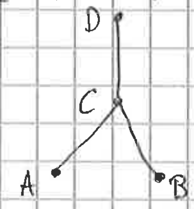
Exposé GT math discrètes

Plan
- I Objects and main question
- II My Motivation
- III First approach: P-partitions
- IV Second approach: cyclic inclusion-exclusion

I. Central object in this talk:

$(P, \leq)$ poset $\rightsquigarrow$ F_P formal series in infinitely many variables $x_1, x_2, \dots$

$$F_P = \sum_{\substack{f : P \to \mathbb{N}_* \\ \text{monotone}}} \prod_{i \in P} x_{f(i)}$$

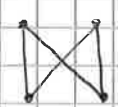


$$F_P(x_1, x_2, \dots) = \sum_{\substack{a,b,c,d \geq 1 \\ a \leq c,\, b \leq c,\, c \leq d}} x_a x_b x_c x_d$$

"quasi-sym function"

$\rightsquigarrow$ quite general model for sum with index set given by inequalities betw. indices (here, we will look only on the linear struct.)

there are some relations

(example : F  $- F$  $- F$  $+ F$  $= 0$)

→ canonical basis? positivity / comb interpretation of coef in the basis ? complete set of relations?
dimension?

II. My motivation: representation theory of symmetric groups.

1. Short introduction to RT

S_n group of permutations of n.

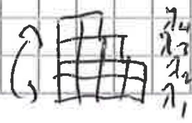
we are interested in representations, that is group morphisms $S_n \to GL(V)$
V ℂ-v.s. dim (V) < ∞

→ all are direct sums of "prime objects" called "irreducible representations"

☺ finite number of irreps, indexed by partitions of n

$\lambda = (\lambda_1, \dots, \lambda_r)$ $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$

$\sum_1 \lambda_i = n$



→ in rep. th., we are particularly interested in traces, called characters.

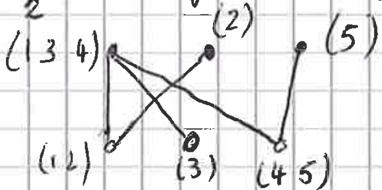
2. A formula for irreduc. character values of S_n

λ partition of n, $\sigma \in S_{\cancel{n}\,k}$ $(k \leq n)$

$$n(n-1)\cdots(n-k+1)\,\frac{\chi^\lambda(\sigma)}{\dim(\lambda)} = \sum_{\substack{\tau_1,\tau_2 \in S_k \\ \tau_1\tau_2 = \sigma}} \varepsilon(\tau_2)\, N_{G_{\tau_1,\tau_2}}(\lambda) \qquad (\text{F. } 2007)$$

$\varepsilon(\tau_2)$: sign

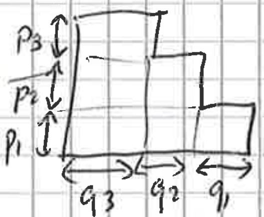
P_{τ_1,τ_2} : def on example $\tau_1 = (1\ 2)(3)(4\ 5)$
$\tau_2 = (1\ 3\ 4)(2)(5)$



Covering relation iff number in commun.

$\rightsquigarrow$ natural surface embedding (map)
(not useful here)

$N_G(\lambda)$: write $\lambda =$



$(V_G = V_0 \sqcup V_1)$

$$\to N_G(\lambda) = \sum_{\substack{f : V_G \to \mathbb{N} \\ \left(\text{if } \overset{v_2}{\underset{v_1}{|}} \in E_G,\ f(v_1) \leq f(v_2)\right)}} \prod_{v \in V_0} p_{f(v)} \prod_{w \in V_1} q_{f(w)}$$

f incr.

(looks like F_P, but with two sets of variables)

does not change relations.

III. First approach: P-partitions (Stanley, 1972)

$\rightsquigarrow$ we extend the objet we are working with

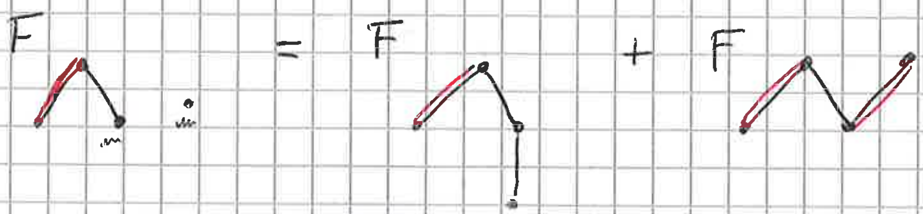
Consider pairs $((P, \leq), S)$ $S \subseteq$ set of covering relations of P.

Def $f : P \to \mathbb{N}^*$ is monotone on $((P,\leq), S)$ if

- f monotone on $(P, \leq)$
- if $(i,j) \in S$, then $f(i) < f(j)$.

Def $F_{(P,S)}(x_1, x_2, \dots) = \sum_{f\ (P,S)\ \text{non}} \prod_{i \in P} x_{f(i)}$

Obvious relation S drawn in red.



iterating this, any $F_{(P,S)}$ writes as a non-negative linear combination

of $B_I =$  $I \subseteq [n-1]$ tells us which edges are red.

Exercise: The B_I's are linearly independent.
(triangular submatrix in monomials)

$\Longrightarrow (B_I)_{I \subseteq [n-1]}$ is a basis of $\mathrm{Vect}((P, \le), S)_{|P|=n}$.

Description of the coeff. in the B_I exp. of F_P.

- assume $s(P) = \{1, \dots, n\}$ (or fix an arbitrary total order on elements of P extending $\le$).

and that $\le_P$ is natural
$i \le_P j \Rightarrow i \le_{\mathbb{N}} j$.

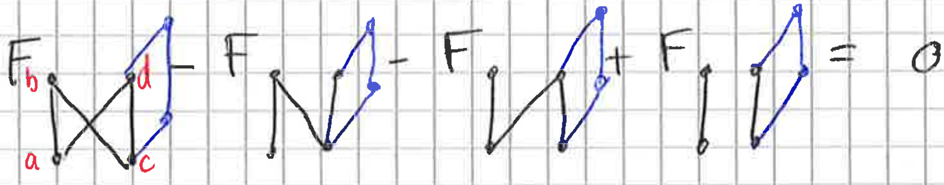
Def: A permutation $\sigma = \sigma_1 \cdots \sigma_n$ is a "linear extension" of P
if $i \le_P j \Rightarrow i$ appears before j in σ

A descent of σ is $i \in [n-1]$ s.t. $\sigma_i \le \sigma_{i+1}$. $\mathrm{Des}(\sigma)$ = set of descents of σ.

Prop: $F_P = \sum_{\sigma \text{ linear ext.}} B_{\mathrm{Des}(\sigma)}$

IV Second approach: cyclic inclusion-exclusion (F. 2008-2014)

Back to the example from I



Proof: All posets have same element set

$\Rightarrow$ All F are sums over functions from P $\to \mathbb{N}$

Fix a function φ and check its contribution is zero.

case analysis (b<a, ... $\square$

Obs: we can add same vertices / edges to all graphs and relation is still valid (in blue)

larger cycle, P poset, C undirected cycle in the Hasse diagram

$$\sum_{E \subseteq E_{1/2}(C)} (-1)^{|E|} F_{P|E} = 0$$

choose an orientation of C and take bottom-top edges.

Prop: $\mathrm{Vect}(F_P) = \mathrm{Vect}(F_T)$ T Hasse diagram without cycles

But F_T not a basis.

Idea: use relation to add edges

Lemma: If P is not of shape  i.e. a complete multipartite poset,

then $\exists\, P_0$ poset with the same element set and C cycle of P_0 such that $P = P_0 \setminus E_{1/2}(C)$

$\Rightarrow$ F_P is a linear combination of $(F_{P'})_{P' \text{ bigger than } P}$

Corollaire $\mathrm{Vect}(F_P) = \mathrm{Vect}(F_M)_{M \text{ multipartite poset}}$

Prop: F_M linearly ind., hence is a basis.

☹ We have signs in the F_M-ext. of F_P.

☺ we did not need to extend our objects.

$\rightarrow$ we could also restrict to bipartite posets (good for our problems with two sets of variables).