Characters of symmetric groups, free cumulants and a combinatorial Hopf algebra

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What is this talk about?

- Irreducible representations of $S_n \simeq$ partitions $\lambda \vdash n$.
- We are interested in normalized character values:

$$\chi^\lambda(\sigma) = \frac{\mathrm{tr}\left(\rho^\lambda(\sigma)\right)}{\dim(V_\lambda)}.$$

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- Link with free probability (Biane, 1998):

$$\chi^\lambda\big((1\ 2\ \dots\ k)\big) \sim_{|\lambda|\to\infty} R_{k+1}(\mu_\lambda).$$

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- These objects live in a combinatorial Hopf algebra.

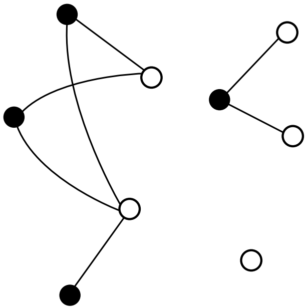
Outline of the talk

1. A combinatorial Hopf algebra
   - Bipartite graphs
   - Relations
   - Polynomial realization
2. Characters of symmetric group and free cumulants
   - Definitions
   - Combinatorial formulas
   - Application to Kerov's polynomials

The ground set of combinatorial objects

We consider:

- unlabelled undirected bipartite graphs,
- without multiple edges,
- without isolated black vertices.



A Hopf algebra structure

We define $\mathcal{H}$ as:

- the space of finite linear combination of graphs;
- the product is defined on the basis by the disjoint union:

$$G \cdot G' = G \sqcup G'$$

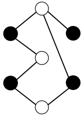
- the coproduct is given by:

$$\Delta(G) = \sum_{E \subset V_\circ(G)} G_E \otimes (G \backslash G_E),$$

  where G_E is the induced graph on the vertices in E and their neighbours.

With some appropriate antipode, $\mathcal{H}$ is a Hopf algebra.

Example of coproduct

$\Delta \Bigg($  $\Bigg) =$

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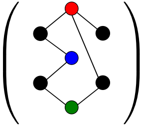
Δ  $=$

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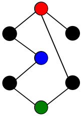
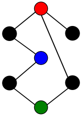
$\Delta \Bigg($  $\Bigg) = 1 \otimes$ 

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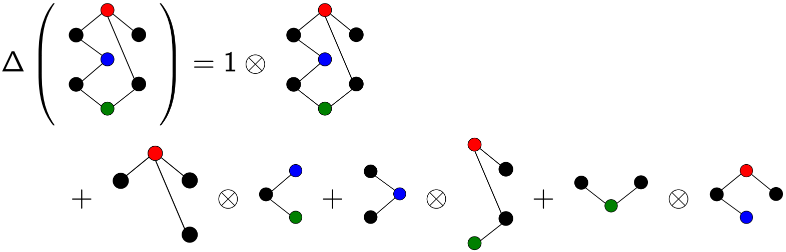

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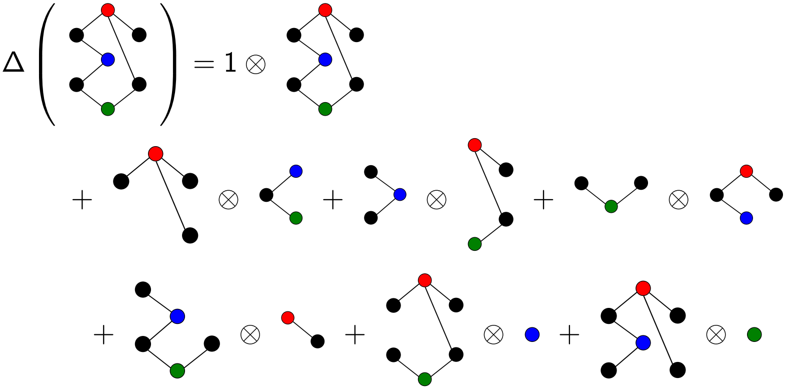

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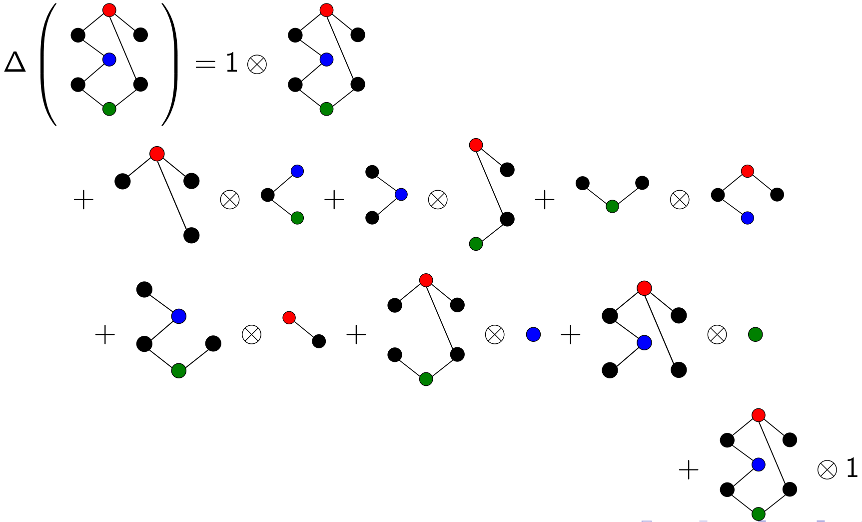
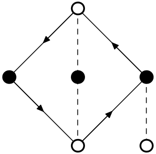


Annihilator elements

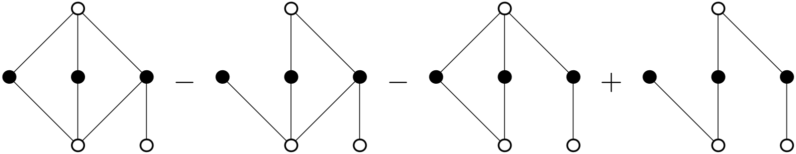
Consider a bipartite graph G endowed with an oriented cycle C.



We define the following element of $\mathcal{H}$

$$\mathcal{A}_{G,C} = \sum_{E \subseteq E_{\circ \to \bullet}(C)} (-1)^{|E|} G \setminus E$$

Example :



Quotient Hopf algebra

Let

$$\mathcal{I} := \mathrm{Vect}(\mathcal{A}_{G,C})$$

Lemma

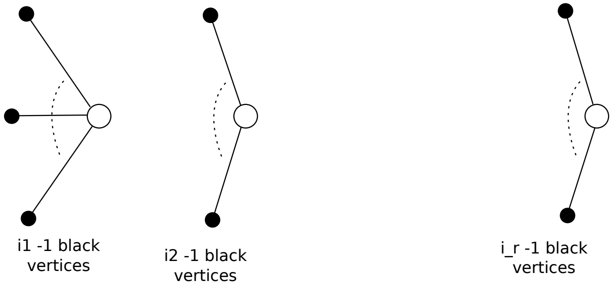
- $\mathcal{I}$ is an ideal of the algebra $\mathcal{H}$.
- $\Delta(\mathcal{I}) \subseteq \mathcal{I} \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{I}$.

Hence $\mathcal{H}/\mathcal{I}$ is a Hopf algebra.

A generating family

Definition

Let $I = (i_1, i_2, \dots, i_r)$ be a composition. Define G_I as the following bipartite graph:



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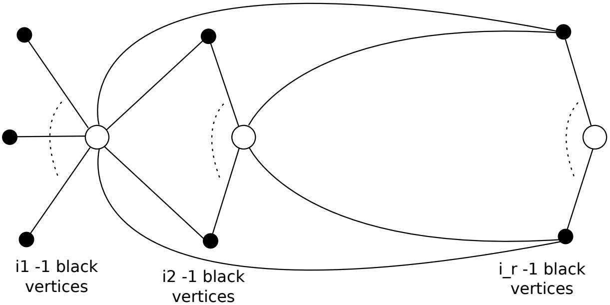

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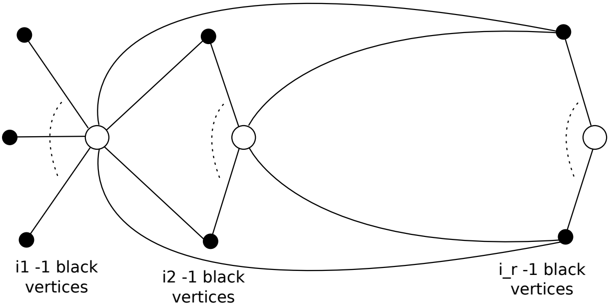

Proposition

$\{G_I, I$ composition$\}$ is a linear generating set of $\mathcal{H}/\mathcal{I}$.

Idea of proof

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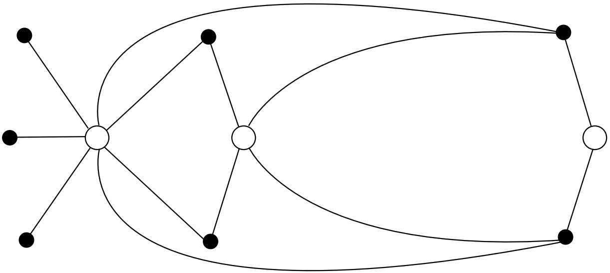

graph G_I

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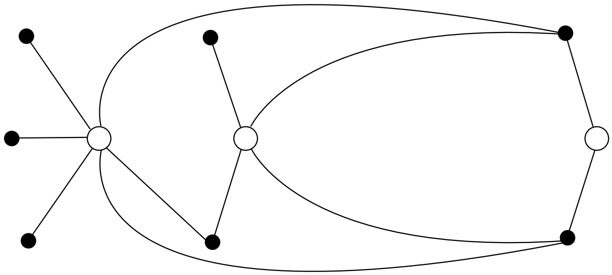

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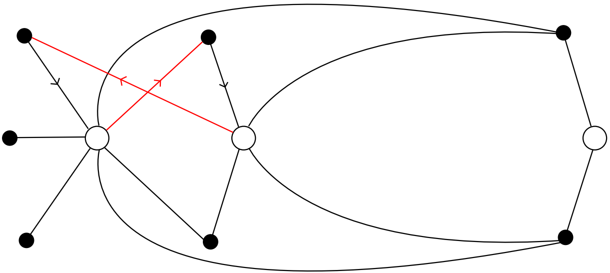

There is a graph G_0 with an oriented cycle C such that

$$G_0 \backslash E_{\circ\!\!\rightarrow\!\!\bullet}(C) = G$$

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Proposition

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Consider a graph $G \neq G_I$.

Lemma: There is a graph G_0 with an oriented cycle C such that:

$$G_0 \backslash E_{\circ \rightarrow \bullet}(C) = G$$

Consequence : in $\mathcal{H}/\mathcal{I}$, $G =$ linear combination of bigger graphs.

$\rightarrow$ we iterate until we obtain a linear combination of G_I's.

Polynomials associated to graphs

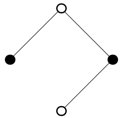
Let $\mathbf{p} = (p_1, p_2, \dots)$ and $\mathbf{q} = (q_1, q_2, \dots)$ two set (infinite) sets of variables.

Let G be a bipartite graph.

$$M_G(\mathbf{p}, \mathbf{q}) = \sum_{\varphi : V_\circ(G) \to \mathbb{N}^\star} \prod_{\circ \in V_\circ} p_{\varphi(\circ)} \prod_{\bullet \in V_\bullet} q_{\psi(\bullet)},$$

where $\psi(\bullet) = \max_{\substack{\circ \text{ neighbour} \\ \text{of } \bullet}} \varphi(\circ)$.

Example:

M  $= \sum_{i,j \geq 1} p_i p_j q_i q_{\max(i,j)}$.

Identification of the two algebras

Lemma

- $M_{G \sqcup G'} = M_G \cdot M_{G'}$.
- $M_{\mathcal{A}_{G,C}} = 0$.

The algebra $\mathrm{Vect}(M_G)$ is a quotient of $\mathcal{H}/\mathcal{I}$.

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The M_{G_I}, where I runs over all compositions are linearly independent.

Hence, $\mathrm{Vect}(M_G) \simeq \mathcal{H}/\mathcal{I}$ and the G_I's form a basis.

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Remark

It is also isomorphic to the quasi-symmetric function ring.

Sketch of proof

Lemma

The M_{G_I}, where I runs over all compositions are linearly independent.

Consider $M_{G_I}(p_1, p_2, \dots, p_r, q_1, q_2, \dots, q_r)$
(we truncate the alphabets to $r = \ell(I) = |V_\circ(G_I)|$ variables)

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We will consider only p-square free monomials.

As total degree in p is r, they are:

$$T_J = p_1 q_1^{j_1 - 1} p_2 q_2^{j_2 - 1} \cdots p_r q_r^{j_r - 1},$$

where J is a composition of n (total number of vertices)

In M_{G_I}, they correspond to bijections $\varphi : V_\circ(G_I) \simeq \{1, \dots, r\}$.

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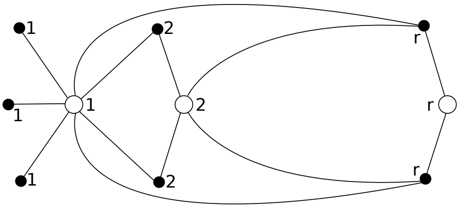

$$M_{G_I} = T_I$$

Sketch of proof

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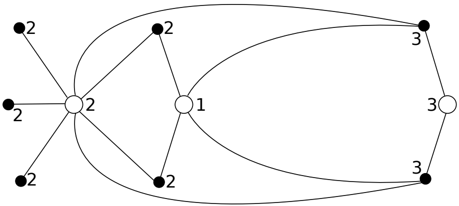


$$M_{G_I} = T_I + \sum_{\substack{|J|=|I|=n,\ \ell(J)=\ell(I)=r \\ J \geq I}} c_{I,J} T_J$$

$$+ \text{ non-}p\text{-square-free terms}$$

$\geq$ stands for the right-dominance order.

Interpretation of the coproduct

Let us write

$$\Delta_G = \sum_i G_1^{(i)} \otimes G_2^{(i)}.$$

Then

$$M_G(p_1, \dots, p_{h+\ell}, q_1, \dots, q_{h+\ell}) = \sum_i M_{G_1^{(i)}}(p_{h+1}, \dots, p_{h+\ell}, q_{h+1}, \dots, q_{h+\ell}) \cdot M_{G_2^{(i)}}(p_1, \dots, p_h, q_1, \dots, q_h)$$

Representation theory of symmetric groups

A representation of S_n is a pair (ρ, V):

- V is a finite dimensional $\mathbb{C}$-vector space;
- ρ is a morphism $S_n \to GL(V)$.

i.e., to each $\sigma \in S_n$, we associate a matrix $\rho(\sigma)$ (we ask that the products are compatible).

To a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n (i.e. $\lambda_1 \geq \lambda_2 \geq \dots$ and $\sum_i \lambda_i = n$), we can associate canonically an (irreducible) representation $(\rho_\lambda, V_\lambda)$.

We are interested in characters (=the trace of the representation matrices):

$$\chi^\lambda(\sigma) = \mathrm{Tr}(\rho_\lambda(\sigma))$$

Central characters

Fix a partition μ of k. Let us define

$$\mathrm{Ch}_\mu : \begin{array}{ccc} \mathcal{Y} & \to & \mathbb{Q}; \\ \lambda & \mapsto & n(n-1)\dots(n-k+1)\frac{\chi^\lambda(\sigma)}{\dim(V_\lambda)}, \end{array}$$

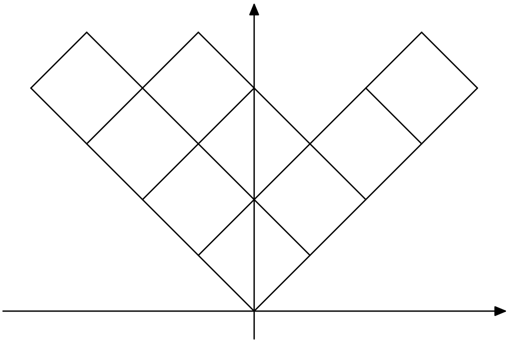
where $n = |\lambda|$
and σ is a permutation in S_n of cycle type $\mu 1^{n-k}$.

Examples:

$$\begin{aligned} \mathrm{Ch}_\mu(\lambda) &= 0 \quad \text{as soon as } |\lambda| < |\mu| \\ \mathrm{Ch}_{1^k}(\lambda) &= n(n-1)\dots(n-k+1) \quad \text{for any } \lambda \vdash n \\ \mathrm{Ch}_{(2)}(\lambda) &= n(n-1)\chi^\lambda((1\ 2)) = \sum_i (\lambda_i)^2 - (\lambda'_i)^2 \\ \mathrm{Ch}_{\mu \cup 1}(\lambda) &= (n - |\mu|)\,\mathrm{Ch}_\mu(\lambda) \quad \text{for any } \lambda \vdash n \end{aligned}$$

Partition, Young diagrams and interlacing coordinates

Consider partition $\lambda = (4, 2, 2, 1)$. We draw the corresponding *Young* diagram (in Russian convention).



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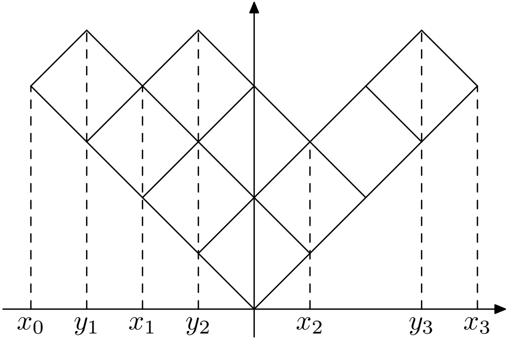


The x_i (resp. y_i) are defined as x-coordinate of inner (resp. outer) corners.

Free cumulants of the transition measure

Transition measure μ_λ:

$$\int_{\mathbb{R}} \frac{d\mu(x)}{z-x} = \frac{\prod_i z-y_i}{\prod z-x_i}$$

Free cumulants:

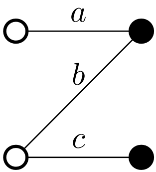
$$R_k(\lambda) := R_k(\mu_\lambda)$$

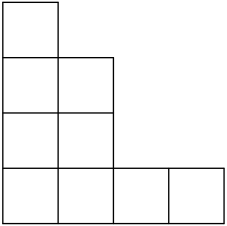
Interesting because:

$$\mathrm{Ch}_{(k)}(\lambda) = R_{k+1}(\lambda) + \text{smaller degree terms in } R$$

Definition of the N_G

Let G be a bipartite graph and λ a partition :





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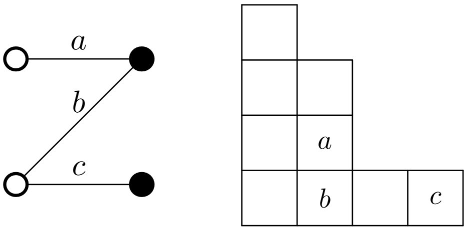

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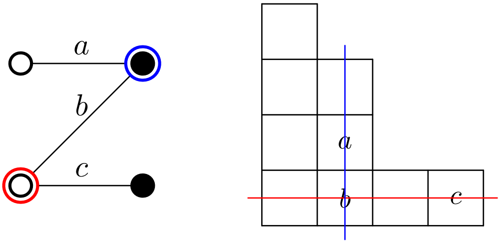

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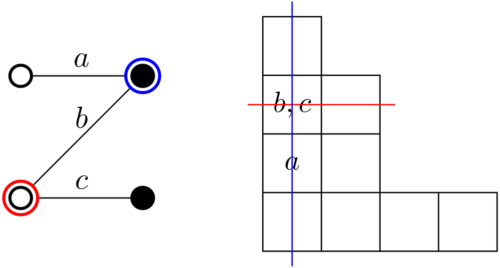

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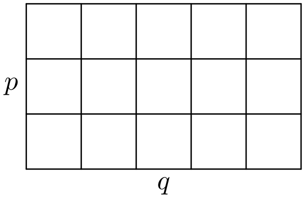
An interesting particular case: rectangular partition

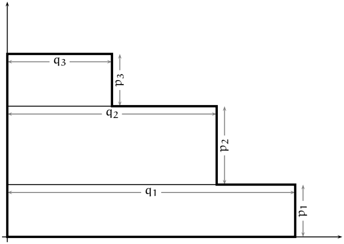
$\lambda =$ 

$$N_G(\lambda) = p^{|V_\circ(G)|} \cdot q^{|V_\bullet(G)|}$$

Indeed, one has to choose independently:

- one row per white vertex ;
- one column per black vertex.

Stanley's coordinates

$\lambda =$ 

$$N_G(\lambda) = M_G(\mathbf{p}, \mathbf{q})$$

As a consequence,

$$\mathrm{Vect}(N_G) \simeq \mathrm{Vect}(M_G) \simeq \mathcal{H}/\mathcal{I}$$

A formula for character values and free cumulants.

Theorem (F. 2006, conjectured by Stanley)

Let $\mu \vdash k$.

$$\mathrm{Ch}_\mu = \sum_C \pm N_{G(C)},$$

where:

- *the sum runs over rooted bipartite maps with k edges and face-length $\mu_1, \mu_2, \dots$*
- *$G(C)$ is the underlying graph of C.*

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- $G(C)$ *is the underlying graph of* C.

Corollary (independant proof, Rattan 2006)

$$R_{k+1} = \sum_T \pm N_T,$$

where T *runs over rooted plane tree with* k *edges.*

Invariants

Theorem

There exists a family F_π of functions $\mathcal{H} \to \mathbb{C}$ indexed by partitions such that:

- $F_\pi(G)$ counts some colorings of white vertices of G with some conditions on numbers of neighbours of set of vertices.
- For any graph G with an oriented cycle C,

$$F_\pi(\mathcal{A}_{G,C}) = 0.$$

- For any partition τ, denote $R_\tau = \prod R_{\tau_i}$. Then,

$$F_\pi(R_\tau) = \delta_{\pi,\tau}.$$

Application

F_π is defined on $\mathcal{H}/\mathcal{I}$ and thus on $Vect(N_G)$.

$$\begin{aligned} F_\pi(\mathrm{Ch}_\mu) &= [R_\pi]\,\mathrm{Ch}_\mu \\ &= \sum_C \pm F_\pi(C) \end{aligned}$$

$\Rightarrow$ we have a combinatorial interpretation of the coefficients of Ch_μ written as a polynomial in R.

- Answer to a question raised by Kerov (2000).
- Already known, but it is a bit simpler than previous proofs.

Extension to Jack polynomials

χ^λ_μ can be defined by:

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By replacing Schur function s_λ by the Jack polynomial $J^{(\alpha)}_\lambda$, one can define a continuous deformation $\mathrm{Ch}^{(\alpha)}_\mu$ of $\mathrm{Ch}_\mu = \mathrm{Ch}^{(1)}_\mu$.

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We know that $\mathrm{Ch}^{(\alpha)}_\mu$ belongs to $Vect(N_G)$. Explicit expression?

A partial result

Case $\alpha = 2$ (zonal polynomials):

Theorem (F., Śniady 2010)

Let $\mu \vdash k$.

$$\mathrm{Ch}_\mu^{(2)} = \sum_M \pm N_{G(M)},$$

*where the sum runs over rooted bipartite maps on **locally oriented surfaces** with k edges and face-length $\mu_1, \mu_2, \dots$*

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$\Longrightarrow$ combinatorial description in terms of the R_ℓ's.

Conjecture for general $\alpha = 1 + \beta$:
Maps are counted with a weight depending on β.

Thanks for listening!

Any Questions?