Polynomial functions on Young diagrams and limit shape of Young diagrams

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Seminar über stochastische Prozesse
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What is this talk about?

- Young diagrams $\lambda \vdash n$: 

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
What is this talk about?

- Young diagrams $\lambda \vdash n$:

- We consider a random Young diagram of size $n$ (the measure is not uniform, it will be described later).

- Question: asymptotic behaviour of the lengths of its first rows, its first columns, ...
What is this talk about?

- Young diagrams $\lambda \vdash n$:

- We consider a random Young diagram of size $n$ (the measure is not uniform, it will be described later).

- Question: asymptotic behaviour of the lengths of its first rows, its first columns, ... 

- Tool: representation theory of symmetric groups.
Outline of the talk

1. Model of random Young diagrams
2. Polynomial functions on Young diagrams
3. How to prove asymptotic results?
4. A few remarks
What is a representation?

\( S_n \): group of permutations of elements 1, 2, \cdots, n.

A representation of \( S_n \) is a couple \((V, \rho)\) where:

- \( V \) is a finite dimensional \( \mathbb{C} \) vector space;
- \( \rho \) is a morphism \( S_n \to GL(V) \).
What is a representation?

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We are interested in **irreducible** representations.
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We are interested in irreducible representations.

Theorem

Irreducible representations of $S_n$ are in bijection with partitions $\lambda \vdash n$ or, equivalently, Young diagrams $\lambda$ with $n$ boxes.

Ex: $n = 10$, $\lambda = (5, 3, 2)$. 

\begin{figure}
\centering
\includegraphics[width=0.3\textwidth]{young_diagram.png}
\caption{Young diagram for $\lambda = (5, 3, 2)$.}
\end{figure}
Characters

Let \((V, \rho)\) be a representation of \(S_n\). We define:

\[ \chi^\rho(\sigma) = \frac{\text{Tr}(\rho(\sigma))}{\dim(\lambda)} \text{ for } \sigma \in S_n. \]

It is the (normalized) character of \(\rho\).

No loss of information!

\((\rho, V) \simeq_{S_n} (\rho', V') \iff \chi^\rho = \chi^{\rho'}\).
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Note that

\[
\chi^\rho(\tau^{-1}\sigma\tau) = \chi^\rho(\sigma).
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\chi^\rho(\tau^{-1}\sigma\tau) = \chi^\rho(\sigma).
$$

Indeed,

$$
\chi^\rho(\tau^{-1}\sigma\tau) = \frac{\text{Tr} (\rho(\tau^{-1}\sigma\tau))}{\dim(\lambda)} = \frac{\text{Tr} (\rho(\tau)^{-1} \cdot \rho(\sigma) \cdot \rho(\tau))}{\dim(\lambda)} = \frac{\text{Tr} (\rho(\sigma))}{\dim(\lambda)}
$$
Characters

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i.e. \(\chi^\rho\) is a central function on \(S_n\).
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Theorem

The characters \(\chi^\lambda\) of irreducible representations \(\rho^\lambda\) of the symmetric group \(S_n\) form an (orthogonal) basis of the space of central functions on \(S_n\).
Some central functions on $\bigcup S_n$

Let $\omega = (\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots, \gamma)$ with

- $\alpha$ and $\beta$ infinite non-increasing sequences;
- $\alpha_i, \beta_i, \gamma \geq 0$;
- $\sum_i \alpha_i + \sum_i \beta_i + \gamma = 1$. 
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- $\alpha$ and $\beta$ infinite non-increasing sequences;
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- $\sum_i \alpha_i + \sum_i \beta_i + \gamma = 1$.

Define

$$p_1(\omega) = 1$$
$$p_k(\omega) = \sum a_i^k + (-1)^{k-1} \sum b_i^k$$
$$p_{(\mu_1, \mu_2, \ldots)}(\omega) = \prod_j p_{\mu_j}(\omega).$$
Some central functions on $\bigcup S_n$

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- $p_1(\omega) = 1$
- $p_k(\omega) = \sum a_i^k + (-1)^{k-1} \sum b_i^k$ for $k \geq 2$.
- $p_{(\mu_1, \mu_2, \ldots)}(\omega) = \prod_j p_{\mu_j}(\omega)$. 
Some central functions on $\bigcup S_n$

Let $\omega = (\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots, \gamma)$.

$$p_1(\omega) = 1$$
$$p_k(\omega) = \sum a_i^k + (-1)^{k-1} \sum b_i^k \quad \text{for } k \geq 2.$$  
$$p_{(\mu_1, \mu_2, \ldots)}(\omega) = \prod_j p_{\mu_j}(\omega).$$

We will consider the following central function on $\bigcup S_n$.

$$F_\omega(\sigma) = p_{t(\sigma)}(\omega),$$

where $t(\sigma)$ is the cycle-type of $\sigma$. 
Examples

\[ F_\omega(\text{Id}) = 1 \]
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\( F_\omega(\text{Id}) = 1 \)

\[ \omega = ((1 - q), (1 - q)q, (1 - q)q^2, \ldots; 0, 0, \ldots; 0) \]

\[ F_\omega((1 2 3)(4 5)) = \left( \sum_{i \geq 0} ((1 - q)q^i)^3 \right) \left( \sum_{i \geq 0} ((1 - q)q^i)^2 \right) \]

\[ = \frac{(1 - q)^5}{(1 - q^3)(1 - q^2)} \]

\[ = F_\omega((1 2 3)(4 5)(6)) \]
Examples

\[ F_\omega(I) = 1 \]

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\[ = \frac{(1 - q)^5}{(1 - q^3)(1 - q^2)} \]

\[ = F_\omega((1 \ 2 \ 3)(4 \ 5)(6)) \]

\[ \omega = (0, 0, \ldots; 0, 0, \ldots; 1) \]

\[ F_\omega(\sigma) = \begin{cases} 1 & \text{if } \sigma = I; \\ 0 & \text{else.} \end{cases} \]
Probability measure on $\mathcal{Y}_n$

Fix a parameter $\omega$ an integer $n$. $F_\omega/S_n$ is a central function on $S_n$. Therefore, as function on $S_n$,

$$F_\omega = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda.$$ 

Note that

$$\sum_{\lambda \vdash n} c_\lambda = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda(\text{Id}) = F_\omega(\text{Id}) = 1.$$
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Proposition

$$\forall \lambda, \quad c_\lambda \geq 0.$$ 

Hence, the $c_\lambda$ define a probability measure $\mathbb{P}_n^\omega$ on $\mathcal{Y}_n$

$$\mathbb{P}_n^\omega(X = \lambda) = c_\lambda.$$
Pursuing the examples

\( \omega = (0, 0, \ldots ; 0, 0, \ldots ; 1) \).

A classical result in representation theory states that:

\[
\mathbb{C}[S_n] \cong S_n \bigoplus_{\lambda \vdash n} V_{\lambda}^{\text{dim}(V_{\lambda})}
\]

We look at the trace of the action of \( \sigma \):

\[
F_\omega = \sum_{\lambda \vdash n} \frac{\text{dim}(V_{\lambda})^2}{n!} \chi_{\lambda}.
\]

Hence \( P_\omega^n(X = \lambda) = \frac{\text{dim}(V_{\lambda})^2}{n!} \). This measure is known as Plancherel measure.
Pursuing the examples

- $\omega = (0, 0, \ldots; 0, 0, \ldots; 1)$.
  A classical result in representation theory states that:
  \[
  \mathbb{C}[S_n] \simeq S_n \bigoplus_{\lambda \vdash n} V_\lambda^{\dim(V_\lambda)}
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  We look at the trace of the action of $\sigma$:
  \[
  F_\omega = \sum_{\lambda \vdash n} \frac{\dim(V_\lambda)^2}{n!} \chi_\lambda.
  \]
  Hence $P_\omega^n(X = \lambda) = \frac{\dim(V_\lambda)^2}{n!}$. This measure is known as Plancherel measure.

- $\omega = ((1 - q), (1 - q)q, (1 - q)q^2, \ldots; 0, 0, \ldots; 0)$.
  $P_\omega^n$ is in this case a $q$-deformation of Plancherel measure already considered by Kerov in another context.
Main theorem of the talk

Theorem

Fix a parameter $\omega$. For each $n$, we pick a random Young diagram $\lambda^{(n)}$ with the distribution $\mathbb{P}_{n,\omega}$. Then, one has the convergences in probability:

$$\forall \ i, \quad \frac{\lambda_i^{(n)}}{n} \to \alpha_i$$

$$\forall \ i, \quad \frac{(\lambda^{(n)})'_i}{n} \to \beta_i$$
Main theorem of the talk

Theorem

Fix a parameter $\omega$. For each $n$, we pick a random Young diagram $\lambda^{(n)}$ with the distribution $\mathbb{P}_{n,\omega}$. Then, one has the convergences in probability:

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- proved by Kerov and Vershik in 1981.
- method of proof used here (which gives also information on the fluctuations): F., Méliot (2010) in the case of $q$-Plancherel measure.
- Generalization with the same argument to all $\mathbb{P}_{n,\omega}$: Méliot 2011.
Examples

- $q = 1/2, \omega = ((1 - q), (1 - q)q, (1 - q)q^2, \ldots; 0, 0, \ldots; 0)$.

\[
\frac{\lambda_i}{n} \to (1/2)^i, \quad \frac{\lambda_i'}{n} \to 0
\]

Here is a random Young diagram of size 200 (computed and drawn by PL Méliot).

\[
\lambda = (101, 51, 28, 8, 7, 3, 1, 1).
\]
Examples

- $q = 1/2$, $\omega = ((1 - q), (1 - q)q, (1 - q)q^2, \ldots ; 0, 0, \ldots ; 0)$.
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  \frac{\lambda_i}{n} \to (1/2)^i, \quad \frac{\lambda'_i}{n} \to 0
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Here is a random Young diagram of size 200 (computed and drawn by PL Méliot).

\[
\lambda = (101, 51, 28, 8, 7, 3, 1, 1).
\]

- $\omega = (0, 0 \ldots ; 0, 0, \ldots ; 1)$ (Plancherel case)
  \[
  \frac{\lambda_i}{n} \to 0, \quad \frac{\lambda'_i}{n} \to 0
  \]

$\Rightarrow$ no big rows and columns (but no precise information!).
Transition

Normalized character values have simple expectations!

Fix $\sigma \in \mathcal{S}_n$. Let us consider the random variable:

$$X_\sigma(\lambda) = \chi^\lambda(\sigma) = \frac{\text{Tr} (\rho_\lambda (\sigma))}{\dim V_\lambda}.$$
Normalized character values have simple expectations!

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Let us compute its expectation:

$$\mathbb{E}_{\mathbb{P}_n^\omega} (X_\sigma) = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda(\sigma) = F_\omega(\sigma),$$

by the very definition of $\mathbb{P}_n^\omega$.!
Normalized character values have simple expectations!

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by the very definition of $\mathbb{P}_n^\omega$!

Strategy: express other functions of Young diagrams in terms of $X_\sigma$. 
Definition of normalized characters

Let us define

$$\text{Ch}_\mu : Y \to \mathbb{C}; \quad \lambda \mapsto |\lambda|(|\lambda| - 1) \ldots (|\lambda| - k + 1) \chi_\lambda(\sigma),$$

where $k = |\mu|$. 

and $\sigma$ is a permutation in $S_{|\lambda|}$ of cycle type $\mu 1^{\lambda - k}$.

In particular,

$$\text{Ch}_\mu(\lambda) = 0 \quad \text{as soon as } |\lambda| < |\mu|$$

Examples:

$$\text{Ch}_{1^k}(\lambda) = |\lambda|(|\lambda| - 1) \ldots (|\lambda| - k + 1)$$

$$\text{Ch}_{(2)}(\lambda) = |\lambda|(|\lambda| - 1)\chi_\lambda((1 \ 2)) = \sum_i (\lambda_i)^2 - (\lambda_i')^2$$

$$\text{Ch}_{\mu \cup 1}(\lambda) = (|\lambda| - |\mu|) \text{Ch}_\mu(\lambda)$$
Product of normalized characters

\[ \text{Ch}_\mu(\lambda) = |\lambda|(|\lambda| - 1) \ldots (|\lambda| - k + 1)\chi^\lambda(\sigma). \]

Proposition

The functions \( \text{Ch}_\mu \), when \( \mu \) runs over all partitions, are linearly independent (over \( \mathbb{C} \)).

Moreover, they span a subalgebra \( \Lambda^* \) of functions on Young diagrams.

Example: \( \text{Ch}_{(2)} \cdot \text{Ch}_{(2)} = 4 \cdot \text{Ch}_{(3)} + \text{Ch}_{(2,2)} + 2 \text{Ch}_{(1,1)}. \)
Frobenius coordinates and their power sums

If $\lambda$ is a Young diagram, define its Frobenius coordinates $(a_i, b_i), 1 \leq i \leq h$ as follows:

$$b_i = \lambda'_i - i + 1/2 > 0, \quad a_i = \lambda_i - i + 1/2 > 0$$
If $\lambda$ is a Young diagram, define its Frobenius coordinates $(a_i, b_i), 1 \leq i \leq h$ as follows:

$$M_k(\lambda) := \sum a_i^k + (-1)^{k-1} \sum b_i^k$$
An algebraic basis of $\Lambda^*$

Theorem (Kerov, Olshanski, 1994)

$(\lambda \mapsto M_k(\lambda))_{k \geq 1}$ is an algebraic basis of $\Lambda^*$.

Example:

$$\text{Ch}_4 = M_4 - 4M_2 \cdot M_1 + \frac{11}{2} M_2.$$ 

Not very explicit formula for the change of basis

$$\text{Ch}_k = [t^{k+1}] \left\{-\frac{1}{k} \prod_{j=1}^k (1 - (j - 1/2)t) \cdot \exp \left( \sum_{j=1}^{\infty} \frac{M_j}{j} t^j \right) \left(1 - (1 - kt)^{-j}\right) \right\}.$$
Asymptotic change of basis

We consider a gradation on $\Lambda^*$:

$$\text{deg}(M_k) = k$$

The highest degree term of the change of basis is easy:

$$\text{Ch}_\mu = \prod_i M_{\mu,i} + \text{smaller degree terms.}$$
Filtration and order of magnitude

Lemma

\[ x \in \Lambda^* \Rightarrow \mathbb{E}_{P_n^\omega}(x) = O(n^{\deg(x)}) \]

Proof:

\[
\mathbb{E}_{P_n^\omega}(\text{Ch}_\mu) = n(n-1) \ldots (n-|\mu|+1)E(\lambda \mapsto \chi^\lambda(\mu))
\]
\[= n(n-1) \ldots (n-|\mu|+1)p_\mu(\omega)
\]
\[= O(n^{\mid \mu \mid})
\]

so the lemma is true for \( x = \text{Ch}_\mu \).
As it is a basis, this is enough.
Asymptotic behaviour of the $M_k$

Using the previous lemma,

$$\mathbb{E}_{\mathbb{P}_n}(M_k) = \mathbb{E}_{\mathbb{P}_n}(\text{Ch}_k) + O(n^{k-1})$$
Asymptotic behaviour of the $M_k$

Using the previous lemma,

$$\mathbb{E}_{\mathcal{P}_n}^\omega (M_k) = \mathbb{E}_{\mathcal{P}_n}^\omega (\text{Ch}_k) + O(n^{k-1})$$

$$= n(n - 1) \ldots (n - k + 1) \mathbb{E}_{\mathcal{P}_n}^\omega \left[ \chi \cdot ((1 \ldots k)) \right] + O(n^{k-1})$$

$$= n^k p_k(\omega) + O(n^{k-1}).$$
Asymptotic behaviour of the $M_k$

Using the previous lemma,

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= n^k p_k(\omega) + O(n^{k-1}).
$$

We can also estimate its variance:

$$
\mathbb{E}_{\mathcal{P}_n}(M_k^2) - \mathbb{E}_{\mathcal{P}_n}(M_k)^2 = \mathbb{E}_{\mathcal{P}_n}(\text{Ch}(k,k)) - \mathbb{E}_{\mathcal{P}_n}(\text{Ch}(k))^2
$$

$$
+ O(n^{2k-1})
$$

$$
= n^{2k} p_{(k,k)}(\omega) - (n^k p_k(\omega))^2 + O(n^{2k-1})
$$

$$
= O(n^{2k-1})
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Asymptotic behaviour of the $M_k$

Using the previous lemma,

$$\mathbb{E}_{\mathcal{P}_n}(M_k) = \mathbb{E}_{\mathcal{P}_n}(\text{Ch}_k) + O(n^{k-1})$$

$$= n(n-1) \ldots (n-k+1) \mathbb{E}_{\mathcal{P}_n}[\chi((1 \ldots k))] + O(n^{k-1})$$

$$= n^k p_k(\omega) + O(n^{k-1}).$$

We can also estimate its variance:

$$\mathbb{E}_{\mathcal{P}_n}(M_k^2) - \mathbb{E}_{\mathcal{P}_n}(M_k)^2 = \mathbb{E}_{\mathcal{P}_n}(\text{Ch}_{k,k}) - \mathbb{E}_{\mathcal{P}_n}(\text{Ch}_k)^2$$

$$+ O(n^{2k-1})$$

$$= n^{2k} p_{k,k}(\omega) - (n^k p_k(\omega))^2 + O(n^{2k-1})$$

$$= O(n^{2k-1})$$

$$\frac{M_k(\lambda)}{n^k}$$ converges in probability towards $p_k(\omega)$. 
Proof of the result

Convergence of lengths of rows and columns

End of the proof of the theorem

\[ \frac{M_k(\lambda)}{nm} \] is the \((k - 1)\)-th moment of the probability measure

\[ X_\lambda = \sum_{i=1}^{d} (a_i^*(\lambda)/n) \delta_{(a_i^*(\lambda)/n)} + (b_i^*(\lambda)/n) \delta_{(-b_i^*(\lambda)/n)}. \]

and \( p_k(\omega) \) the \((k - 1)\)-th moment of

\[ X_\omega = \gamma \delta_0 + \sum_{i \geq 1} \alpha_i \delta_{\alpha_i} + \beta_i \delta_{-\beta_i}. \]
End of the proof of the theorem

\[ M_k(\lambda) \frac{n^m}{!m} \] is the \((k - 1)\)-th moment of the probability measure

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\[ X_\omega = \gamma \delta_0 + \sum_{i \geq 1} \alpha_i \delta_\alpha_i + \beta_i \delta_{-\beta_i}. \]

We have convergence in probability of the repartition function at each point \( x \neq 0, \alpha_i, -\beta_i \).

\( \Rightarrow \) easy to check that it implies the theorem.
Fluctuations of the $M_k$

- it is easy to compute the dominant term of $\text{Var}(M_k)$ and $\text{Cov}(M_k, M_l)$.

- One can show that the fluctuations of the $M_k$’s form a Gaussian vector (using joint cumulants).
  It requires combinatorial computations in $\Lambda^*$ and some technical tools (introduced by P. Śniady)
Consider the case $\beta_1 = \beta_2 = \cdots = \gamma = 0$. Let

$$Y_i = \sqrt{n} \left( \frac{\lambda_i}{n} - \alpha_i \right).$$

Then

$$M_k(\lambda) \sim \sum_i \lambda_i^k = n^k \left( \sum \alpha_i^k + \frac{k}{\sqrt{n}} \sum \alpha_i^{k-1} Y_i + \ldots \right)$$

i.e.

$$\forall \ k, \ k \sum \alpha_i^{k-1} Y_i = \text{fluctuations}(M_k).$$

We can recover the $Y_i$ only if the $\alpha_i$ are distinct!
If the $\alpha_i$ are distinct, it’s working. For instance, if $\omega = ((1 - q), (1 - q)q, (1 - q)q^2, \ldots ; 0, 0, \ldots ; 0),$

Theorem (F., Méliot 2010)

Denote $Y_{n,q,i}$ the rescaled deviation

$$\sqrt{n} \left( \frac{\lambda_i}{n} - (1 - q) q^{i-1} \right).$$

Then we have convergence of the finite-dimensional laws of the random process $(Y_{n,q,i})_{i \geq 1}$ towards those of a gaussian process $(Y_{q,i})_{i \geq 1}$ with:

$$\mathbb{E}[Y_{q,i}] = 0 \quad ; \quad \mathbb{E}[Y_{q,i}^2] = (1 - q) q^{i-1} - (1 - q)^2 q^{2(i-1)} \quad ;$$

$$\text{cov}(Y_{q,i}, Y_{q,j}) = -(1 - q)^2 q^{i+j-2}.$$

Otherwise, we need another method (cf. Pierre-Loïc’s talk).
Remark on the Plancherel case

- The theorem does not give much information.

- There is a deterministic limit shape (Logan, Shepp 77 and Kerov, Vershik 77). The fluctuations around this shape is known (Ivanov, Kerov, Olshanski 2002).

- They used the same kind of method, replacing power sums of Frobenius coordinates by **free cumulant of the transition measure** (in fact, we have adapted their ideas!).

End of the talk. Thanks!