Cyclic inclusion-exclusion and the kernel of P-partitions

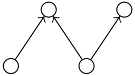
Valentin Féray

Institut für Mathematik, Universität Zürich

FPSAC 28, Vancouver, July 4th – July 8th 2016

Universität Zürich[UZH]

Main object

Take an unlabelled acyclic directed graph $G_{\text{ex}} =$  .

Definition

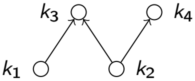
A function $f : V_G \to \mathbb{N}$ is order-preserving if

$$(i,j) \in E_G \;\Rightarrow\; f(i) \leq f(j).$$

We consider the multivariate generating function in $x_1, x_2, \dots$

$$\Gamma(G) = \sum_{\substack{f : V \to \mathbb{N} \\ f \text{ order-preserving}}} \prod_{v \in V} x_{f(v)} \quad \in \mathrm{QSym}\,.$$

Main object

Take an unlabelled acyclic directed graph $G_{\text{ex}} =$ 

.

Definition

A function $f : V_G \to \mathbb{N}$ is order-preserving if

$$(i,j) \in E_G \;\Rightarrow\; f(i) \leq f(j).$$

We consider the multivariate generating function in $x_1, x_2, \dots$

$$\Gamma(G) = \sum_{\substack{f:V\to\mathbb{N} \\ f \text{ order-preserving}}} \prod_{v\in V} x_{f(v)} \quad \in \mathrm{QSym}\,.$$

On the example above:

$$\Gamma(G_{\text{ex}}) = \sum_{\substack{k_1,k_2,k_3,k_4 \\ k_1\leq k_3,\; k_2\leq k_3,\; k_2\leq k_4}} x_{k_1} x_{k_2} x_{k_3} x_{k_4}\,.$$

Variant: have some strict inequalities.

Background and result

- ordered preserving function $\leftrightarrow$ weak P-partition (Stanley, 72).
- multivariate GF considered in a seminal paper of Gessel, 84.
  $\rightarrow$ introduces QSym and gives the fundamental expansion of $\Gamma(G)$.
- when do we have $\Gamma(G_1) = \Gamma(G_2)$? (MaNamara, Ward, 13).
- Surjectivity of Γ as a linear map (Billera, Reiner, Stanley, 05).
- Γ as a Hopf algebra morphism. Extension to non-commutative variable and to *finite topologies* (Foissy, Malvenuto, 15).

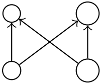
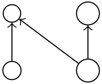
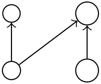
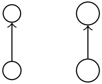
Background and result

- ordered preserving function $\leftrightarrow$ weak P-partition (Stanley, 72).
- multivariate GF considered in a seminal paper of Gessel, 84.
  $\rightarrow$ introduces QSym and gives the fundamental expansion of $\Gamma(G)$.
- when do we have $\Gamma(G_1) = \Gamma(G_2)$? (MaNamara, Ward, 13).
- Surjectivity of Γ as a linear map (Billera, Reiner, Stanley, 05).
- Γ as a Hopf algebra morphism. Extension to non-commutative variable and to *finite topologies* (Foissy, Malvenuto, 15).

Main result (informal)

A simple combinatorial description of the kernel of Γ.
Extension to the non-commutative/bipartite framework.

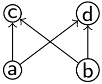
A first simple relation

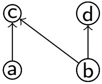
$G_0 =$  , $G_1 =$  , $G_2 =$  , $G_{1,2} =$  .

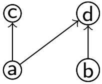
Lemma

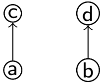
$$\Gamma(G_0) - \Gamma(G_1) - \Gamma(G_2) + \Gamma(G_{1,2}) = 0$$

A first simple relation

$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_{1,2} =$ 

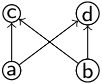
.

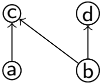
Lemma

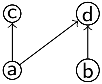
$$\Gamma(G_0) - \Gamma(G_1) - \Gamma(G_2) + \Gamma(G_{1,2}) = 0$$

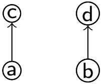
Proof: All graphs have the same vertex-set $V = \{a, b, c, d\}$. A function $f : V \to \mathbb{N}$ contributes $\prod x_{f(v)}$ to $\Gamma(G_i)$ if it is G_i order preserving.
We show that the total contribution of any f to LHS is 0.

A first simple relation

$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_{1,2} =$ 

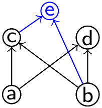
.

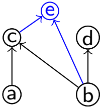
Lemma

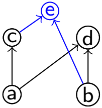
$$\Gamma(G_0) - \Gamma(G_1) - \Gamma(G_2) + \Gamma(G_{1,2}) = 0$$

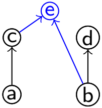
Proof: All graphs have the same vertex-set $V = \{a, b, c, d\}$. A function $f : V \to \mathbb{N}$ contributes $\prod x_{f(v)}$ to $\Gamma(G_i)$ if it is G_i order preserving.
We show that the total contribution of any f to LHS is 0.

- if $f(a) > f(c)$ or $f(b) > f(d)$, then f contributes always zero.
- if $f(a) \leq f(d)$, then f has the same contribution to $\Gamma(G_0)$ and $\Gamma(G_1)$ on one hand, and to $\Gamma(G_2)$ and $\Gamma(G_{1,2})$ on the other hand.
- idem if $f(b) \leq f(c)$.
- Otherwise, $f(a) \leq f(c) < f(b) \leq f(d) < f(a)$. Contradiction! □

$G_0 =$ 

, $G_1 =$ 

, $G_2 =$ 

, $G_{1,2} =$ 

.

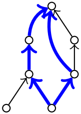
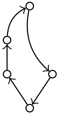
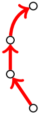
Lemma

$$\Gamma(G_0) - \Gamma(G_1) - \Gamma(G_2) + \Gamma(G_{1,2}) = 0$$

The relation still holds with additional vertices and edges or with longer cycles. . .

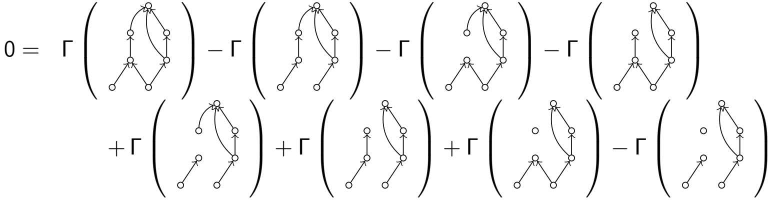
General cyclic inclusion-exclusion

Let G be an acyclic digraph and C a cycle in the undirected version of G, C^+ the edges of C read from **bottom-to-top** when following C.

$G_{ex} =$  , $C_{ex} =$  , $C^+ =$  .

Proposition: cyclic inclusion-exclusion (CIE) relations

$$\sum_{D \subseteq C^+} (-1)^{|D|} \Gamma(G \setminus D) = 0.$$



Main result (commutative non-restricted case)

Theorem

The cyclic-inclusion relations form a complete set of relations among the $\Gamma(G)$*. In other words, the elements of the form*

$$\sum_{D \subseteq C^+} (-1)^{|D|} (G \setminus D)$$

span the kernel of the linear operator Γ*.*

First approach: removing edges

Proposition

Span($\Gamma(G)$) *is spanned by the* $\Gamma(F)$, *where* F *run over forests.*

Proof: choose cycles and apply CIE relation until you cannot find any cycle any more.

First approach: removing edges

Proposition

Span($\Gamma(G)$) *is spanned by the* $\Gamma(F)$, *where* F *run over forests.*

Proof: choose cycles and apply CIE relation until you cannot find any cycle any more.

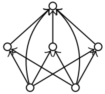
Problem

The $\Gamma(F)$ are not independent; the expansion of $\Gamma(G)$ obtained by iterating CIE relations depend on the choices we make.

Second approach: adding edges

Definition

Let $I = (i_1, \dots, i_\ell)$ be a composition. We define G_I as the graph with vertex set $V = \biguplus_{j=1}^{\ell} V_j$ with $|V_j| = i_j$ and edge set $E = \biguplus_{j<k} V_j \times V_k$.

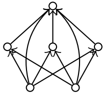


$G_{(2,3,1)}$

Second approach: adding edges

Definition

Let $I = (i_1, \dots, i_\ell)$ be a composition. We define G_I as the graph with vertex set $V = \biguplus_{j=1}^{\ell} V_j$ with $|V_j| = i_j$ and edge set $E = \biguplus_{j<k} V_j \times V_k$.



$G_{(2,3,1)}$

Lemma

If G is not one of the G_I, then there exists a graph G_0 and a cycle C in G_0 such that $G = G_0 \setminus C^+$ (i.e. G is the smallest graph in the CIE relation of (G_0, C)).

By iterating CIE relations, we can write any $\Gamma(G)$ as a linear combination of $\Gamma(G_I)$.

Proposition

The $\Gamma(G_I)$ forms a $\mathbb{Z}$-basis of QSym.

End of the proof

Take an element $\sum c_G\, G$ in the kernel of Γ, i.e.

$$\sum c_G\, \Gamma(G) = 0.$$

End of the proof

Take an element $\sum c_G\, G$ in the kernel of Γ, i.e.

$$\sum c_G\, \Gamma(G) = 0.$$

Using CIE relations, we can write the LHS as a linear combination of $\Gamma(G_I)$.

End of the proof

Take an element $\sum c_G\, G$ in the kernel of Γ, i.e.

$$\sum c_G\, \Gamma(G) = 0.$$

Using CIE relations, we can write the LHS as a linear combination of $\Gamma(G_I)$.

But since $\Gamma(G_I)$ are linearly independent, all coefficients in this expression must be 0.

End of the proof

Take an element $\sum c_G\, G$ in the kernel of Γ, i.e.

$$\sum c_G\, \Gamma(G) = 0.$$

Using CIE relations, we can write the LHS as a linear combination of $\Gamma(G_I)$.

But since $\Gamma(G_I)$ are linearly independent, all coefficients in this expression must be 0.

I.e. we have gone from $\sum c_G\, G$ to 0 by adding/substracting elements of the form

$$\sum_{D \subseteq C^+} (-1)^{|D|}\, (G \setminus D). \qquad \square$$

Comments

We get a bit more than the kernel:

- the surjectivity of Γ (already observed by Stanley, 05);
- a basis $\Gamma(G_I)$ and an "algorithm" to write any function $\Gamma(G)$ in this basis;

Comments

We get a bit more than the kernel:

- the surjectivity of Γ (already observed by Stanley, 05);
- a basis $\Gamma(G_I)$ and an "algorithm" to write any function $\Gamma(G)$ in this basis;

Quite robust approach:

- extends to non-commutative variables (using labelled graphs and WQSym);
- also true for the restriction to bipartite graphs ;
- and for bipartite graphs in the non-commutative setting (here, showing that the relevant family is a basis of WQSym is highly non-trivial).

Applications

- The question popped up in studying symmetric group character polynomials...

Applications

- The question popped up in studying symmetric group character polynomials...

- Cyclic inclusion-exclusion relations can be applied to any sum of the form

$$\sum_{\substack{i_1,\dots,i_r \\ \text{some inequalities}}} \cdots$$

e.g. Boussicault, F., FPSAC 2009.

Applications

- The question popped up in studying symmetric group character polynomials...
- Cyclic inclusion-exclusion relations can be applied to any sum of the form

$$\sum_{\substack{i_1,\dots,i_r \\ \text{some inequalities}}} \cdots$$

e.g. Boussicault, F., FPSAC 2009.

The fact that CIE form a complete set of relations tell you that you are not missing something when studying this kind of sum with CIE...

Applications

- The question popped up in studying symmetric group character polynomials...

- Cyclic inclusion-exclusion relations can be applied to any sum of the form

$$\sum_{\substack{i_1,\dots,i_r \\ \text{some inequalities}}} \cdots$$

  e.g. Boussicault, F., FPSAC 2009.

  The fact that CIE form a complete set of relations tell you that you are not missing something when studying this kind of sum with CIE...

- Proof reveals nice bases of QSym/WQSym, on which $\Gamma(G)$ expands naturally.

Open questions/ future work

- In the non-commutative framework: can we find a basis $\boldsymbol{F_I}$ of WQSym such that all $\Gamma(\boldsymbol{G})$ expand positively on $\boldsymbol{F_I}$? There is one for bipartite $\boldsymbol{G}$.
- Does this help to classify graphs with the same image by Γ?
- Similar results for other objects : chromatic symmetric function, Billera-Jia-Reiner quasi-symmetric functions for matroids, . . .

Open questions/ future work

- In the non-commutative framework: can we find a basis $\boldsymbol{F_I}$ of WQSym such that all $\Gamma(\boldsymbol{G})$ expand positively on $\boldsymbol{F_I}$? There is one for bipartite $\boldsymbol{G}$.
- Does this help to classify graphs with the same image by Γ?
- Similar results for other objects : chromatic symmetric function, Billera-Jia-Reiner quasi-symmetric functions for matroids, ...

Thank you for your attention!