Jack polynomials and multirectangular coordinates

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Symmetric functions

= "polynomials" in infinitely many variables $x_1, x_2, x_3, \dots$
that are invariant by permuting indices

- Augmented monomial basis:

$$\tilde{m}_\lambda = \sum_{\substack{i_1,\dots,i_\ell \geq 1 \\ \text{distinct}}} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

Example: $\tilde{m}_{(2,1,1)} = 2x_1^2 x_2 x_3 + 2x_1 x_2^2 x_3 + 2x_1 x_2 x_3^2 + 2x_1^2 x_2 x_4 + \dots$

- Power-sum basis:

$$p_r = x_1^r + x_2^r + \dots, \quad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$$

Schur functions

(s_λ) is another basis of the symmetric function ring.

Several equivalent definitions:

- $s_\lambda = \sum_T x^T$, sum over semi standard Young tableaux ;
- orthogonal basis (for *Hall scalar product*) + triangular over (augmented) monomial basis ;
- with determinants. . .

-> Encode irreducible characters of symmetric and general linear groups.

Jack polynomials

Deformation of Schur functions with a positive real parameter α.

$$(J_\lambda^{(\alpha)}) \text{ basis}, \qquad J_\lambda^{(1)} = \mathrm{cst}_\lambda \cdot s_\lambda$$

Several equivalent definitions:

- $J_\lambda = \sum_T \psi_T(\alpha) x^T$, sum over semi standard Young tableaux ;
- orthogonal basis (for a deformation of *Hall scalar product*) + triangular over (augmented) monomial basis.

For $\alpha = 1/2, 2$, they also have a representation-theoretical interpretation (in terms of Gelfand pairs) but not in general !

Polynomiality in α with non-negative coefficients

Both definitions involve rational functions in α. Nevertheless, ...

Macdonald-Stanley conjecture ($\sim$ 90)

The coefficients of Jack polynomials in augmented monomial basis are polynomials in α with non-negative integer coefficients.

Notation: $[\tilde{m}_\tau]J_\lambda$.

Polynomiality in α with non-negative coefficients

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Knop-Sahi theorem (97)

The coefficients of Jack polynomials in augmented monomial basis are polynomials in α with non-negative integer coefficients.

Notation: $[\tilde{m}_\tau]J_\lambda$.

KS give a combinatorial interpretation of $[\tilde{m}_\tau]J_\lambda$ as a weighted enumeration of *admissible* tableaux.

A function on the set of all Young diagrams

Definition

Let μ be a partition of k. Define

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \begin{cases} \binom{n-k+m_1(\mu)}{m_1(\mu)} \cdot z_\mu \cdot [p_{\mu 1^{n-k}}] J_\lambda^{(\alpha)} & \text{if } n = |\lambda| \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{Ch}_\mu^{(\alpha)}$ is a function on all Young diagrams.

z_μ: standard explicit numerical factor.

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$\mathrm{Ch}_\mu^{(\alpha)}$ is a function on all Young diagrams.

Specialization: if $|\mu| < |\lambda|$,

$$\mathrm{Ch}_\mu^{(1)}(\lambda) = \frac{|\lambda|!}{(|\lambda| - |\mu|)!} \cdot \frac{\chi^\lambda_{\mu 1^{n-k}}}{\dim(V_\lambda)}.$$

Introduced by S. Kerov, G. Olshanski in the case $\alpha = 1$ (to study random diagrams with Plancherel measure), by M. Lassalle in the general case.

A function on the set of all Young diagrams

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Proposition (Kerov/Olshanski for $\alpha = 1$, Lassalle in general)

For any r, the map

$$(\lambda_1, \dots, \lambda_r) \mapsto \mathrm{Ch}_\mu^{(\alpha)}\left((\lambda_1, \dots, \lambda_r)\right)$$

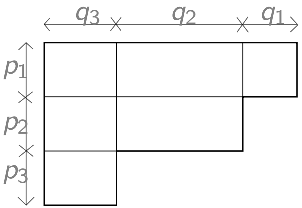
is a polynomial in λ_1, ..., λ_r. Besides, it is symmetric in $\lambda_1 - 1/\alpha$, ..., $\lambda_r - r/\alpha$.

In other words, $\mathrm{Ch}_\mu^{(\alpha)}$ is a shifted symmetric function.

Multirectangular coordinates (R. Stanley)

Consider two lists $\mathbf{p}$ and $\mathbf{q}$ of positive integers of the same length m.
We associate to them the partition

$$\lambda(\mathbf{p},\mathbf{q}) = \Big(\underbrace{q_1+\dots+q_m, \dots, q_1+\dots+q_m}_{p_1 \text{ times}}, \underbrace{q_2+\dots+q_m, \dots, q_2+\dots+q_m}_{p_2 \text{ times}}, \dots \Big).$$



Young diagram of $\lambda(\mathbf{p},\mathbf{q})$

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Proposition (Lassalle, F., Dołęga)

Let μ be a partition of k. $\mathrm{Ch}_\mu^{(\alpha)}(\lambda(\mathbf{p},\mathbf{q}))$ is a polynomial in

$$p_1, p_2, \ldots, q_1, q_2, \ldots, \alpha$$

Lassalle gave an algorithm to compute those polynomials.

Multirectangular coordinates (R. Stanley)

Consider two lists **p** and **q** of positive integers of the same length m. We associate to them the partition

$$\lambda(\mathbf{p},\mathbf{q}) = \Big(\underbrace{q_1+\cdots+q_m,\ldots,q_1+\cdots+q_m}_{p_1 \text{ times}},\underbrace{q_2+\cdots+q_m,\ldots,q_2+\cdots+q_m}_{p_2 \text{ times}},\ldots\Big).$$

Conjecture (M. Lassalle, still open)

Let μ be a partition of k. $(-1)^k \operatorname{Ch}^{(\alpha)}_\mu(\lambda(\mathbf{p},\mathbf{q}))$ is a polynomial in

$$p_1, p_2, \ldots, -q_1, -q_2, \ldots, \alpha - 1$$

with non-negative integer coefficients.

Link between the two questions ?

Knop-Sahi theorem and Lassalle conjecture do not seem related.

Two (main) differences:

Knop-Sahi	Lassalle
monomial coefficients	power-sum coefficients
look at one coefficient	coefficients as shifted symmetric functions

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Two (main) differences:

Knop-Sahi	Lassalle
monomial coefficients	power-sum coefficients
look at one coefficient	coefficients as shifted symmetric functions

Idea: look at monomial coefficients as shifted symmetric functions.

Monomial coefficients as shifted symmetric functions

Definition

Let μ be a partition of k. Define

$$\mathrm{Ko}_\mu^{(\alpha)}(\lambda) = \begin{cases} \binom{n-k+m_1(\mu)}{m_1(\mu)} \cdot z_\mu \cdot [\tilde{m}_{\mu 1^{n-k}}] J_\lambda^{(\alpha)} & \text{if } n = |\lambda| \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

Monomial coefficients as shifted symmetric functions

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Proposition

$\mathrm{Ko}_\mu^{(\alpha)}$ is a shifted symmetric function.

Proof: Easy $\mathrm{Ko}_\mu^{(\alpha)} = \sum_{\nu \vdash k} L_{\mu,\nu} \, \mathrm{Ch}_\nu^{(\alpha)}$ (with $L_{\mu,\nu}$ defined by $p_\nu = \sum_{\mu \vdash k} L_{\mu,\nu} \tilde{m}_\mu$).

Then apply Lassalle proposition.

A new conjecture

Proposition

$\mathrm{Ko}_{\mu}^{(\alpha)}(\mathbf{p} \times \mathbf{q})$ is a polynomial in $\mathbf{p}$, $\mathbf{q}$ and α.

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- when specialized to non-negative values of $\mathbf{p}$ and $\mathbf{q}$, $\mathrm{Ko}_\mu^{(\alpha)}(\mathbf{p} \times \mathbf{q})$ is a polynomial in α with nonnegative coefficients (KS) ;
- But it does not have non-negative coefficients as polynomial in $\mathbf{p}$, $\mathbf{q}$ and α.

A new conjecture

Proposition

$\mathrm{Ko}_\mu^{(\alpha)}(\mathbf{p} \times \mathbf{q})$ is a polynomial in $\mathbf{p}$, $\mathbf{q}$ and α.

Conjecture (F., Alexandersson)

In the falling factorial basis in $\mathbf{p}$ and $\mathbf{q}$, $\mathrm{Ko}_\mu^{(\alpha)}(\mathbf{p} \times \mathbf{q})$ has non-negative integer coefficients.

falling factorial: $(n)_k := n(n-1)\dots(n-k+1)$.

falling factorial basis: $\Big((p_1)_{i_1}(p_2)_{i_2}\dots(q_1)_{j_1}(q_2)_{j_2}\dots\alpha^k\Big)$.

It is stronger than positivity in Knop-Sahi theorem (and does not follow from their combinatorial interpretation) !

Evidence for our conjecture

- Computer exploration: the conjecture holds for $|\mu| \leq 9$ and 4 rectangles ;
- Proof for $\mu = (k)$;
- Proof for $\alpha = 1$ (and $\alpha = 2$) ;

Nice combinatorics in these case.

Combinatorial formulas for $\alpha = 1$

For $\alpha = 1$, there is a combinatorial formula for $\mathrm{Ch}_\mu^{(1)}$:

Theorem (F. 2007; F., Śniady 2008 ; conj. by Stanley 2006)

Let μ a partition of k. Fix a permutation π in S_k of type μ. Then

$$\mathrm{Ch}_\mu(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau = \pi}} \varepsilon(\tau) N_{\sigma,\tau}(\mathbf{p}, \mathbf{q}).$$

$N_{\sigma,\tau}$: combinatorial polynomial with non-negative integer coefficients.
$\Rightarrow$ Lassalle conjecture holds for $\alpha = 1$.

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Proposition

Fix a set-partition $\Pi = \{\Pi_1, \dots, \Pi_\ell\}$ whose block sizes are given by μ. Then

$$\mathrm{Ko}_\mu^{(1)}(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau \in S_\Pi}} \varepsilon(\tau) N_{\sigma,\tau}(\mathbf{p}, \mathbf{q}),$$

where $S_\Pi = S_{\Pi_1} \times S_{\Pi_2} \cdots \times S_{\Pi_\ell}$.

$\mathrm{Ko}^{(\alpha)}_{(k)}$ is FF non-negative.

Observation: $\mathrm{Ko}^{(1)}_{(k)}(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\sigma,\tau \in S_k \\ \text{no restriction}}} \varepsilon(\tau) N_{\sigma,\tau}(\mathbf{p},\mathbf{q}).$

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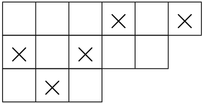
This formula simplifies to:

$$\mathrm{Ko}^{(1)}_{(k)}(\lambda) = \sum_{\substack{E \subset \lambda,\ |E|=k \\ \text{for any column } C,\ |E \cap C| \leq 1}} \prod_{\substack{R \text{ row} \\ \text{of } \lambda}} |E \cap R|!$$

Example:  has weight 4 in $\mathrm{Ko}^{(1)}_{(5)}((6,5,3))$.

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Example:

			×		×
×		×			
	×				

has weight $(\alpha+1)^2$ in $\mathrm{Ko}^{(\alpha)}_{(5)}((6,5,3))$.

Proposition (deformation for a general α)

$$\mathrm{Ko}^{(\alpha)}_{(k)}(\lambda) = \sum_{\substack{E \subset \lambda,\ |E|=k \\ \text{for any column } C,\ |E \cap C| \leq 1}} \prod_{\substack{R \text{ row} \\ \text{of } \lambda}} (1+\alpha) \cdots (1 + (|E \cap R| - 1)\alpha)$$

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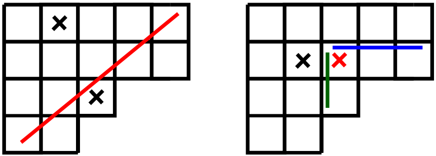
Sketch of proof: Start from KS combinatorial interpretation

$$\mathrm{Ko}^{(\alpha)}_{(k)} \overset{\mathrm{KS}}{=} \sum_{\substack{F\subset\lambda,\ |F|=k \\ \text{for any column } C,\ |F\cap C|\leq 1 \\ \text{forbidden pattern}}} \mathrm{wt}_\alpha(F).$$



forbidden pattern   weight: $3\alpha + 2$

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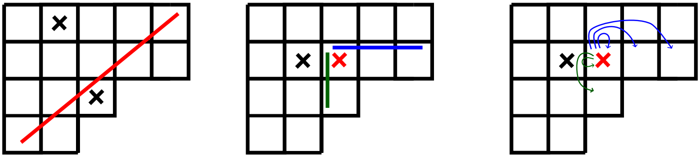

forbidden pattern — weight: $3\alpha + 2$ — Apply this rule

With well chosen rules to solve conflicts (two elements in the same column), each E is obtained with the good weight.

Proposition

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- This formula implies that $\mathrm{Ko}^{(\alpha)}_{(k)}$ is non negative in the falling factorial basis, while KS combinatorial interpretation does not.
- The reason is that our combinatorial interpretation is dilatation invariant.
- Maybe our conjecture suggests the existence of a dilatation-invariant combinatorial interpretation of $[\tilde{m}_{\mu 1^{n-k}}]J^{(\alpha)}_\lambda$ (so far, we have one only for hooks!).

Back to $\alpha = 1$

$$\mathrm{Ko}_\mu^{(1)}(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau \in S_\Pi}} \varepsilon(\tau) N_{\sigma,\tau}(\mathbf{p},\mathbf{q}),$$

…use explicit expression of $N_{\sigma,\tau}(\mathbf{p},\mathbf{q})$ + sum manipulations …

It is enough to prove

Question 1

Fix three set partitions T, U and Π of the same set and define $S_T = S_{T_1} \times \cdots \times S_{T_l}$. Then

$$\sum_{\substack{\sigma \in S_T, \tau \in S_U \\ \sigma\tau \in S_\Pi}} \varepsilon(\tau) \geq 0.$$

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$$\sum_{\substack{\sigma \in S_T, \tau \in S_U \\ \sigma\tau \in S_\Pi}} \varepsilon(\tau) \geq 0.$$

Natural idea: construct an involution by multiplication by a well-chosen transposition. This does not work here! (we have one example where $\tau = \mathrm{id}$ or $\tau = (1\ 2)(3\ 4)(5\ 6)$)

Other approach

$$\mathrm{Ko}_\mu^{(1)}(\lambda) = [\tilde{m}_{\mu 1^{n-k}}] J_\lambda^{(1)} = [m_{\mu 1^{n-k}}] \frac{n!}{(n-k)! \prod_i m_i(\mu)!} \frac{s_\lambda}{\dim(\lambda)}$$
$$= \frac{n!}{(n-k)! \prod_i m_i(\mu)! \dim(\lambda)} \# \mathrm{SSYT}(\lambda, \mu 1^{n-k})$$

Ex:

5	7	8	
2	3	4	6
1	1	1	2

$\in \mathrm{SSYT}((4,4,3),(3,2,1^6))$.

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$\mathrm{SSYT}(\lambda, \mu 1^{n-k}) \longleftrightarrow \{(\nu, T, T') \text{ with } T \in \mathrm{SSYT}(\nu, \mu), T' \in SYT(\lambda/\nu)\}$

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Shifted Schur function (Okounkov, Olshanski)

Using shifted Schur function (1/2)

In the previous slide, we have seen:

Proposition

$\mathrm{Ko}_\mu^{(1)}$ is a non-negative linear combination of $s_\nu^\#$.

Is $s_\nu^\#$ non-negative in the falling factorial basis ?

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$$s_\mu^\#(\mathbf{p} \times \mathbf{q}) = \sum_{\sigma,\tau \in S_k} \chi^\mu(\sigma\,\tau)\,\varepsilon(\tau)\,N_{\sigma,\tau}(\mathbf{p},\mathbf{q}),$$

where χ^μ is an irreducible character of the symmetric group.

Using shifted Schur functions (2/2)

$$s^{\#}_{\mu}(\mathbf{p} \times \mathbf{q}) = \sum_{\sigma,\tau \in S_k} \chi^{\mu}(\sigma\,\tau)\,\varepsilon(\tau) N_{\sigma,\tau}(\mathbf{p}, \mathbf{q}),$$

... use explicit expression of $N_{\sigma,\tau}(\mathbf{p}, \mathbf{q})$ + sum manipulations ...

It is enough to prove

Question

For any two set partitions T, U of $[n]$ and integer partition μ of n,

$$\sum_{\sigma \in S_T, \tau \in S_U} \varepsilon(\tau)\,\chi^{\mu}(\sigma\,\tau) \geq 0.$$

Using shifted Schur functions (2/2)

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. . . use explicit expression of $N_{\sigma,\tau}(\mathbf{p},\mathbf{q})$ + sum manipulations . . .

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Proof by easy representation-theoretical manipulations.

Another conjecture

Shifted Schur function can be deformed to shifted Jack polynomials $J^{\#}_{(\alpha)}$.

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Conjecture (F., Alexandersson)

In the falling factorial basis in $\mathbf{p}$ and $\mathbf{q}$, $\alpha^{\ell(\mu)} J^{\sharp}{}^{(\alpha)}_{\mu}(\mathbf{p} \times \mathbf{q})$ has non-negative integer coefficients.

Proposition

$\mathrm{Ko}^{(\alpha)}_{\mu}$ is a linear combination of $\alpha^{\ell(\mu)} J^{\sharp}{}^{(\alpha)}_{\mu}(\mathbf{p} \times \mathbf{q})$. The coefficients in this linear combination are rational functions in α with non-negative coefficients in the numerator and denominator.

This new conjecture does not imply our other conjecture !

Conclusion

A bridge between KS theorem and Lassalle's conjecture:

- Our conjecture involves shifted symmetric functions and multirectangular coordinates and implies KS theorem ;
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Future work:

- What about (shifted) Macdonald polynomials and multirectangular coordinates?