Cyclic inclusion/exclusion

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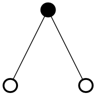
What is this talk about?

To a bipartite graph, we will associate a formal series:



$$\mapsto \sum_{i,j} p_i p_j q_{\max(i,j)}$$

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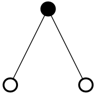

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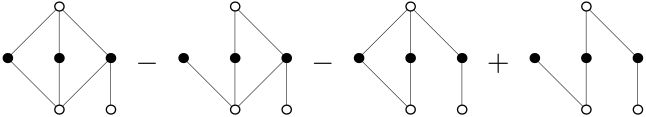
Question: which linear combination of graphs are sent to 0?

Example :  $\mapsto 0.$

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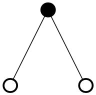

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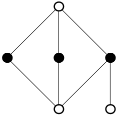
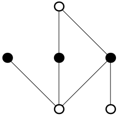
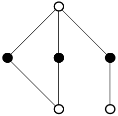
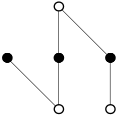
Example :  $-$  $-$  $+$  $\mapsto 0.$

Motivation : computation of irreducible character values of symmetric groups.

Definition of N

Let **p**, **q** be two infinite set of variables.

$$G =$$

$$N(G) =$$

General formula:

$$N(G) = \qquad \in \mathbb{Q}[\mathbf{p}, \mathbf{q}],$$

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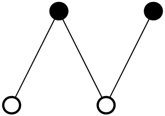
$G =$ 

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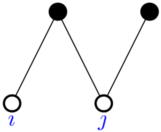
$G =$ 

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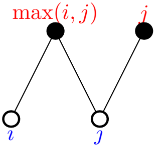
$G =$ 

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with $\psi(\bullet) = \max$  $\varphi(\circ)$.

N as an algebra morphism

Let $\mathcal{BG}$ be the $\mathbb{Q}$ vector space of linear combination of bipartite graphs.
It is an algebra

$$G \cdot G' = G \sqcup G'.$$

N defines a morphism of algebra

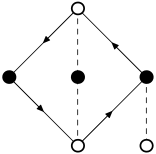
$$\begin{array}{rcl} \mathcal{BG} & \to & \mathbb{Q}[\mathbf{p}, \mathbf{q}] \\ G & \mapsto & \displaystyle\sum_{\varphi : V_\circ \to \mathbb{N}} \prod_{\circ \in V_\circ} p_{\varphi(\circ)} \prod_{\bullet \in V_\bullet} q_{\psi(\bullet)}, \end{array}$$

Question

What is the kernel of N?

Cyclic inclusion/exclusion

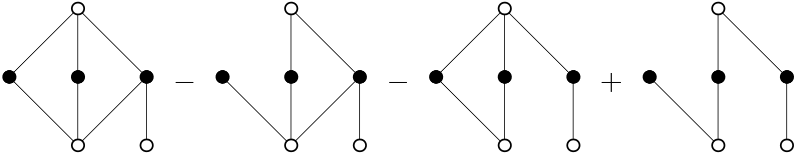
Consider a bipartite graph G endowed with an oriented cycle C.



We define the following element of $\mathcal{BG}$:

$$\mathcal{A}_{G,C} = \sum_{E \subseteq E_{\circ \rightarrow \bullet}(C)} (-1)^{|E|} G \setminus E$$

Example :



Kernel of N

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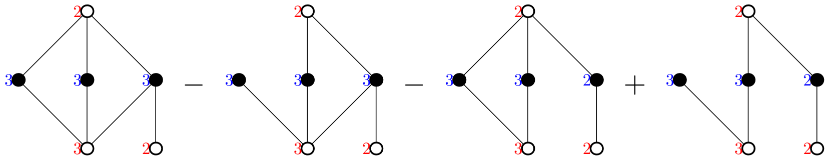
Sketch of proof. We look at the contribution of a fixed function $\varphi : V_\circ(G) \to \mathbb{N}$.



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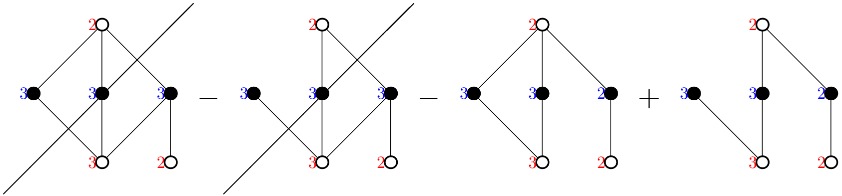

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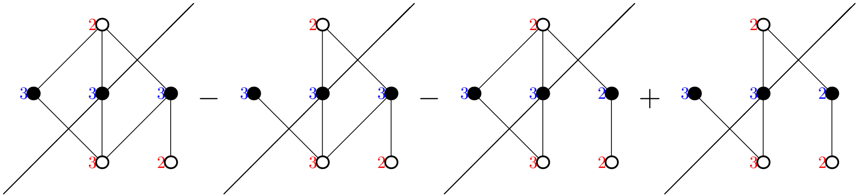


Presentation of the next few slides

We will show that N defines an injective morphism

$$\mathcal{BG}/\langle \mathcal{A}_{G,C} \rangle \hookrightarrow \mathbb{Z}[[\mathbf{p},\mathbf{q}]].$$

Method:

- Construct a family of graphs G_I.
- Show that G_I is a generating family in the quotient

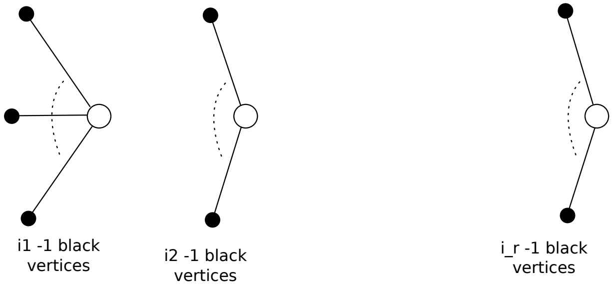
$$\mathcal{BG}/\langle \mathcal{A}_{G,C} \rangle.$$

- Prove that $N(G_I)$ is linearly independant in $\mathbb{Z}[[\mathbf{p},\mathbf{q}]]$.

A generating family

Definition

Let $I = (i_1, i_2, \ldots, i_r)$ be a composition. Define G_I as the following bipartite graph:



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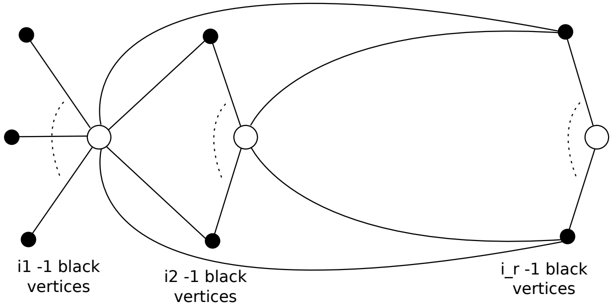

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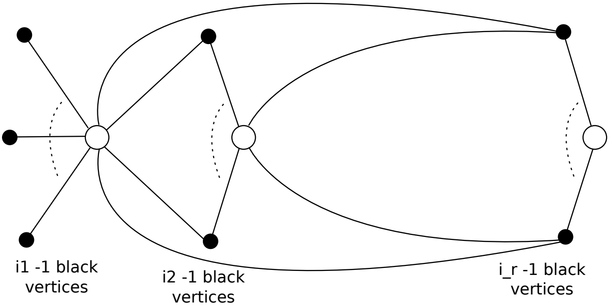

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$\{G_I, I$ composition$\}$ is a linear generating set of $\mathcal{BG}/\mathcal{I}$.

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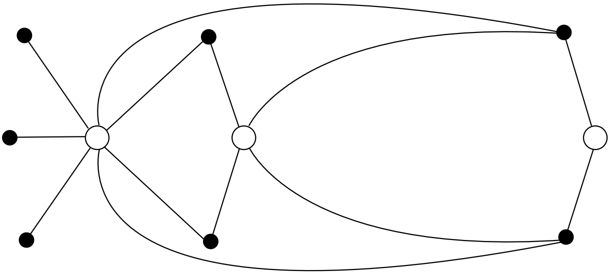

graph G_I

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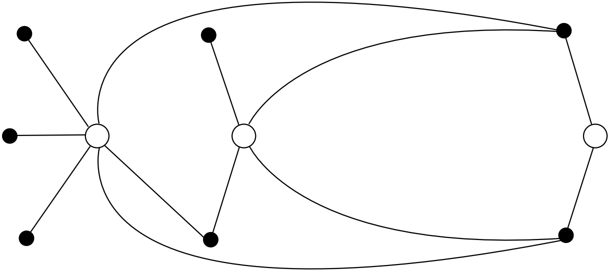

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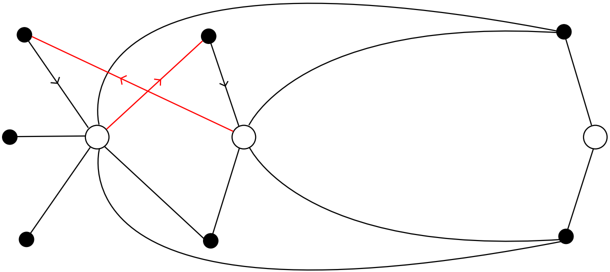

There is a graph G_0 with an oriented cycle C such that

$$G_0 \backslash E_{\circ \to \bullet}(C) = G$$

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Consider a graph $G \neq G_I$.

Lemma: There is a graph G_0 with an oriented cycle C such that:

$$G_0 \backslash E_{\circ \to \bullet}(C) = G$$

Consequence : in $\mathcal{BG}/\mathcal{I}$, $G =$ linear combination of bigger graphs.

$\rightarrow$ we iterate until we obtain a linear combination of G_I's.

Independence

Lemma

The $N(G_I)$, where I runs over all compositions are linearly independent.

Gradation:
$N(G_I)$ is a homogenous polynomial of degree $r = \ell(I)$ in $\mathbf{p}$ and of total degree $n = |I|$.

Independence

Lemma

The $N(G_I)$, where I runs over compositions of length r and size n are linearly independent.

Consider $M_{G_I}(p_1, p_2, \dots, p_r, q_1, q_2, \dots, q_r)$
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We will consider only p-square free monomials.

As total degree in p is r, they are:

$$T_J = p_1 q_1^{j_1 - 1} p_2 q_2^{j_2 - 1} \cdots p_r q_r^{j_r - 1},$$

where J is a composition of n and length r.

In $N(G_I)$, they correspond to bijections $\varphi : V_\circ(G_I) \simeq \{1, \dots, r\}$.

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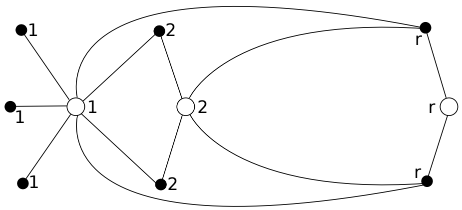

$$M_{G_I} = T_I$$

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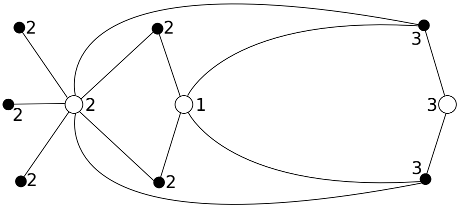


$$M_{G_I} = T_I + \sum_{\substack{|J|=|I|=n,\ \ell(J)=\ell(I)=r \\ J \geq I}} c_{I,J} T_J$$

$$+ \text{non-}p\text{-square-free terms}$$

$\geq$ stands for the right-dominance order.

One set of variables

We considered a morphism

$$\mathcal{BG} \longrightarrow \mathbb{Q}[[\mathbf{p}, \mathbf{q}]]$$

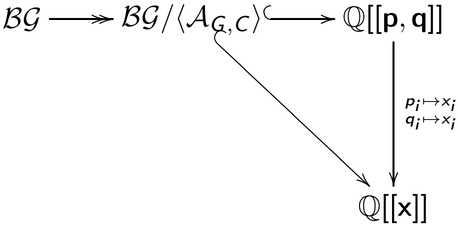
One set of variables

It factorizes *via*

$$\mathcal{BG} \twoheadrightarrow \mathcal{BG}/\langle \mathcal{A}_{G,C} \rangle \hookrightarrow \mathbb{Q}[[\mathbf{p}, \mathbf{q}]]$$

One set of variables

Same kernel if we identify p_i and q_i!



One set of variables

We can describe the image

$$
\begin{array}{ccccc}
\mathcal{BG} & \twoheadrightarrow & \mathcal{BG}/\langle \mathcal{A}_{G,C} \rangle & \hookrightarrow & \mathbb{Q}[[\mathbf{p},\mathbf{q}]] \\
 & & \downarrow \simeq & & \downarrow {\scriptstyle p_i \mapsto x_i,\ q_i \mapsto x_i} \\
 & & \mathrm{QSym}(\mathbf{x}) & \hookrightarrow & \mathbb{Q}[[\mathbf{x}]]
\end{array}
$$

QSym($\mathbf{x}$): ring of quasi-symmetric function in $\mathbf{x}$.
Example: $M_{1,2}(x_1, x_2, x_3) = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$.

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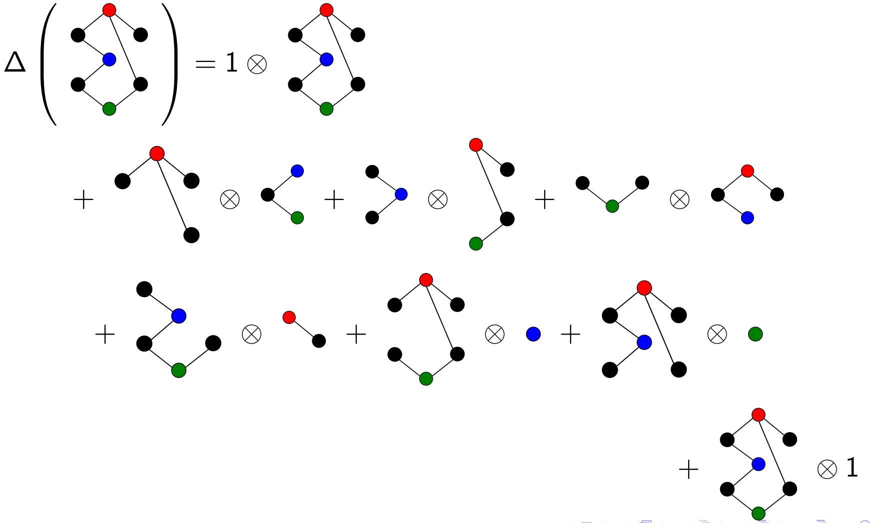
$\mathcal{BG} \to$ QSym is a Hopf algebra morphism!

Coproduct on $\mathcal{BG}$



Some variants

- $N'(G) := \sum_{\substack{\varphi : V_G \to \mathbb{N} \\ (\circ,\bullet)\in E_G \Rightarrow \varphi(\circ)\leq\varphi(\bullet)}} \left(\prod_{v\in V_G} x_{\varphi_G} \right)$. Then

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- The definition above is naturally extended to acyclic directed graphs.
- One can also consider labeled graphs and polynomials in non-commutative variables (QSym is replaced by WQSym).
$$\boldsymbol{N}(\boldsymbol{G}) = \sum_{\substack{f:[n]\to G \\ f \nearrow}} a_{f(1)} \dots a_{f(n)},$$
where the a's are *non-commutative* variables.

Does cyclic inclusion/exclusion always span the kernel?

Same method as before:

1. Construct a family of (labelled/unlabelled) (bipartite/directed acyclic) graphs.
2. Show that it is a generating family in the quotient

$$\mathcal{G}_\star / \langle \mathcal{A}_{G,C} \rangle.$$

3. Prove that the corresponding functions are linearly independant.

3 is hard in non-commutative setting.

The family of graphs in the bipartite labelled setting

We consider set compositions (or ordered set-partitions) $\boldsymbol{I}$.
Example: $I = 35 \mid 14 \mid 8 \mid 267$.
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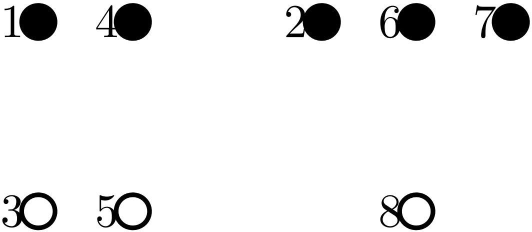

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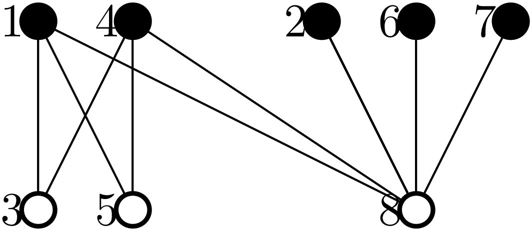

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Conjecture

$\boldsymbol{N}(G_I)$, where I runs over all set compositions, is a basis of WQSym.

Equivalently, the $\mathcal{A}_{G,C}$ span the kernel of $\boldsymbol{N}$.

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Thanks for listening!