

Cyclic inclusion/exclusion

Valentin Féray

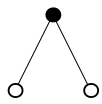
LaBRI, CNRS, Bordeaux

Journées Combinatoire Algébrique
du GDR-IM, Rouen



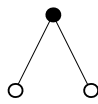
What is this talk about?

To a bipartite graph, we will associate a formal series:


$$\mapsto \sum_{i,j} p_i p_j q_{\max(i,j)}$$

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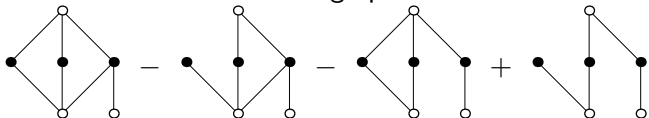
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$$\mapsto \sum_{i,j} p_i p_j q_{\max(i,j)}$$

Question: which linear combination of graphs are sent to 0?

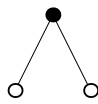
Example :



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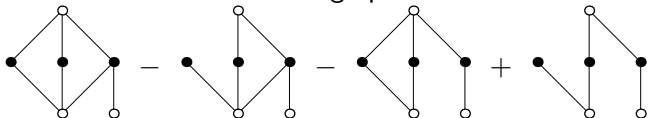
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Motivation : computation of irreducible character values of symmetric groups.

Definition of N

Let \mathbf{p}, \mathbf{q} be two infinite set of variables.

$$G =$$

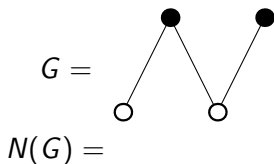
$$N(G) =$$

General formula:

$$N(G) = \dots \in \mathbb{Q}[\mathbf{p}, \mathbf{q}],$$

Definition of N

Let \mathbf{p}, \mathbf{q} be two infinite set of variables.



General formula:

$$N(G) = \sum_{\sigma \in \mathcal{S}_G} \prod_{(i,j) \in \sigma} p_i q_j \in \mathbb{Q}[\mathbf{p}, \mathbf{q}],$$

Definition of N

Let \mathbf{p}, \mathbf{q} be two infinite set of variables.

$$G = \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ \textit{i} \quad \textit{j} \\ \backslash \quad / \\ \bullet \end{array}$$

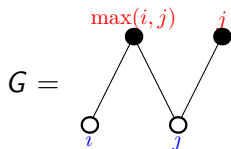
$$N(G) = \sum_{i,j} p_i p_j$$

General formula:

$$N(G) = \sum_{\varphi: V_o \rightarrow \mathbb{N}} \prod_{o \in V_o} p_{\varphi(o)} \in \mathbb{Q}[\mathbf{p}, \mathbf{q}],$$

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$$N(G) = \sum_{i,j} p_i p_j q_j q_{\max(i,j)}$$

General formula:

$$N(G) = \sum_{\varphi: V_0 \rightarrow \mathbb{N}} \prod_{o \in V_0} p_{\varphi(o)} \prod_{\bullet \in V_\bullet} q_{\psi(\bullet)} \in \mathbb{Q}[\mathbf{p}, \mathbf{q}],$$

with $\psi(\bullet) = \max_{o \rightarrow \bullet} \varphi(o)$.

N as an algebra morphism

Let \mathcal{BG} be the \mathbb{Q} vector space of linear combination of bipartite graphs. It is an algebra

$$G \cdot G' = G \sqcup G'.$$

N defines a morphism of algebra

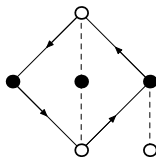
$$\begin{aligned} \mathcal{BG} &\rightarrow \mathbb{Q}[\mathbf{p}, \mathbf{q}] \\ G &\mapsto \sum_{\varphi: V_0 \rightarrow \mathbb{N}} \prod_{o \in V_0} p_{\varphi(o)} \prod_{\bullet \in V_1} q_{\psi(\bullet)}, \end{aligned}$$

Question

What is the kernel of N ?

Cyclic inclusion/exclusion

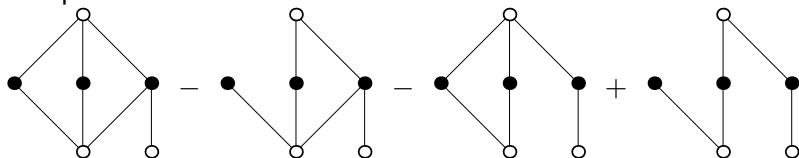
Consider a bipartite graph G endowed with an oriented cycle C .



We define the following element of \mathcal{BG} :

$$\mathcal{A}_{G,C} = \sum_{E \subseteq E_{\rightarrow \bullet}(C)} (-1)^{|E|} G \setminus E$$

Example :



Kernel of N

Proposition

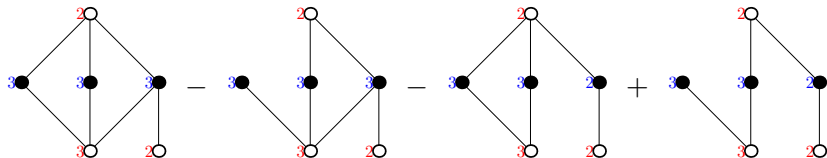
$$N(\mathcal{A}_{G,C}) = 0.$$

Kernel of N

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Sketch of proof. We look at the contribution of a fixed function $\varphi : V_o(G) \rightarrow \mathbb{N}$.



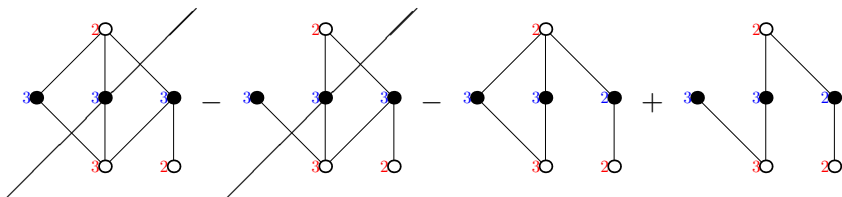
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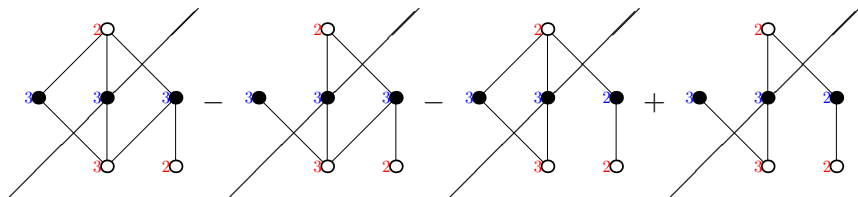
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Presentation of the next few slides

We will show that N defines an injective morphism

$$\mathcal{BG}/\langle \mathcal{A}_{G,C} \rangle \hookrightarrow \mathbb{Z}[[\mathbf{p}, \mathbf{q}]].$$

Method:

- Construct a family of graphs G_I .
- Show that G_I is a generating family in the quotient

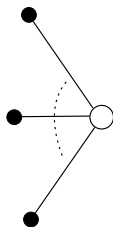
$$\mathcal{BG}/\langle \mathcal{A}_{G,C} \rangle.$$

- Prove that $N(G_I)$ is linearly independent in $\mathbb{Z}[[\mathbf{p}, \mathbf{q}]]$.

A generating family

Definition

Let $I = (i_1, i_2, \dots, i_r)$ be a composition. Define G_I as the following bipartite graph:



$i_1 - 1$ black
vertices



$i_2 - 1$ black
vertices

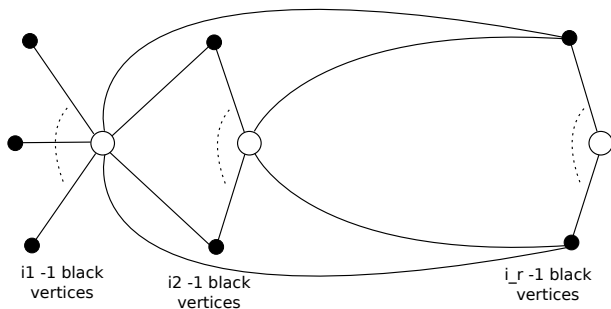


$i_r - 1$ black
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A generating family

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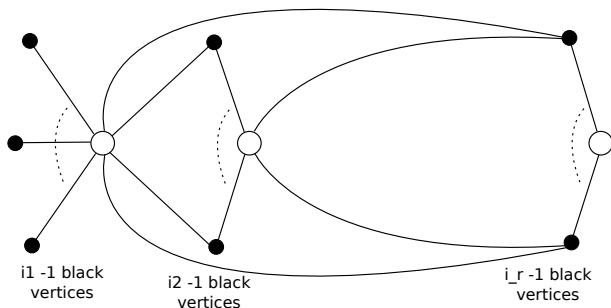
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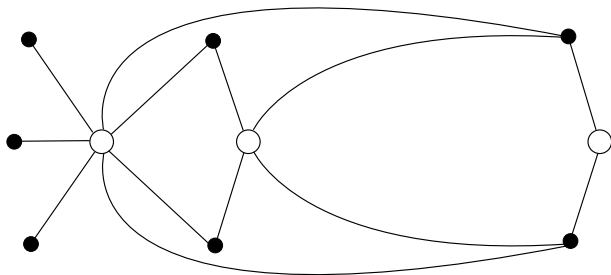
Proposition

$\{G_I, I \text{ composition}\}$ is a linear generating set of \mathcal{BG}/\mathcal{I} .

Idea of proof

Proposition

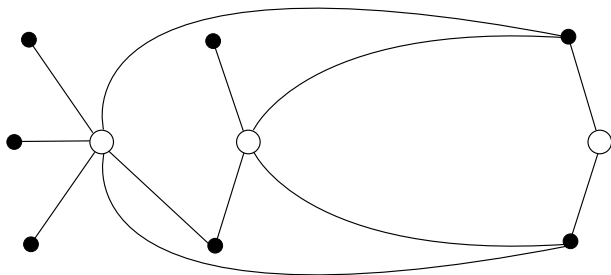
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graph G_I

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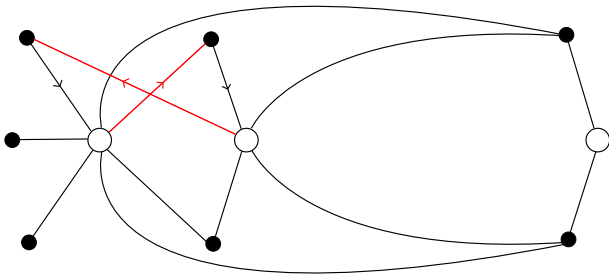


Consider a graph $G \neq G_I$

Idea of proof

Proposition

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There is a graph G_0 with an oriented cycle C such that

$$G_0 \setminus E_{\rightarrow}(C) = G$$

Idea of proof

Proposition

$\{G_I, I \text{ composition}\}$ is a linear generating set of \mathcal{BG}/\mathcal{I} .

Consider a graph $G \neq G_I$.

Lemma: There is a graph G_0 with an oriented cycle C such that:

$$G_0 \setminus E_{\rightarrow}(C) = G$$

Consequence : in \mathcal{BG}/\mathcal{I} , $G =$ linear combination of bigger graphs.

→ we iterate until we obtain a linear combination of G_I 's.

Independence

Lemma

The $N(G_I)$, where I runs over all compositions are linearly independent.

Gradation:

$N(G_I)$ is a homogenous polynomial of degree $r = \ell(I)$ in \mathfrak{p} and of total degree $n = |I|$.

Independence

Lemma

The $N(G_I)$, where I runs over compositions of *length r and size n* are linearly independent.

Consider $M_{G_I}(p_1, p_2, \dots, p_r, q_1, q_2, \dots, q_r)$

(we truncate the alphabets to $r = \ell(I) = |V_\circ(G_I)|$ variables)

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(we truncate the alphabets to $r = \ell(I) = |V_o(G_I)|$ variables)

We will consider only p -square free monomials.

As total degree in p is r , they are:

$$T_J = p_1 q_1^{j_1-1} p_2 q_2^{j_2-1} \dots p_r q_r^{j_r-1},$$

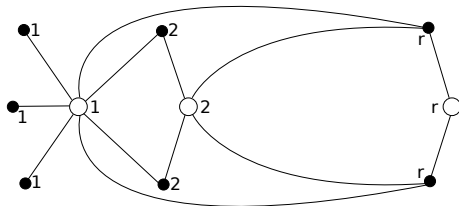
where J is a composition of n and length r .

In $N(G_I)$, they correspond to bijections $\varphi : V_o(G_I) \simeq \{1, \dots, r\}$.

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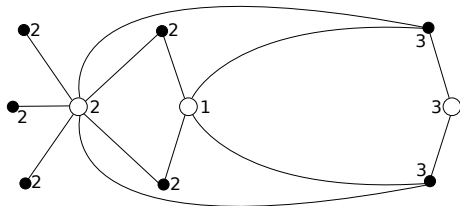


$$M_{G_I} = T_I$$

Independence

Lemma

The $N(G_I)$, where I runs over compositions of length r and size n are linearly independent.



$$M_{G_I} = T_I + \sum_{\substack{|J|=|I|=n, \ell(J)=\ell(I)=r \\ J \geq I}} c_{I,J} T_J$$

+ non- p -square-free terms

\geq stands for the right-dominance order.

One set of variables

We considered a morphism

$$BG \longrightarrow \mathbb{Q}[[p, q]]$$

One set of variables

It factorizes *via*

$$\mathcal{BG} \twoheadrightarrow \mathcal{BG}/\langle \mathcal{A}_{G,c} \rangle \hookrightarrow \mathbb{Q}[[\mathbf{p}, \mathbf{q}]]$$

One set of variables

Same kernel if we identify p_i and q_i !

$$\begin{array}{ccc}
 BG & \twoheadrightarrow & BG / \langle \mathcal{A}_{G,c} \rangle \hookrightarrow & \mathbb{Q}[[\mathbf{p}, \mathbf{q}]] \\
 & & \searrow & \downarrow \begin{array}{l} p_i \mapsto x_i \\ q_i \mapsto x_i \end{array} \\
 & & & \mathbb{Q}[[\mathbf{x}]]
 \end{array}$$

One set of variables

We can describe the image

$$\begin{array}{ccccc}
 \mathcal{BG} & \twoheadrightarrow & \mathcal{BG} / \langle \mathcal{A}_{G,C} \rangle^c & \hookrightarrow & \mathbb{Q}[[\mathbf{p}, \mathbf{q}]] \\
 & & \downarrow \cong & & \downarrow \begin{array}{l} p_i \mapsto x_i \\ q_i \mapsto x_i \end{array} \\
 & & \mathbb{Q}\text{Sym}(\mathbf{x}) & \hookrightarrow & \mathbb{Q}[[\mathbf{x}]]
 \end{array}$$

$\mathbb{Q}\text{Sym}(\mathbf{x})$: ring of quasi-symmetric function in \mathbf{x} .

Example: $M_{1,2}(x_1, x_2, x_3) = x_1x_2^2 + x_1x_3^2 + x_2x_3^2$.

One set of variables

We can describe the image

$$\begin{array}{ccccc}
 \mathcal{BG} & \longrightarrow & \mathcal{BG} / \langle \mathcal{A}_{G,C} \rangle^c & \hookrightarrow & \mathbb{Q}[[\mathbf{p}, \mathbf{q}]] \\
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$\mathcal{BG} \rightarrow \mathcal{QSym}$ is a Hopf algebra morphism!

Coproduct on \mathcal{BG}

$$\Delta \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = 1 \otimes \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 1} \\ \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \\ \text{Diagram 9} \end{array} \otimes \begin{array}{c} \text{Diagram 10} \\ \text{Diagram 11} \\ \text{Diagram 12} \\ \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \\ \text{Diagram 19} \\ \text{Diagram 20} \\ \text{Diagram 21} \\ \text{Diagram 22} \end{array} + \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} \otimes 1$$

The diagrams are as follows:

- Diagram 1: A graph with 5 vertices and 6 edges. The top vertex is red, the middle-left is blue, and the bottom is green. The edges connect (red, blue), (red, middle-right), (blue, middle-right), (blue, middle-left), (middle-left, middle-right), and (middle-right, green).
- Diagram 2: A graph with 3 vertices and 2 edges. The top vertex is red, and the two bottom vertices are black.
- Diagram 3: A graph with 3 vertices and 2 edges. The top vertex is blue, and the two bottom vertices are black.
- Diagram 4: A graph with 3 vertices and 2 edges. The top vertex is red, and the two bottom vertices are black.
- Diagram 5: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is green, and the bottom-right is black.
- Diagram 6: A graph with 4 vertices and 3 edges. The top vertex is red, the middle-left is blue, and the bottom is green. The edges connect (red, blue), (red, middle-right), (blue, middle-right), (blue, middle-left), and (middle-left, middle-right).
- Diagram 7: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is blue, and the bottom-right is black.
- Diagram 8: A graph with 2 vertices and 1 edge. The top vertex is red, and the bottom vertex is black.
- Diagram 9: A graph with 3 vertices and 2 edges. The top vertex is red, the middle-left is blue, and the bottom-right is black.
- Diagram 10: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is blue, and the bottom-right is black.
- Diagram 11: A graph with 2 vertices and 1 edge. The top vertex is red, and the bottom vertex is black.
- Diagram 12: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is blue, and the bottom-right is black.
- Diagram 13: A graph with 2 vertices and 1 edge. The top vertex is blue, and the bottom vertex is black.
- Diagram 14: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is blue, and the bottom-right is black.
- Diagram 15: A single green vertex.
- Diagram 16: A graph with 5 vertices and 6 edges. The top vertex is red, the middle-left is blue, and the bottom is green. The edges connect (red, blue), (red, middle-right), (blue, middle-right), (blue, middle-left), (middle-left, middle-right), and (middle-right, green).
- Diagram 17: A graph with 3 vertices and 2 edges. The top vertex is red, and the two bottom vertices are black.
- Diagram 18: A graph with 3 vertices and 2 edges. The top vertex is blue, and the two bottom vertices are black.
- Diagram 19: A graph with 3 vertices and 2 edges. The top vertex is red, and the two bottom vertices are black.
- Diagram 20: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is green, and the bottom-right is black.
- Diagram 21: A graph with 4 vertices and 3 edges. The top vertex is red, the middle-left is blue, and the bottom is green. The edges connect (red, blue), (red, middle-right), (blue, middle-right), (blue, middle-left), and (middle-left, middle-right).
- Diagram 22: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is blue, and the bottom-right is black.
- Diagram 23: A graph with 2 vertices and 1 edge. The top vertex is red, and the bottom vertex is black.
- Diagram 24: A graph with 3 vertices and 2 edges. The top vertex is black, the middle-left is blue, and the bottom-right is black.

Some variants

- $N'(G) := \sum_{\substack{\varphi: V_G \rightarrow \mathbb{N} \\ (o, \bullet) \in E_G \Rightarrow \varphi(o) \leq \varphi(\bullet)}} \left(\prod_{v \in V_G} x_{\varphi_G} \right)$. Then

$$\text{Ker}(N') = \text{Ker}(N).$$

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- The definition above is naturally extended to acyclic directed graphs.
- One can also consider labeled graphs and polynomials in non-commutative variables (QSym is replaced by WQSym).

$$N(\mathbf{G}) = \sum_{\substack{f: [n] \rightarrow G \\ f \nearrow}} a_{f(1)} \cdots a_{f(n)},$$

where the a 's are *non-commutative* variables.

Does cyclic inclusion/exclusion always span the kernel?

Same method as before:

- 1 Construct a family of (labelled/unlabelled) (bipartite/directed acyclic) graphs.
- 2 Show that it is a generating family in the quotient

$$\mathcal{G}_*/\langle \mathcal{A}_{G,C} \rangle.$$

- 3 Prove that the corresponding functions are linearly independent.

3 is hard in non-commutative setting.

The family of graphs in the bipartite labelled setting

We consider set compositions (or ordered set-partitions) I .

Example: $I = 35 \mid 14 \mid 8 \mid 267$.

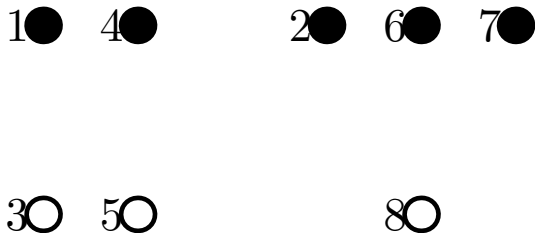
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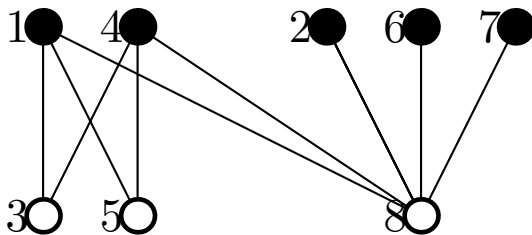


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I need some help here!

Proposition

The graphs G_I generate the quotient

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Conjecture

$\mathcal{N}(G_I)$, where I runs over all set compositions, is a basis of WQSym .

Equivalently, the $\mathcal{A}_{G,C}$ span the kernel of \mathcal{N} .

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Thanks for listening!