Classical and mean field limit of field-particle systems

Roscoff, February 5th 2014
Outline

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that should reduce classically to Newton equations

$$\begin{cases}
\frac{d\xi}{dt}(t) = \frac{1}{m} \pi(t) \\
\frac{d\pi}{dt}(t) = -\nabla V(\xi(t))
\end{cases}.$$
Mean field limit
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- Formally, we are looking to reduce a very big phase-space (the $n$ particle one), to a single-particle phase space; in the limit $n \to \infty$. 
Mean field theory for many bosons

Consider the system of $n$ non-relativistic bosons described by the following Hamiltonian of $L^2(\mathbb{R}^d)$:

$$H = \sum_{j=1}^{n} -\Delta x_j + 1 \sum_{i<j} V(x_i - x_j).$$

We expect that when $n$ is very large the dynamics of each particle should be dictated by the mean field Hartree equation:

$$i \frac{\partial}{\partial t} \phi_t = -\Delta \phi_t + (V \ast |\phi_t|^2) \phi_t.$$
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Overview

The physical setting

Particles interacting with fields
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H = \frac{1}{2M} \int (\nabla \psi)^*(x) \nabla \psi(x) dx + \int \omega(k)a^*(k) a(k) dk + \lambda \int \varphi(x) \psi^*(x)\psi(x) dx
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with \(\omega(k) = \sqrt{k^2 + \mu^2}\), \(M > 0, \mu \geq 0\), coupling constant \(\lambda > 0\) and

\[
\varphi(x) = \int \frac{\chi(k)}{\sqrt{\omega}} (a(k) e^{ikx} + a^*(k) e^{-ikx}) dk.
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(The precise meaning of the mean field limit in this system will be explained in detail.)
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$$H = \frac{1}{2m} \left( p - eA(q) \right)^2 + \hbar \sum_{\lambda=1,2} \int \omega(k) a^* (k, \lambda) a(k, \lambda) dk ,$$
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where $p = -i\sqrt{\hbar}\nabla$, $q = \sqrt{\hbar}x$, $\omega(k) = c|k|$ and

$$A(x) = \sum_{\lambda=1,2} \int \sqrt{\frac{\hbar}{\omega(k)}} e_{\lambda}(k) \chi(k) (a(k, \lambda)e^{ik\cdot x} + a^*(k, \lambda)e^{-ik\cdot x}) dk .$$
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\begin{align*}
\partial_t B + \nabla \times E &= 0 \\
\partial_t E - \nabla \times B &= -j \\
\dot{\xi} &= v \\
\dot{v} &= e[(\varphi \ast E)(\xi) + v \times (\varphi \ast B)(\xi)]
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\[
\begin{align*}
\nabla \cdot E &= \rho \\
\nabla \cdot B &= 0
\end{align*}
\]

\[
\begin{align*}
j &= ev\varphi(\xi - x) \\
\rho &= e\varphi(\xi - x)
\end{align*}
\]
Mean field theory for many particles
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No information on rate of convergence (apart for small times, in some systems); nor on fluctuations around the mean field solution.
Hepp method.
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PROs: Informations on the evolution of fluctuations (between coherent states). Bound on the rate of convergence of reduced density matrices.

CONs: Specific initial states has to be considered, namely factorized and coherent ones (also partially factorized and linear combinations of the above [F., 1305]). Even if, due to symmetries, the natural setting of the system is $L^2(\mathbb{R}^{nd})$, the whole Fock space $\mathcal{F}(L^2(\mathbb{R}^d))$ has to be considered to prove convergence (since the method relies on the Weyl operators of the whole Fock space).
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**CONs:** No information on the fluctuations, nor on the rate of convergence.
Other approaches.

- Recently, other methods have been developed that deserve further mention. The first has been developed by Pickl [2011], and it is based on counting the particles that are not condensed at some time $t$; provided we started with an initial state completely factorized, he proves that this number goes to zero when $n \to \infty$, hence obtaining convergence for the reduced density matrix.

- Another approach is due to Lewin et al. [2013]. They are able to describe the evolution of fluctuations around Hartree states, instead of coherent ones, in the limit $n \to \infty$. Their result implies as well the convergence of reduced density matrices and gives a bound on the rate of convergence.

- Using the construction of a truncated Fock space (and of a map that substitutes the Weyl operators) [see Lewin et al., 2012], they restrict the analysis to a space isomorphic to $L_2(\mathbb{R}^d)$, instead of considering the whole Fock space as in the Hepp method.

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Overview Mathematical results

Particles interacting with fields

Mean field limit of the Nelson model. Ginibre et al. [2006] studied the partial limit of the Nelson model without cut off, using the Hepp method. In [F., 1301], I analyzed the complete mean field limit of the Nelson model, but with cut off. I will describe the results in detail in the following section.

Classical limit of Particle QED.

No result yet!
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- **Classical limit of Particle QED.** No result yet!
The Nelson model
The mean field limit as $\lambda \to 0$

Recall the Nelson Hamiltonian:

$$H = \frac{1}{2M} \int (\nabla \psi)^*(x) \nabla \psi(x) dx + \int \omega(k) a^*(k) a(k) dk + \lambda \int \varphi(x) \psi^*(x) \psi(x) dx.$$
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Consider a state $\phi_{n_1,n_2}$ such that $\langle \phi_{n_1,n_2} , (N_1 + N_2) \phi_{n_1,n_2} \rangle \sim n_1 + n_2$; we would like to describe its dynamics in the limit $n_1, n_2 \to \infty$ as a mean field theory, with the particles coupled as described above.
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In order to do that $n_1$ and $n_2$ has to be related, and it turns out that they have also to be related to the coupling constant $\lambda$, as $n_1 \sim n_2 \sim \lambda^{-2}$. So the mean field limit is also a weak coupling limit $\lambda \to 0$. 

Marco Falconi (CHL, Univ. Rennes1) Classical and mean field limit of field-particle systems Roscoff, February 5th 2014 17 / 36
Quantum dynamics

Proposition

\( H \) is essentially self-adjoint on \( \mathcal{F}_s(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_s(L^2(\mathbb{R}^3)) \).
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Consider the subspace $\mathcal{H}_{n_1} = L^2(\mathbb{R}^{3n_1}) \otimes \mathcal{F}_s(L^2(\mathbb{R}^3))$ of the whole Fock space with fixed number $n_1$ of non-relativistic particles, $H\big|_{n_1}$ the restriction of $H$ to that subspace.
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$$\|H_i|_{n_1}\phi\|^2 \leq \varepsilon^2 \|H_0|_{n_1}\phi\|^2 + C(\varepsilon, n_1)\|\phi\|^2.$$
\[ H \big|_{n_1} \text{ is self-adjoint with domain } D(H_0 \big|_{n_1}), \quad \forall n_1 \in \mathbb{N}. \]
$H_{n_1}$ is self-adjoint with domain $D(H_0|_{n_1})$, $\forall n_1 \in \mathbb{N}$.

Since $\varepsilon$ does not depend on $n_1$, we can define $H$ as the direct sum:

$$H = \bigoplus_{n_1=0}^{\infty} H_{n_1}.$$
Classical dynamics

Recall the classical equation,

\[
\begin{align*}
\left( i \partial_t + \frac{1}{2M} \Delta \right) u &= (2\pi)^{-3/2} (\tilde{\chi} \ast A) u \\
(\partial_t^2 - \Delta + \mu^2) A &= -(2\pi)^{-3/2} \tilde{\chi} \ast |u|^2.
\end{align*}
\]
Recall the classical equation,

\[
\begin{cases}
  
  
  (i \partial_t + \frac{1}{2M} \Delta) u = (2\pi)^{-3/2} (\tilde{\chi} * A) u \\
  
  (\partial_t^2 - \Delta + \mu^2) A = -(2\pi)^{-3/2} \tilde{\chi} * |u|^2 
\end{cases}
\]

A can be written as

\[
A(x) = \int \frac{1}{\sqrt{\omega}} \left( \alpha(k)e^{ikx} + \bar{\alpha}(k)e^{-ikx} \right) dk .
\]
The system of equations for $u$ and $\alpha$ then becomes:

$$\begin{align*}
    i\partial_t u &= -\frac{1}{2M} \Delta u + (2\pi)^{-3/2} (\tilde{\chi} * A) u \\
    i\partial_t \alpha &= \omega \alpha + (\omega)^{-1/2} \chi(|u|^2).
\end{align*}$$
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\begin{align*}
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\end{align*}
\]

**Proposition**

Let $(u_0, \alpha_0)$ in $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. Then the equation above admits an unique global solution $(u(t), \alpha(t)) \in C^0(\mathbb{R}, L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3))$. 
Evolution of fluctuations

Let $u, \alpha \in L^2(\mathbb{R}^3)$. Define the Weyl operator:

$$C(u, \alpha) = \exp\{\psi^*(u) - \psi(u)\} \otimes \exp\{a^*(\alpha) - a(\alpha)\}$$

where

$$\psi(u) = \int \overline{u}(x) \psi(x) \, dx,$$

$$\psi^*(u) = (\psi(u))^*,$$

analogous for $a^*(\alpha)$.
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Marco Falconi (CHL, Univ. Rennes1) Classical and mean field limit of field-particle systems Roscoff, February 5th 2014 22 / 36
Evolution of fluctuations

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$$C(u, \alpha) = \exp\{\psi^*(u) - \psi(u)\} \otimes \exp\{a^*(\alpha) - a(\alpha)\}$$

($\psi(u) = \int \bar{u}(x)\psi(x)dx$, $\psi^*(u) = (\psi(u))^*$, analogous for $a^#(\alpha)$).
Then we can define the quantum evolution between coherent states as:

\[ W(t, s) = C^*(u(t)/\lambda, \alpha(t)/\lambda) \exp\{-i(t - s)H\} C(u(s)/\lambda, \alpha(s)/\lambda)e^{i\Lambda(t,s)} ; \]
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\]

where \((u(t), \alpha(t))\) is the solution of the classical equation with initial state \((u(s), \alpha(s))\) and

\[
\Lambda(t, s) = -\frac{1}{2} (2\pi)^{-3/2} \lambda^{-2} \int_s^t \int_{\mathbb{R}^3} (\tilde{\chi} * A)(\tau) \tilde{u}(\tau) u(\tau) dx d\tau.
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More difficult to define is its putative limit when $\lambda \rightarrow 0$: we call it $U_2(t, s)$. 
More difficult to define is its putative limit when $\lambda \to 0$: we call it $U_2(t, s)$. $U_2(t, s)$ is formally generated by the time-dependent Hamiltonian:

$$H_2(t) = \frac{1}{2M} \int (\nabla \psi)^*(x) \nabla \psi(x) dx + \int \omega a^*(k)a(k) dk + \left[ \int \left( \frac{1}{2}(2\pi)^{-3/2} (\tilde{\chi} * A(t))\psi^*\psi + u(t)\varphi\psi^* \right) dx \right. h.c.$$
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$U_2(t, s)$ can be rigorously defined by means of a truncated Dyson series in the interaction representation.
Theorem 1 ([F., 1301])

Let $\phi \in \mathcal{H}$. Then

$$\lim_{\lambda \to 0} W(t, s)\phi = U_2(t, s)\phi$$

in the strong topology of $\mathcal{H}$; uniformly in $t$ and $s$ on compact intervals.
Theorem 1 ([F., 1301])

Let $\phi \in \mathcal{H}$. Then

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in the strong topology of $\mathcal{H}$; uniformly in $t$ and $s$ on compact intervals.

$U_2(t, s)$ describes the evolution of quantum fluctuations operators $\psi^\# - u(s)/\lambda$ and $a^\# - \alpha(s)/\lambda$ in the limit $\lambda \to 0$. 
Mean field limit of quantum variables

- The quantum variables of the system are the annihilation and creation operators for the two species of particles: $\psi^\#(x)$ and $a^\#(k)$. 

The multiplicative factor $\lambda$ is necessary, otherwise $\psi^\#$ and $a^\#$ would diverge in the limit as $\sqrt{n_1} \sim \lambda^{-1}$ and $\sqrt{n_2} \sim \lambda^{-1}$ respectively.
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- We expect that, in some sense, in the limit $\lambda \to 0$: $\lambda \psi_t^\#(x) \sim u^\#(t, x)$, $\lambda a_t^\#(k) \sim \alpha^\#(t, k)$. 
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- We may consider the mean field limit for initial states that are coherent ($C(u_0/\lambda, \alpha_0/\lambda)\Omega$) or factorized ($u_0^{\otimes n_1} \otimes \alpha_0^{\otimes n_2}$) or factorized in the first species and coherent in the second one ($u_0^{\otimes n_1} \otimes C(\alpha_0/\lambda)\Omega_2$).
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- The quantum variables of the system are the annihilation and creation operators for the two species of particles: $\psi^#(x)$ and $a^#(k)$.

- We expect that, in some sense, in the limit $\lambda \to 0$: $\lambda \psi_t^#(x) \sim u^#(t, x)$, $\lambda a_t^#(k) \sim \alpha^#(t, k)$.
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- Each one of these states has the property that $\langle \psi, (N_1 + N_2)\psi \rangle \sim n_1 + n_2 \sim \lambda^{-2}$.
Let $\psi_#(x) = \exp\{itH\} \psi_#(x) \exp\{-itH\}$, $a_#(k) = \exp\{itH\} a_#(k) \exp\{-itH\}$. 
Let \( \psi_t^#(x) = \exp\{itH\} \psi^#(x) \exp\{-itH\} \), \( a_t^#(k) = \exp\{itH\} a^#(k) \exp\{-itH\} \).

Also, let \((u(t), \alpha(t))\) be the classical solution corresponding to initial datum \((u_0, \alpha_0)\) with \( \|u_0\|_2 = \|\alpha_0\|_2 = 1 \).
Let $\psi_t^#(x) = \exp\{itH\} \psi^#(x) \exp\{-itH\}$, $a_t^#(k) = \exp\{itH\} a^#(k) \exp\{-itH\}$. Also, let $(u(t), \alpha(t))$ be the classical solution corresponding to initial datum $(u_0, \alpha_0)$ with $\|u_0\|_2 = \|\alpha_0\|_2 = 1$.

**Theorem 2**

As functions of $L^2(\mathbb{R}^3)$, we have the following convergence:

$$\lim_{\lambda \to 0} \langle C(u_0/\lambda, \alpha_0/\lambda)\Omega, \lambda \psi_t^#(x) C(u_0/\lambda, \alpha_0/\lambda)\Omega \rangle = u^#(t, x) ;$$

$$\lim_{\lambda \to 0} \langle C(u_0/\lambda, \alpha_0/\lambda)\Omega, \lambda a_t^#(k) C(u_0/\lambda, \alpha_0/\lambda)\Omega \rangle = \alpha^#(t, k) .$$
The convergence result holds for arbitrary products of normal ordered annihilation and creation operators.
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We obtain the same convergence and rate if we replace $C(u_0/\lambda, \alpha_0/\lambda)\Omega$ with $u_0^\otimes n_1 \otimes C(\alpha_0/\lambda)\Omega_2$.
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Up to a normalization factor, the result above (for products with the same number of creation and annihilation operators of each type) is equivalent to the convergence of reduced density matrices in the Hilbert-Schmidt norm.
The result for factorized vectors $u_0^{\otimes n_1} \otimes \alpha_0^{\otimes n_2}$ deserves special attention.
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The result for factorized vectors $u_0 \otimes^{n_1} \alpha_0 \otimes^{n_2}$ deserves special attention. Let $(u_\theta(t), \alpha_\theta(t))$ be the classical solution corresponding to initial datum $(u_0, \exp\{ -i\theta \} \alpha_0)$.

**Theorem 3**

As functions of $L^2(\mathbb{R}^6)$ and $L^2(\mathbb{R}^3)$ respectively, we have the following convergence:

$$\lim_{\lambda \to 0} \langle u_0 \otimes^{n_1(\lambda)} \alpha_0 \otimes^{n_2(\lambda)}, \lambda^2 \psi_t^*(x) \psi_t(y) u_0 \otimes^{n_1(\lambda)} \alpha_0 \otimes^{n_2(\lambda)} \rangle = \int_0^{2\pi} \bar{u}_\theta(t,x) u_\theta(t,y) \frac{d\theta}{2\pi}$$

$$\lim_{\lambda \to 0} \langle u_0 \otimes^{n_1(\lambda)} \alpha_0 \otimes^{n_2(\lambda)}, \lambda a_t^\#(k) u_0 \otimes^{n_1(\lambda)} \alpha_0 \otimes^{n_2(\lambda)} \rangle = \int_0^{2\pi} \alpha^\#_\theta(t,k) \frac{d\theta}{2\pi}.$$
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Up to a normalization factor, the result above (for products with the same number of creation and annihilation operators of each type) is equivalent to the convergence of reduced density matrices in the Hilbert-Schmidt norm.

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Up to a normalization factor, the result above (for products with the same number of creation and annihilation operators of each type) is equivalent to the convergence of reduced density matrices in the Hilbert-Schmidt norm.

The quantum evolution does not preserve the number $n_2$ of relativistic particles; this affects the classical limit, for initial states that have a fixed number of relativistic particles, in an unexpected fashion.
Mean field limit for a general class of states
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The Nelson model: Wigner measures

Mean field limit for a general class of states

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We say that a family of density matrices $(\rho_\lambda)_{\lambda \in (0,\bar{\lambda})}$ on a Fock space $\mathcal{F}_s(\mathcal{L})$ converges to a measure $\mu$ on $\mathcal{L}$
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We say that a family of density matrices $(\rho_\lambda)_{\lambda \in (0, \bar{\lambda})}$ on a Fock space $\mathcal{F}_s(\mathcal{L})$ converges to a measure $\mu$ on $\mathcal{L}$ if a probability measure on $\mathcal{L}$ exists such that for all $\xi \in \mathcal{L}$:

$$\lim_{\lambda \to 0} \text{Tr} \left[ \rho_\lambda \exp \left\{ i \left( a^\dagger (\xi) + a(\xi) \right) / \sqrt{2} \right\} \right] = \int_{\mathcal{L}} e^{i \sqrt{2} \text{Re} \langle \xi, \zeta \rangle} \, d\mu(\zeta).$$
Proposition

Let \( (\rho_\lambda)_{\lambda \in (0, \lambda]} \) be a family of normal states on \( \mathcal{F}_s(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_s(L^2(\mathbb{R}^3)) \) (satisfying some regularity properties) that converges to a probability measure \( \mu_0 \) of \( \mathcal{L} := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) when \( \lambda \to 0 \).
Proposition

Let $(\rho_\lambda)_{\lambda \in (0, \bar{\lambda})}$ be a family of normal states on $\mathcal{F}_s(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_s(L^2(\mathbb{R}^3))$ (satisfying some regularity properties) that converges to a probability measure $\mu_0$ of $\mathcal{Z} := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ when $\lambda \rightarrow 0$. Also, let $\Phi(t, s)$ be the classical flux, i.e. $(u(t), \alpha(t)) = \Phi(t, s)(u(s), \alpha(s))$, and $\mu_t = \Phi(t, 0)_* \mu_0$. 
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Proposition

Let $\left(\rho_\lambda\right)_{\lambda \in (0, \bar{\lambda})}$ be a family of normal states on $\mathcal{F}_s(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_s(L^2(\mathbb{R}^3))$ (satisfying some regularity properties) that converges to a probability measure $\mu_0$ of $Z := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ when $\lambda \to 0$. Also, let $\Phi(t, s)$ be the classical flux, i.e. $(u(t), \alpha(t)) = \Phi(t, s)(u(s), \alpha(s))$, and $\mu_t = \Phi(t, 0)_* \mu_0$. Then in the $L^1_s(L^2(\mathbb{R}^3p_1) \otimes L^2_s(\mathbb{R}^3p_1))$-norm, for all $p_1, p_2 \in \mathbb{N}$:

$$\lim_{\lambda \to 0} \gamma_{\lambda}^{p_1, p_2}(t) = \frac{1}{\int_{\mathcal{X}} |Z_1|^{2p_1} |Z_2|^{2p_2} d\mu_t(Z)} \int_{\mathcal{X}} |Z_1^{\otimes p_1} \otimes Z_2^{\otimes p_2} \rangle \langle Z_1^{\otimes p_1} \otimes Z_2^{\otimes p_2}| d\mu_t(Z).$$
Future developments
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Classical limit of particle QED.
Mean field limit of the Nelson model without cut off.
Scattering in the mean field limit.
Future developments

- Classical limit of particle QED.
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References


Thank you.