and 2^ℓ.

Duplicator has a winning strategy:

- if Spoiler plays near a corner or an already chosen dot, play at the same distance;
- if Spoiler plays far from corners/other dots, do the same.

For $\ell = 3$:





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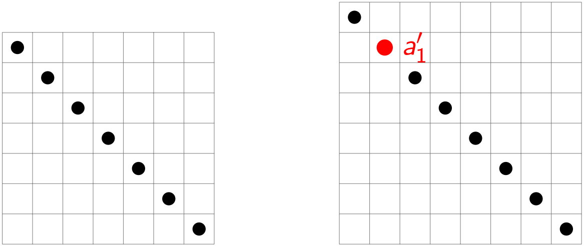

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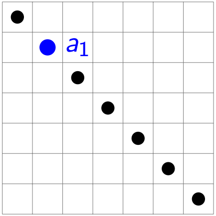

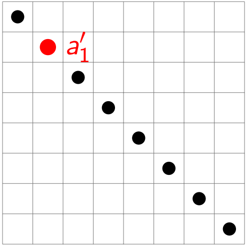

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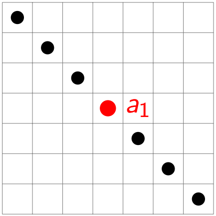

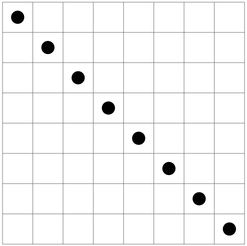

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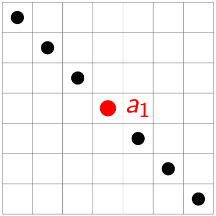

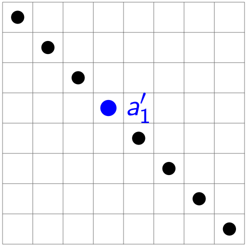

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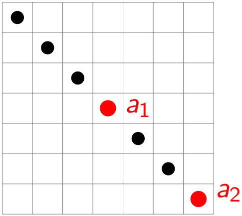

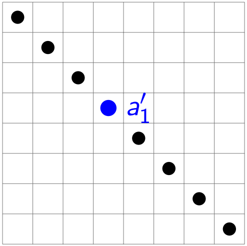

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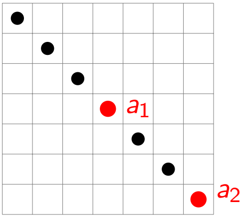

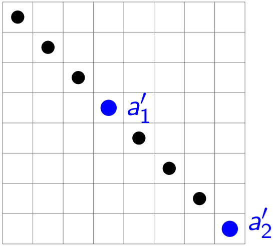

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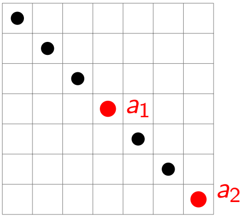

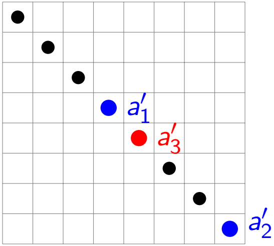

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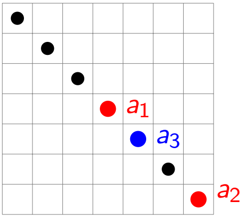

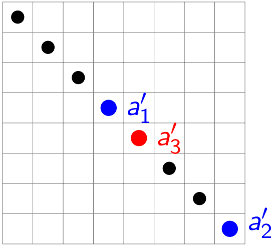

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A non-expressivity result

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Since Duplicator wins the game $\mathrm{EF}(\delta_{2^\ell-1}, \delta_{2^\ell})$, either both $\delta_{2^\ell-1}$ and δ_{2^ℓ} satisfy Ψ, or none of them does.

But $\delta_{2^\ell-1}$ has a fixed point, while δ_{2^ℓ} does not. □

Transition

Reminder of expressibility of some properties:

Having a fixed point expressible in TOOB but not in TOTO;

Containing a 231-pattern expressible in TOTO but not in TOOB.

Question

What are the properties expressible in both TOTO and TOOB?

We expect not to find many, since, intuitively, the two logics consider permutations with orthogonal point of views. . .

Some properties expressible in TOTO and TOOB (1/3)

Easy examples of properties expressible in both logics:

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Some properties expressible in TOTO and TOOB (1/3)

Easy examples of properties expressible in both logics:

- Being in some conjugacy class $\mathcal{C}_\lambda$ or in a finite union of those (or the negation);
- Being an identity permutation of any size (also called increasing permutations).

  TOOB Each element is sent to itself ($\forall x,\ xRx$);

  TOTO Values and positions order coincide ($\forall x, y,\ (x <_V y) \Leftrightarrow (x <_P y)$).

Some properties expressible in TOTO and TOOB (2/3)

"Being a transposition" is also expressible in both logics!

- In TOOB, it's trivial:

  There exist two points which are sent one to the other and all other points are fixed.

Some properties expressible in TOTO and TOOB (2/3)

“Being a transposition” is also expressible in both logics!

- In TOOB, it’s trivial:

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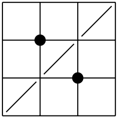
- To express it in TOTO, we observe that the diagram of a transposition has a very particular form.



There exist two points x and y which form an inversion and, for all other points, value and position order coincide, and the following holds: z is smaller (resp. bigger) than both x and y in position order iff it also is in value order.

Some properties expressible in TOTO and TOOB (3/3)

Set $\mathcal{D}_\lambda = \bigcup_k \mathcal{C}_{\lambda \cup (1^k)}$, i.e. the set of permutations whose non-fixed points form a permutation of type λ.

Lemma

Being in $\mathcal{D}_\lambda$ is expressible in both logics.

Easy generalization of the expressibility of "being a transposition" (which corresponds to $\lambda = (2)$.)

Main theorem(s)

Main theorem – weak form (Albert, Bouvel & F. 18)

Let (P) be a property expressible in both TOOB and TOTO. Then

- either all permutations with sufficiently large support verify (P),
- or there is a bound on the size of the support of permutations verifying (P).

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Main theorem – strong form (Albert, Bouvel & F. 18)

Consider the boolean algebra

$$\mathfrak{A} = \big\langle \mathcal{C}_\lambda, \mathcal{D}_\lambda, \, \lambda \text{ partition} \big\rangle,$$

i.e. the smaller collection of sets containing all $\mathcal{C}_\lambda$'s, all $\mathcal{D}_\lambda$'s, and closed by taking unions, intersections and complements.
Then being in some set $\mathcal{A}$ is expressible in both TOOB and TOTO if and only if $\mathcal{A}$ is in $\mathfrak{A}$.

Strategy of proof (1/4)

Let (P) be a property expressible in both TOOB and TOTO, and let ℓ be the quantifier depth of its TOTO FO formula.

Lemma (changing length of long full cycles)

If (P) *is satisfied by* some *full cycles of length* $n_1 \geq 2^\ell$, *then* all *full cycles of length at least* 2^ℓ *satisfy* (P).

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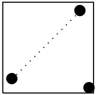

size n_1

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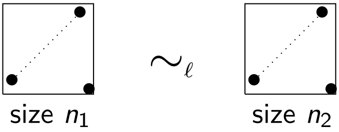

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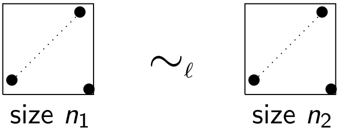

whenever $n_2 \geq 2^\ell$. Thus the full-cycle on the RHS satisfies (P). Using again invariance of (P) by conjugacy, all full cycles of all sizes $n_2 \geq 2^\ell$ satisfy (P).

Strategy of proof (2/4)

Another lemma of the same flavour. Take $k \geq 2^{\ell-1}$.

Lemma (Changing repeated short cycles into a long cycle)

(P) *is satisfies by* some/all *permutations of type* $(2, \dots, 2)$ *(k times),*
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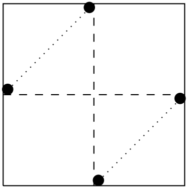

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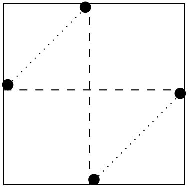
$\sim_\ell$



sizes of the segment differ by 1

type $(2k+1)$;

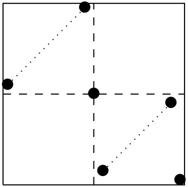
(Easy to adapt to $(m, \ldots, m)$ instead of $(2, \ldots, 2)$ for $\boldsymbol{m \geq 2}$.)

Strategy of proof (3/4)

Lemma (absorbing a small cycle)

(P) *is satisfied by* some/all *full cycles of length* $n \geq 2^\ell$,
iff some/all *permutations of type* $(n-2,1)$ *also satisfy* (P).

Proof (for $n \equiv 2 \pmod 4$):



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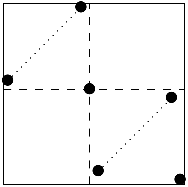

two segments of the same size

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Going from the weak to the strong form is relatively easy.

Other results (1/2)

Stack sorting operator S defined by: $S(LnR) = S(L)S(R)n$.

Knuth ('68) $S(\sigma) = \text{id}$ iff $\sigma \in \text{Av}(231)$;

West ('93) characterization of σ's s.t. $S^2(\sigma) = \text{id}$;

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Theorem (Albert, Bouvel & F. '18)

For all $\ell \geq 1$, there is a constructible TOTO *formula expressing the property $S^\ell(\sigma) = \text{id}$.*

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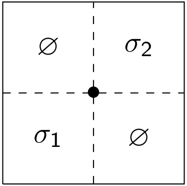
there is a fixed point $\Leftrightarrow$ there is a dot x with no dots at its top left, nor at its bottom right.



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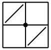
Theorem (Albert, Bouvel & F. '18)

"Having a fixed point" is not expressible in TOTO($\mathcal{C}$) *if and only if* $\mathcal{C}$ *contains either all decreasing permutations or all permutations of the form*  .

$+$ some extension (but no complete characterization) for longer cycles.

Logic and random permutations

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Conjecture (Albert, Bouvel & F. '18)

There is a convergence law for TOTO for uniform random permutations.

Namely, let $\boldsymbol{\sigma}_n$ be a uniform random permutation of size n. Then for any TOTO FO-formula (P),

$$\mathbb{P}\big(\boldsymbol{\sigma}_n \text{ satisfies } (P)\big)$$

has a limit as $n \to \infty$.

Example: $\mathbb{P}(\boldsymbol{\sigma}_n \text{ has an adjacency*}) \to \frac{1}{e^2}$.

*Adjacency: two consecutive entries with consecutive values (in any order).

Thank you for
your attention!