On generalizations of partition theorems of Schur and Andrews to overpartitions

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Abstract

In this paper we give three new proofs of Schur's theorem for overpartitions using recurrences and generating functions. We also prove two new theorems on overpartitions with difference conditions. These generalize two partition identities of Andrews.

1 Introduction

A partition of n is a non-increasing sequence of natural numbers whose sum is n. An overpartition of n is a partition of n in which the first occurence of a number may be overlined. For example, there are 14 overpartitions of 4: 4, $\overline{4}$, $3+1$, $\overline{3}+1$, $3+\overline{1}$, $\overline{3}+\overline{1}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$ and $\overline{1}+1+1+1$.

The following theorem is known as "Schur's theorem for overpartitions" [10]:

Theorem 1.1. *Let $A(k,n)$ denote the number of overpartitions of n into parts congruent to 1 or 2 modulo 3 with k non-overlined parts. Let $B(k,n)$ denote the number of overpartitions of n with k non-overlined parts, where parts differ by at least 3 if the smaller is overlined OR both parts are divisible by 3, and parts differ by at least 6 if the smaller is overlined AND both parts are divisible by 3. Then $A(k,n) = B(k,n)$.*

The case $k = 0$ is Schur's celebrated partition theorem [12]. Several proofs of Schur's theorem have been given using a variety of different techniques such as bijective mappings [8, 7], the method of weighted words [1], and recurrences [2, 4, 6].

Theorem 1.1 was discovered using the method of weighted words [10] and was subsequently proved bijectively [11]. In the first part of this paper we give three new proofs of Theorem 1.1 using recurrences. Though they are based on ideas originally due to Andrews, the equations and techniques used to solve them are different and more intricate.

Andrews used his ideas about recurrences not only to prove Schur's partition theorem, but also to generalize it in two different ways [3, 5]. These generalizations are indexed by the numbers $N = 2^n - 1$, Schur's theorem corresponding to $N = 3$.

In the second part of this paper we take a first step toward the generalization of Andrews' two theorems to overpartitions by proving the following:

Theorem 1.2. *Let $C(k,n)$ denote the number of overpartitions of n into parts congruent to 1, 2 or 4 modulo 7, with k non-overlined parts. Let $D(k,n)$ denote the number of overpartitions of n with k non-overlined parts of the form $n = \lambda_1 + ... + \lambda_s$, where*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 7\chi(\lambda_{i+1} \textit{ is overlined}) \textit{ if } \lambda_{i+1} \equiv 1,2,4 \ (\textit{mod } 7), \\ 5 + 7\chi(\lambda_{i+1} \textit{ is overlined}) \textit{ if } \lambda_{i+1} \equiv 3 \ (\textit{mod } 7), \\ 3 + 7\chi(\lambda_{i+1} \textit{ is overlined}) \textit{ if } \lambda_{i+1} \equiv 5,6 \ (\textit{mod } 7), \\ 8 + 7\chi(\lambda_{i+1} \textit{ is overlined}) \textit{ if } \lambda_{i+1} \equiv 0 \ (\textit{mod } 7), \end{cases}$$

where $\chi(\lambda_{i+1}$ *is overlined*$) = 1$ *if* λ_{i+1} *is overlined and* 0 *otherwise. Then* $C(k,n) = D(k,n)$.

Theorem 1.3. *Let* $U(k,n)$ *denote the number of overpartitions of* n *into parts congruent to* 3, 5 *or* 6 *modulo* 7, *with* k *non-overlined parts. Let* $V(k,n)$ *denote the number of overpartitions of* n *with* k *non-overlined parts of the form* $n = \lambda_1 + ... + \lambda_s$, *where*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 7\chi(\lambda_{i+1} \text{ is overlined}) \text{ if } \lambda_i \equiv 3,5,6 \ (mod\ 7) \\ 5 + 7\chi(\lambda_{i+1} \text{ is overlined}) \text{ if } \lambda_i \equiv 4 \ (mod\ 7), \\ 3 + 7\chi(\lambda_{i+1} \text{ is overlined}) \text{ if } \lambda_i \equiv 1,2 \ (mod\ 7), \\ 8 + 7\chi(\lambda_{i+1} \text{ is overlined}) \text{ if } \lambda_i \equiv 0 \ (mod\ 7), \end{cases}$$

and $\lambda_s \neq 1, \overline{1}, 2, \overline{2}, 4, \overline{4}, 7, \overline{7}$. *Then* $U(k,n) = V(k,n)$.

When $k = 0$, Theorems 1.2 and 1.3 reduce to the case $N = 7$ in Andrews' two generalizations of Schur's partition theorem mentioned above [3, 5]. Let us illustrate Theorems 1.2 and 1.3 with an example. For Theorem 1.2, the overpartitions of 4 counted by $D(k,4)$ are: 4, $\overline{4}$, $3+1$, $\overline{3}+1$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $1+1+1+1$ and $\overline{1}+1+1+1$. The overpartitions of 4 into parts congruent to 1,2 or 4 modulo 7 are: 4, $\overline{4}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$ and $\overline{1}+1+1+1$. In both cases, we have 1 overpartition with 0 non-overlined parts, 3 overpartitions with 1 non-overlined part, 3 overpartitions with 2 non-overlined parts, 2 overpartitions with 3 non-overlined parts and 1 overpartition with 4 non-overlined parts. For Theorem 1.3, the overpartitions of 8 counted by $V(k,8)$ are: 8, $\overline{8}$, $5+3$ and $\overline{5}+3$. The overpartitions of 4 into parts congruent to 3, 5 or 6 modulo 7 are: $5+3$, $\overline{5}+3$, $5+\overline{3}$ and $\overline{5}+\overline{3}$. In both cases, we have 1 overpartition with 0 non-overlined parts, 2 overpartitions with 1 non-overlined part, and 1 overpartition with 2 non-overlined parts.

The proofs of Theorems 1.2 and 1.3 again use recurrences, but the details are even more considerable for $N = 7$ than for $N = 3$. Consequently, it is not clear whether it is feasible to employ these techniques to give an overpartition identity for all $N = 2^n - 1$.

2 Three new proofs of Schur's theorem for overpartitions

In this section we give three new proofs of Theorem 1.1. The first one uses recurrences based on the smallest part of the overpartition, and the other two use recurrences based on the largest part.

2.1 Proof using recurrences based on the smallest part

Let $b_j(k,m,n)$ denote the number of overpartitions counted by $B(k,n)$ having m parts such that the smallest part is $> j$.

Lemma 2.1.

$$b_0(k,m,n) - b_1(k,m,n) = b_0(k,m-1,n-3m+2) + b_0(k-1,m-1,n-1), \tag{2.1}$$

$$b_1(k,m,n) - b_2(k,m,n) = b_1(k,m-1,n-3m+1) + b_1(k-1,m-1,n-2), \tag{2.2}$$

$$b_2(k,m,n) - b_3(k,m,n) = b_3(k,m-1,n-3m) + b_0(k-1,m-1,n-3m), \tag{2.3}$$

$$b_3(k,m,n) = b_0(k,m,n-3m). \tag{2.4}$$

Proof: We observe that $b_{i-1}(k,m,n) - b_i(k,m,n)$ is the number of overpartitions counted by $b_{i-1}(k,m,n)$ such that the smallest part is equal to i. We begin by treating (2.1) : If $\lambda_m = \overline{1}$, then $\lambda_{m-1} \geq 4$. In that case we remove the $\overline{1}$ and subtract 3 from each remaining part. The number of parts is reduced to $m-1$, the number of non-overlined parts is still k, and the number partitioned is now $n - 3m + 2$. So we have an overpartition counted by $b_0(k, m-1, n-3m+2)$. If $\lambda_m = 1$, then $\lambda_{m-1} \geq 1$. In that case we remove λ_m. The number of parts is reduced to $m-1$, the number of non-overlined parts is reduced to $k-1$, and the number partitioned is now $n-1$. We have an overpartition counted by $b_0(k-1, m-1, n-1)$. The equations (2.2), (2.3) and (2.4) are proved in the same way. □ □

For $|x| < 1$, $|d| < 1$ $|q| < 1$, we define

$$f_i(x,d,q) = f_i(x) = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} b_i(k,m,n)x^m d^k q^n. \tag{2.5}$$

We want to find $f_0(1)$, which is the generating function for overpartitions counted by $B(k,n)$. Equations (2.1), (2.2), (2.3) and (2.4) imply:

$$f_0(x) - f_1(x) = xqf_0(xq^3) + dxqf_0(x), \tag{2.6}$$

$$f_1(x) - f_2(x) = xq^2 f_1(xq^3) + dxq^2 f_1(x), \tag{2.7}$$

$$f_2(x) - f_3(x) = xq^3 f_3(xq^3) + dxq^3 f_0(xq^3), \tag{2.8}$$

$$f_3(x) = f_0(xq^3). \tag{2.9}$$

Thus by (2.6),

$$f_1(x) = (1 - dxq)f_0(x) - xqf_0(xq^3). \tag{2.10}$$

By (2.8) and (2.9),

$$f_2(x) = (1 + dxq^3)f_0(xq^3) + xq^3 f_0(xq^6). \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.7), we obtain:

$$\begin{aligned}(1 - dxq)(1 - dxq^2)f_0(x) = {} & (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^6)f_0(xq^3) \\ & + xq^3(1 - xq^3)f_0(xq^6).\end{aligned}$$

The following lemma for $N = 3$ will complete the proof. We prove it for general N because the case $N = 7$ will be used in the proof of Theorem 1.2.

Lemma 2.2. *Let N be a positive integer and $f_0(x)$ a function such that $f_0(0) = 1$ and*

$$\begin{aligned}(1 - dxq)(1 - dxq^2)f_0(x) = {} & (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^{N+3})f_0(xq^N) \\ & + xq^3(1 - xq^N)f_0(xq^{2N}).\end{aligned} \tag{2.12}$$

Then

$$f_0(1) = \prod_{k=0}^{\infty} \frac{(1 + q^{Nk+1})(1 + q^{Nk+2})}{(1 - dq^{Nk+1})(1 - dq^{Nk+2})}.$$

Proof: Let

$$F(x) = f_0(x)\prod_{k=0}^{\infty}\frac{(1 - dxq^{Nk+1})}{(1 - xq^{Nk})}.$$

Then by (2.12),

$$\begin{aligned}(1 - x)(1 - dxq^2)F(x) = {} & (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^{N+3})F(xq^N) \\ & + xq^3(1 - dxq^{N+1})F(xq^{2N}).\end{aligned}$$

Let $F(x) = \sum_{n=0}^{\infty} A_n x^n$. Then $A_0 = F(0) = f_0(0) = 1$ and

$$\begin{aligned}& A_n - A_{n-1} - dq^2A_{n-1} + dq^2A_{n-2} = \\ & q^{Nn}A_n + (q^{N(n-1)+1} + q^{N(n-1)+2} + dq^{N(n-1)+3})A_{n-1} \\ & - (dq^{N(n-2)+3} + dq^{N(n-1)+3})A_{n-2} + q^{2N(n-1)+3}A_{n-1} \\ & - dq^{N(2n-3)+4}A_{n-2}.\end{aligned}$$

Simplifying, we obtain:

$$\begin{aligned}(1-q^{Nn})A_n &= (1+dq^2+q^{N(n-1)+2})(1+q^{N(n-1)+1})A_{n-1}\\ &\quad - dq^2(1+q^{N(n-1)+1})(1+q^{N(n-2)+1})A_{n-2}.\end{aligned}$$

Let $A_n = a_n \prod_{k=0}^{n-1}(1+q^{Nk+1})$. Then

$$(1-q^{Nn})a_n = (1+dq^2+q^{N(n-1)+2})a_{n-1} - dq^2 a_{n-2}.$$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Thus

$$(1-x)(1-dxq^2)f(x) = (1+xq^2)f(xq^N).$$

Thus $f(0) = a_0 = 1$ and

$$f(x) = \prod_{k=0}^{\infty} \frac{(1+xq^{Nk+2})}{(1-xq^{Nk})(1-dxq^{Nk+2})} f(0) = \prod_{k=0}^{\infty} \frac{(1+xq^{Nk+2})}{(1-xq^{Nk})(1-dxq^{Nk+2})}.$$

Next,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{A_n x^n}{\prod_{k=0}^{n-1}(1+q^{Nk+1})} = \prod_{k=0}^{\infty} \frac{(1+xq^{Nk+2})}{(1-xq^{Nk})(1-dxq^{Nk+2})}.$$

By Appell's Comparison Theorem [9, p. 101],

$$\begin{aligned}\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} \frac{A_n x^n}{\prod_{k=0}^{n-1}(1+q^{Nk+1})} &= \frac{A_\infty}{\prod_{k=0}^{\infty}(1+q^{Nk+1})}\\ &= \prod_{k=0}^{\infty} \frac{(1+q^{Nk+2})}{(1-q^{Nk+N})(1-dq^{Nk+2})}.\end{aligned}$$

Thus

$$A_\infty = \prod_{k=0}^{\infty} \frac{(1+q^{Nk+2})(1+q^{Nk+1})}{(1-q^{Nk+N})(1-dq^{Nk+2})}.$$

Next,

$$f_0(x) = \prod_{k=0}^{\infty} \frac{(1-xq^{Nk})}{(1-dxq^{Nk+1})} \sum_{n=0}^{\infty} A_n x^n = (1-x)\prod_{k=0}^{\infty} \frac{(1-xq^{Nk+N})}{(1-dxq^{Nk+1})} \sum_{n=0}^{\infty} A_n x^n.$$

We apply Appell's Comparison Theorem again and we obtain:

$$f_0(1) = \prod_{k=0}^{\infty} \frac{(1+q^{Nk+1})(1+q^{Nk+2})}{(1-dq^{Nk+1})(1-dq^{Nk+2})}.$$

□ □

We apply Lemma 2.2 with $N = 3$ (we have $f_0(0) = 1$ by (2.5)). We obtain that

$$f_0(1) = \prod_{n=0}^{\infty} \frac{(1+q^{3n+1})(1+q^{3n+2})}{(1-dq^{3n+1})(1-dq^{3n+2})}.$$

So $f_0(1)$ is the generating function for overpartitions with parts congruent to 1 or 2 modulo 3, which proves Schur's theorem for overpartitions.

2.2 Proof using recurrences based on the largest part

Let $\pi_m(k,n)$ denote the number of overpartitions counted by $B(k,n)$ such that the largest part is $\le m$ and overlined. Let $\phi_m(k,n)$ denote the number of overpartitions of n counted by $B(k,n)$ such that the largest part is $\le m$ and non-overlined.

Notice that for every $m,n,k \ge 1$,

$$\pi_m(k-1,n) = \phi_m(k,n) \tag{2.13}$$

because we can either overline the largest part or not.

Lemma 2.3.

$$\pi_{3m+1}(k,n) = \pi_{3m}(k,n) + \phi_{3m+1}(k,n-3m-1) + \pi_{3m-2}(k,n-3m-1), \tag{2.14}$$

$$\pi_{3m+2}(k,n) = \pi_{3m+1}(k,n) + \phi_{3m+2}(k,n-3m-2) + \pi_{3m-1}(k,n-3m-2), \tag{2.15}$$

$$\pi_{3m+3}(k,n) = \pi_{3m+2}(k,n) + \phi_{3m+2}(k,n-3m-3) + \pi_{3m-1}(k,n-3m-3). \tag{2.16}$$

$$\phi_{3m+1}(k,n) = \phi_{3m}(k,n) + \phi_{3m+1}(k-1,n-3m-1) + \pi_{3m-2}(k-1,n-3m-1), \tag{2.17}$$

$$\phi_{3m+2}(k,n) = \phi_{3m+1}(k,n) + \phi_{3m+2}(k-1,n-3m-2) + \pi_{3m-1}(k-1,n-3m-2), \tag{2.18}$$

$$\phi_{3m+3}(k,n) = \phi_{3m+2}(k,n) + \phi_{3m+2}(k-1,n-3m-3) + \pi_{3m-1}(k-1,n-3m-3). \tag{2.19}$$

Proof: We give a proof of (2.14). The other equations can be proved in the same way. We break the set of overpartitions enumerated by $\pi_{3m+1}(k,n)$ into two sets, those with largest part less than $3m+1$ and those with largest part equal to $3m+1$. The first one is enumerated by $\pi_{3m}(k,n)$. The second is enumerated by $\phi_{3m+1}(k,n-3m-1) + \pi_{3m-2}(k,n-3m-1)$. To see this, we remove the largest part, so the number partitioned becomes $n-3m-1$. The largest part was overlined so the number of remaining non-overlined parts is still k. If the second part is overlined, it has to be $\le 3m-2$ and we obtain an overpartition counted by $\pi_{3m-2}(k,n-3m-1)$. If it is not overlined, it has to be $\le 3m+1$ and we obtain an overpartition counted by $\phi_{3m+1}(k,n-3m-1)$. $\square$ $\square$

For all m,n,k, let $\psi_m(k,n) = \pi_m(k,n) + \phi_m(k,n)$.

Adding (2.14) and (2.17), and using (2.13), we obtain

$$\psi_{3m+1}(k,n) = \psi_{3m}(k,n) + \psi_{3m+1}(k-1,n-3m-1) + \psi_{3m-2}(k,n-3m-1). \tag{2.20}$$

Adding (2.15) and (2.18), and using (2.13), we obtain

$$\psi_{3m+2}(k,n) = \psi_{3m+1}(k,n) + \psi_{3m+2}(k-1,n-3m-2) + \psi_{3m-1}(k,n-3m-2). \tag{2.21}$$

Adding (2.16) and (2.19), and using (2.13), we obtain

$$\psi_{3m+3}(k,n) = \psi_{3m+2}(k,n) + \psi_{3m+2}(k-1,n-3m-3) + \psi_{3m-1}(k,n-3m-3). \tag{2.22}$$

We define, for $m \ge 1$, $|q|<1$, $|d|<1$,

$$a_m(q,d) = 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \psi_m(k,n) q^n d^k,$$

and we set $a_0(q,d) = a_{-1}(q,d) = a_{-2}(q,d) = 1$ and $a_{-m}(q,d) = 0$ for $m \ge 3$.

As $m \to \infty$, $a_m(q,d) \to a(q,d)$ where $a(q,d)$ is the generating function for overpartitions counted by $B(k,n)$.

By (2.20), (2.21) and (2.22), we obtain:

$$(1-dq^{3m+1})a_{3m+1}(q,d)=a_{3m}(q,d)+q^{3m+1}a_{3m-2}(q,d), \tag{2.23}$$

$$(1-dq^{3m+2})a_{3m+2}(q,d)=a_{3m+1}(q,d)+q^{3m+2}a_{3m-1}(q,d), \tag{2.24}$$

$$a_{3m+3}(q,d)=(1+dq^{3m+3})a_{3m+2}(q,d)+q^{3m+3}a_{3m-1}(q,d). \tag{2.25}$$

Substituting (2.24) and (2.25) into (2.23), we obtain:

$$\begin{aligned}&(1-dq^{3m+1})(1-dq^{3m+2})a_{3m+2}(q,d)=\\&(1+q^{3m+1}+q^{3m+2}+dq^{3m}-dq^{6m}-dq^{6m+3})a_{3m-1}(q,d)\\&+q^{3m}(1-q^{3m})a_{3m-4}(q,d).\end{aligned} \tag{2.26}$$

Let $\alpha_m(q,d)=a_{3m+2}(q,d)$.
Then, $\alpha_{-1}(q,d)=1$, $\alpha_{-2}(q,d)=0$ and by (2.26) we have:

$$\begin{aligned}&(1-dq^{3m+1})(1-dq^{3m+2})\alpha_m(q,d)=\\&(1+q^{3m+1}+q^{3m+2}+dq^{3m}-dq^{6m}-dq^{6m+3})\alpha_{m-1}(q,d)\\&+q^{3m}(1-q^{3m})\alpha_{m-2}(q,d).\end{aligned}$$

The following lemma for $N=3$ will complete the proof. We prove it for general N because we use the case $N=7$ in the proof of Theorem 1.3.

Lemma 2.4. *Let N be a positive integer and $(\alpha_n(q,d))_{n\in\mathbb{N}}$ be a sequence such that $\alpha_0(q,d)=1$, $\alpha_1(q,d)=\frac{(1+q^{N-1}+q^{N-2}-dq^{2N-3})}{(1-dq^{N-1})(1-dq^{N-2})}$ and:*

$$\begin{aligned}&(1-dq^{Nm-1})(1-dq^{Nm-2})\alpha_m(q,d)=\\&(1+q^{Nm-1}+q^{Nm-2}+dq^{Nm-3}-dq^{N(2m-1)-3}-dq^{2Nm-3})\alpha_{m-1}(q,d)\\&+q^{Nm-3}(1-q^{N(m-1)})\alpha_{m-2}(q,d).\end{aligned} \tag{2.27}$$

Then

$$\alpha_\infty(q,d)=\lim_{m\to\infty}\alpha_m(q,d)=\prod_{k=1}^{\infty}\frac{(1+q^{Nk-1})(1+q^{Nk-2})}{(1-dq^{Nk-1})(1-dq^{Nk-2})}.$$

Proof: Let

$$\beta_m(q,d)=\alpha_m(q,d)\prod_{k=1}^{m}\frac{(1-dq^{Nk-1})}{(1-q^{Nk})}.$$

Then by (2.27), we obtain

$$\begin{aligned}&(1-dq^{Nm-2})(1-q^{Nm})\beta_m(q,d)=\\&(1+q^{Nm-1}+q^{Nm-2}+dq^{Nm-3}-dq^{N(2m-1)-3}-dq^{2Nm-3})\beta_{m-1}(q,d)\\&+q^{Nm-3}(1-dq^{N(m-1)-1})\beta_{m-2}(q,d).\end{aligned} \tag{2.28}$$

For $|x|<1$, let

$$f(x)=\sum_{m=0}^{\infty}\beta_m(q,d)x^m.$$

From (2.28) we deduce

$$\begin{aligned}(1-x)f(x)=&(dq^{-2}+1+xq^{N-2})(1+xq^{N-1})f(xq^N)\\&-dq^{-2}(1+xq^{N-1})(1+xq^{2N-1})f(xq^{2N}).\end{aligned}$$

Let

$$f(x) = F(x) \prod_{k=1}^{\infty} (1 + xq^{Nk-1}).$$

Thus

$$(1-x)F(x) = (dq^{-2} + 1 + xq^{N-2})F(xq^N) - dq^{-2}F(xq^{2N}).$$

Let

$$F(x) = \sum_{n=0}^{\infty} s_n x^n.$$

Then $s_0 = F(0) = 1$ and

$$s_n = \frac{(1 + q^{Nn-2})}{(1 - dq^{Nn-2})(1 - q^{Nn})} s_{n-1}.$$

So

$$s_n = s_0 \prod_{k=1}^{n} \frac{(1 + q^{Nk-2})}{(1 - dq^{Nk-2})(1 - q^{Nk})} = \prod_{k=1}^{n} \frac{(1 + q^{Nk-2})}{(1 - dq^{Nk-2})(1 - q^{Nk})}.$$

We have:

$$f(x) = \sum_{m=0}^{\infty} \beta_m(q,d)x^m = \prod_{k=1}^{\infty} (1 + xq^{Nk-1}) \sum_{n=0}^{\infty} s_n x^n.$$

Thus

$$\lim_{x \to 1^-} (1-x) \sum_{m=0}^{\infty} \beta_m(q,d)x^m = \prod_{k=1}^{\infty} (1 + q^{Nk-1}) \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} s_n x^n.$$

Using Appell's Comparison Theorem we deduce that

$$\beta_\infty(q,d) = \prod_{k=1}^{\infty} (1 + q^{Nk-1}) s_\infty = \prod_{k=1}^{\infty} \frac{(1 + q^{Nk-1})(1 + q^{Nk-2})}{(1 - dq^{Nk-2})(1 - q^{Nk})}.$$

Finally

$$\alpha_m(q,d) = \prod_{k=1}^{m} \frac{(1 - q^{Nk})}{(1 - dq^{Nk-1})} \beta_m(q,d).$$

So

$$\alpha_\infty(q,d) = \prod_{k=1}^{\infty} \frac{(1 + q^{Nk-1})(1 + q^{Nk-2})}{(1 - dq^{Nk-1})(1 - dq^{Nk-2})},$$

which completes the proof of Lemma 2.4. □ □

We apply Lemma 2.4 for $N = 3$ to $(\alpha_{m-1})_{m \in \mathbb{N}}$ and we obtain:

$$\alpha_\infty(q,d) = \prod_{k=1}^{\infty} \frac{(1 + q^{3k-1})(1 + q^{3k-2})}{(1 - dq^{3k-1})(1 - dq^{3k-2})} = \prod_{k=0}^{\infty} \frac{(1 + q^{3k+1})(1 + q^{3k+2})}{(1 - dq^{3k+1})(1 - dq^{3k+2})}$$

which is the generating function for overpartitions with parts congruent to 1 or 2 modulo 3. It completes this second proof of Theorem 1.1.

2.3 Proof with the largest part and parts counted twice

The beginning of this proof follows the same principle as the previous proof, except that parts congruent to 0 modulo 3 are counted twice. This actually gives a refinement of Theorem 1.1.

Let $\pi_m(M,k,n)$ denote the number of overpartitions counted by $B(k,n)$ having M parts, where parts divisible by 3 are counted twice, such that the largest part is $\leq m$ and overlined. Let $\phi_m(M,k,n)$ denote the number of overpartitions counted by $B(k,n)$ having M parts, where parts divisible by 3 are counted twice, such that the largest part is $\leq m$ and non-overlined.

Notice that for every $M,m,n,k \geq 1$,

$$\pi_m(M,k-1,n) = \phi_m(M,k,n) \tag{2.29}$$

because we can either overline the largest part or not.

Lemma 2.5.

$$\begin{aligned}\pi_{3m+1}(M,k,n) = \pi_{3m}(M,k,n) + \phi_{3m+1}(M-1,k,n-3m-1)\\ + \pi_{3m-2}(M-1,k,n-3m-1),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\pi_{3m+2}(M,k,n) = \pi_{3m+1}(M,k,n) + \phi_{3m+2}(M-1,k,n-3m-2)\\ + \pi_{3m-1}(M-1,k,n-3m-2),\end{aligned} \tag{2.31}$$

$$\begin{aligned}\pi_{3m+3}(M,k,n) = \pi_{3m+2}(M,k,n) + \phi_{3m+2}(M-2,k,n-3m-3)\\ + \pi_{3m-1}(M-2,k,n-3m-3).\end{aligned} \tag{2.32}$$

$$\begin{aligned}\phi_{3m+1}(M,k,n) = \phi_{3m}(M,k,n) + \phi_{3m+1}(M-1,k-1,n-3m-1)\\ + \pi_{3m-2}(M-1,k-1,n-3m-1),\end{aligned} \tag{2.33}$$

$$\begin{aligned}\phi_{3m+2}(M,k,n) = \phi_{3m+1}(M,k,n) + \phi_{3m+2}(M-1,k-1,n-3m-2)\\ + \pi_{3m-1}(M-1,k-1,n-3m-2),\end{aligned} \tag{2.34}$$

$$\begin{aligned}\phi_{3m+3}(M,k,n) = \phi_{3m+2}(M,k,n) + \phi_{3m+2}(M-2,k-1,n-3m-3)\\ + \pi_{3m-1}(M-2,k-1,n-3m-3).\end{aligned} \tag{2.35}$$

Proof: We give a proof of (2.30). The other equations can be proved in the same way. We break the set of overpartitions enumerated by $\pi_{3m+1}(M,k,n)$ into two sets, those with largest part less than $3m+1$ and those with largest part equal to $3m+1$. The first one is enumerated by $\pi_{3m}(M,k,n)$. The second is enumerated by $\phi_{3m+1}(M-1,k,n-3m-1) + \pi_{3m-2}(M-1,k,n-3m-1)$. To see this, we remove the largest part, so the number partitioned becomes $n-3m-1$. The largest part was overlined so the number of remaining non-overlined parts is still k and the number of parts is now $M-1$. If the second part is overlined, it has to be $\leq 3m-2$ and we obtain an overpartition counted by $\pi_{3m-2}(M-1,k,n-3m-1)$. If it is not overlined, it has to be $\leq 3m+1$ and we obtain an overpartition counted by $\phi_{3m+1}(M-1,k,n-3m-1)$. □ □

For all M,m,n,k, let $\psi_m(M,k,n) = \pi_m(M,k,n) + \phi_m(M,k,n)$.

Adding (2.30) and (2.33), and using (2.29), we obtain

$$\begin{aligned}\psi_{3m+1}(M,k,n) = \psi_{3m}(M,k,n) + \psi_{3m+1}(M-1,k-1,n-3m-1)\\ + \psi_{3m-2}(M-1,k,n-3m-1).\end{aligned} \tag{2.36}$$

Adding (2.31) and (2.34), and using (2.29), we obtain

$$\begin{aligned}\psi_{3m+2}(M,k,n) = \psi_{3m+1}(M,k,n) + \psi_{3m+2}(M-1,k-1,n-3m-2)\\ + \psi_{3m-1}(M-1,k,n-3m-2).\end{aligned} \tag{2.37}$$

Adding (2.32) and (2.35), and using (2.29), we obtain

$$\begin{aligned}\psi_{3m+3}(M,k,n) = \psi_{3m+2}(M,k,n) + \psi_{3m+2}(M-2,k-1,n-3m-3)\\ + \psi_{3m-1}(M-2,k,n-3m-3).\end{aligned} \tag{2.38}$$

We define, for $|q| < 1$, $|d| < 1$, $|t| < 1$,

$$a_m(q,d,t) = 1 + \sum_{M=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \psi_m(M,n,k) q^n d^k t^M.$$

As $m \to \infty$, $a_m(q,d,t) \to a(q,d,t)$ where $a(q,d,t)$ is the generating function for overpartitions counted by $B(k,n)$ having M parts where parts divisible by 3 are counted twice.

By (2.36), (2.37) and (2.38), we obtain

$$(1-dtq^{3m+1})a_{3m+1}(q,d,t) = a_{3m}(q,d,t) + tq^{3m+1}a_{3m-2}(q,d,t), \tag{2.39}$$

$$(1-dtq^{3m+2})a_{3m+2}(q,d,t) = a_{3m+1}(q,d,t) + tq^{3m+2}a_{3m-1}(q,d,t), \tag{2.40}$$

$$a_{3m+3}(q,d,t) = (1+dt^2q^{3m+3})a_{3m+2}(q,d,t) + t^2q^{3m+3}a_{3m-1}(q,d,t). \tag{2.41}$$

By (2.39), (2.40) and (2.41) we obtain:

$$\begin{aligned}&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m+2}(q,d,t) =\\ &(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m-1}(q,d,t)\\ &+t^2q^{3m}(1-q^{3m})a_{3m-4}(q,d,t).\end{aligned} \tag{2.42}$$

Replacing m by $m-1$ and t by tq^3 in (2.42), we obtain:

$$\begin{aligned}&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m-1}(q,d,tq^3) =\\ &(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m-4}(q,d,tq^3)\\ &+t^2q^{3m+3}(1-q^{3m-3})a_{3m-7}(q,d,tq^3).\end{aligned} \tag{2.43}$$

We want to prove that $a_{3m+3}(q,d,t)$ satisfies the same equation (2.43) as $a_{3m-1}(q,d,tq^3)$. Using (2.41) we have

$$\begin{aligned}&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m+3}(q,d,t)\\ &-(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m}(q,d,t)\\ &-t^2q^{3m+3}(1-q^{3m-3})a_{3m-3}(q,d,t)\\ =\;&(1-dtq^{3m+1})(1-dtq^{3m+2})\times\\ &[(1+dt^2q^{3m+3})a_{3m+2}(q,d,t)+t^2q^{3m+3}a_{3m-1}(q,d,t)]\\ &-(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})\times\\ &[(1+dt^2q^{3m})a_{3m-1}(q,d,t)+t^2q^{3m}a_{3m-4}(q,d,t)]\\ &-t^2q^{3m+3}(1-q^{3m-3})[(1+dt^2q^{3m-3})a_{3m-4}(q,d,t)+t^2q^{3m-3}a_{3m-7}(q,d,t)].\end{aligned} \tag{2.44}$$

Substituting (2.42) into (2.44) we obtain after simplification:

$$\begin{aligned}&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m+3}(q,d,t)\\ &-(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m}(q,d,t)\\ &-t^2q^{3m+3}(1-q^{3m-3})a_{3m-3}(q,d,t)\\ =\;&t^2q^{3m+3}[(1-dtq^{3m-1})(1-dtq^{3m-2})a_{3m-1}(q,d,t)\\ &-(1+tq^{3m-1}+tq^{3m-2}+dt^2q^{3m-3}-dt^2q^{6m-3}-dt^2q^{6m-6})a_{3m-4}(q,d,t)\\ &-t^2q^{3m-3}(1-q^{3m-3})a_{3m-7}(q,d,t)]\\ =\;&0,\end{aligned}$$

by (2.42) in which we have replaced m by $m-1$.
So $a_{3m+3}(q,d,t)$ satisfies the same recurrence equation as $a_{3m-1}(q,d,tq^3)$.
Furthermore, by (2.39), (2.40) and (2.41), we obtain:

$$a_3(q,d,t) = \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} = \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} a_{-1}(q,d,tq^3), \tag{2.45}$$

and

$$\begin{aligned} a_6(q,d,t) &= \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} \frac{(1+tq^4+tq^5-dt^2q^9)}{(1-dtq^4)(1-dtq^5)} \\ &= \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} a_2(q,d,tq^3). \end{aligned} \tag{2.46}$$

Using the recurrence equation satisfied by $a_{3m+3}(q,d,t)$ and $a_{3m-1}(q,d,tq^3)$ and (2.45) and (2.46), by mathematical induction, we have for all $m \geq 0$:

$$a_{3m+3}(q,d,t) = \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} a_{3m-1}(q,d,tq^3).$$

So, if we let $m \to \infty$, we obtain:

$$\lim_{m\to\infty} a_m(q,d,t) = \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} \lim_{m\to\infty} a_m(q,d,tq^3). \tag{2.47}$$

Iteration of (2.47) shows that:

$$\lim_{m\to\infty} a_m(q,d,t) = \prod_{n=0}^{\infty} \frac{(1+tq^{3n+1})(1+tq^{3n+2})}{(1-dtq^{3n+1})(1-dtq^{3n+2})}. \tag{2.48}$$

This completes the proof.

3 Two new theorems on overpartitions with difference conditions

3.1 Proof of Theorem 1.2

Let $d_i(k,m,n)$ denote the number of overpartitions counted by $D(k,n)$ having m parts such that the smallest part is bigger than i. We have the following equations:

Lemma 3.1.

$$d_0(k,m,n) - d_1(k,m,n) = d_0(k,m-1,n-7m+6) + d_0(k-1,m-1,n-1), \tag{3.1}$$

$$d_1(k,m,n) - d_2(k,m,n) = d_1(k,m-1,n-7m+5) + d_1(k-1,m-1,n-2), \tag{3.2}$$

$$d_2(k,m,n) - d_3(k,m,n) = d_0(k,m-1,n-14m+11) + d_0(k-1,m-1,n-7m+4), \tag{3.3}$$

$$d_3(k,m,n) - d_4(k,m,n) = d_3(k,m-1,n-7m+3) + d_3(k-1,m-1,n-4), \tag{3.4}$$

$$d_4(k,m,n) - d_5(k,m,n) = d_0(k,m-1,n-14m+9) + d_0(k-1,m-1,n-7m+2), \tag{3.5}$$

$$d_5(k,m,n) - d_6(k,m,n) = d_1(k,m-1,n-14m+8) + d_1(k-1,m-1,n-7m+1), \tag{3.6}$$

$$d_6(k,m,n) - d_7(k,m,n) = d_0(k,m-1,n-21m+14) + d_0(k-1,m-1,n-14m+7), \tag{3.7}$$

$$d_7(k,m,n) = d_0(k,m,n-7m). \tag{3.8}$$

Proof: We prove (3.3). The other equations are proved in the same way. Now $d_2(k,m,n) - d_3(k,m,n)$ denotes the overpartitions with the smallest part λ_s equal to 3. If λ_s is overlined, then $\lambda_{s-1} \geq 15$. In that case we remove 14 from each part and we obtain $d_0(k, m-1, n-14m+11)$. If λ_s is not overlined, then $\lambda_{s-1} \geq 8$. In that case we remove 7 from each part and we obtain $d_0(k-1, m-1, n-7m+4)$. □ □

For $|x| < 1$, $|d| < 1$ $|q| < 1$, we define

$$f_i(x, d, q) = f_i(x) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} d_i(k, m, n) x^m d^k q^n. \tag{3.9}$$

We want to find $f_0(1)$, which is the generating function for overpartitions counted by $D(k, n)$.
By (3.1) - (3.8), we have:

$$f_0(x) - f_1(x) = xqf_0(xq^7) + dxqf_0(x), \tag{3.10}$$
$$f_1(x) - f_2(x) = xq^2 f_1(xq^7) + dxq^2 f_1(x), \tag{3.11}$$
$$f_2(x) - f_3(x) = xq^3 f_0(xq^{14}) + dxq^3 f_0(xq^7), \tag{3.12}$$
$$f_3(x) - f_4(x) = xq^4 f_3(xq^7) + dxq^4 f_3(x), \tag{3.13}$$
$$f_4(x) - f_5(x) = xq^5 f_0(xq^{14}) + dxq^5 f_0(xq^7), \tag{3.14}$$
$$f_5(x) - f_6(x) = xq^6 f_1(xq^{14}) + dxq^6 f_1(xq^7), \tag{3.15}$$
$$f_6(x) - f_7(x) = xq^7 f_0(xq^{21}) + dxq^7 f_0(xq^{14}), \tag{3.16}$$
$$f_7(x) = f_0(xq^7). \tag{3.17}$$

By (3.10), we know that

$$f_1(x) = (1 - dxq)f_0(x) - xqf_0(xq^7). \tag{3.18}$$

By (3.10), (3.11) and (3.12), we have

$$\begin{aligned} f_3(x) =&(1 - dxq)(1 - dxq^2)f_0(x) \\ &+ (-xq - xq^2 - dxq^3 + dx^2q^3 + dx^2q^{10})f_0(xq^7) \\ &+ xq^3(-1 + xq^7)f_0(xq^{14}). \end{aligned} \tag{3.19}$$

Summing equations (3.10) - (3.16), and using (3.18), (3.19) and (3.17) to replace respectively $f_1(x)$, $f_3(x)$ and $f_7(x)$ by expressions using only f_0, we obtain

$$\begin{aligned} &(1 - dxq)(1 - dxq^2)(1 - dxq^4)f_0(x) = \\ &[1 + xq + xq^2 + xq^4 + dxq^3 + dxq^5 + dxq^6 - dx^2q^3 - dx^2q^5 - dx^2q^6 - dx^2q^{10} \\ &- dx^2q^{12} - dx^2q^{13} - d^2x^2q^7 - d^2x^2q^{14} + d^2x^3q^7 + d^2x^3q^{14} + d^2x^3q^{21}]f_0(xq^7) \\ &+ (1 - xq^7)[xq^3 + xq^5 + xq^6 + dxq^7 - dx^2q^7 - dx^2q^{14} - dx^2q^{21}]f_0(xq^{14}) \\ &+ (1 - xq^7)(1 - xq^{14})xq^7 f_0(xq^{21}). \end{aligned} \tag{3.20}$$

Let

$$F(x) = f_0(x) \prod_{n=0}^{\infty} \frac{(1 - dxq^{7n+4})}{(1 - xq^{7n})}.$$

Then by (3.20) we obtain

$$\begin{aligned}
&(1-dxq)(1-dxq^2)(1-x)F(x) = \\
&[1+xq+xq^2+xq^4+dxq^3+dxq^5+dxq^6-dx^2q^3-dx^2q^5-dx^2q^6-dx^2q^{10} \\
&-dx^2q^{12}-dx^2q^{13}-d^2x^2q^7-d^2x^2q^{14}+d^2x^3q^7+d^2x^3q^{14}+d^2x^3q^{21}]F(xq^7) \\
&+(1-dxq^{11})[xq^3+xq^5+xq^6+dxq^7-dx^2q^7-dx^2q^{14}-dx^2q^{21}]F(xq^{14}) \\
&+(1-dxq^{11})(1-dxq^{18})xq^7F(xq^{21}).
\end{aligned}$$

Let $F(x) = \sum_{n=0}^{\infty} A_n x^n$. Then $A_0 = F(0) = f_0(0) = 1$ by (3.9) and

$$\begin{aligned}
&(1-q^{7n})A_n = \\
&(1+q^{7n-3})[1+dq+dq^2+q^{7n-6}+q^{7n-5}+dq^{7n-4}+q^{14n-11}]A_{n-1} \\
&-(1+q^{7n-3})(1+q^{7n-10})[dq+dq^2+d^2q^3+dq^{7n-11}+dq^{7n-4}]A_{n-2} \\
&+(1+q^{7n-3})(1+q^{7n-10})(1+q^{7n-17})d^2q^3A_{n-3}.
\end{aligned}$$

Let $A_n = a_n \prod_{k=0}^{n-1}(1+q^{7k+4})$. Then $a_0 = A_0 = 1$ and

$$\begin{aligned}
(1-q^{7n})a_n =& [1+dq+dq^2+q^{7n-6}+q^{7n-5}+dq^{7n-4}+q^{14n-11}]a_{n-1} \\
&-[dq+dq^2+d^2q^3+dq^{7n-11}+dq^{7n-4}]a_{n-2}+d^2q^3a_{n-3}.
\end{aligned}$$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We obtain that $f(0) = 1$ and

$$\begin{aligned}
&(1-x)(1-dxq)(1-dxq^2)f(x) \\
&=(1+xq+xq^2+dxq^3-dx^2q^3-dx^2q^{10})f(xq^7)+xq^3f(xq^{14}).
\end{aligned}$$

Let $G(x) = f(x)\prod_{n=0}^{\infty}(1-xq^{7n})$. Thus $G(0) = 1$ and

$$\begin{aligned}
(1-dxq)(1-dxq^2)G(x) =& (1+xq+xq^2+dxq^3-dx^2q^3-dx^2q^{10})G(xq^7) \\
&+xq^3(1-xq^7)G(xq^{14}).
\end{aligned} \tag{3.21}$$

To solve (3.21), we apply Lemma 2.2 with $N = 7$. We obtain that

$$G(1) = \prod_{n=0}^{\infty} \frac{(1+q^{7n+1})(1+q^{7n+2})}{(1-dq^{7n+1})(1-dq^{7n+2})}.$$

So we have

$$(1-x)f(x) = (1-x)\sum_{n=0}^{\infty} a_n x^n = G(x)\prod_{n=1}^{\infty}\frac{1}{(1-xq^{7n})}.$$

By Appell's Comparison Theorem we obtain

$$a_\infty = G(1)\prod_{n=0}^{\infty}\frac{1}{(1-q^{7n+7})}.$$

So

$$A_\infty = a_\infty \prod_{k=0}^{\infty}(1+q^{7k+4}) = \prod_{n=0}^{\infty}\frac{(1+q^{7n+1})(1+q^{7n+2})(1+q^{7n+4})}{(1-q^{7n+7})(1-dq^{7n+1})(1-dq^{7n+2})}.$$

We have

$$f_0(x) = \prod_{n=0}^{\infty}\frac{(1-xq^{7n+7})}{(1-dxq^{7n+4})}(1-x)F(x) = \prod_{n=0}^{\infty}\frac{(1-xq^{7n+7})}{(1-dxq^{7n+4})}(1-x)\sum_{n=0}^{\infty}A_n x^n.$$

We apply Appell's Comparison Theorem again and we obtain

$$\begin{aligned} f_0(1) &= \prod_{n=0}^{\infty} \frac{(1-q^{7n+7})}{(1-dq^{7n+4})} A_\infty \\ &= \prod_{n=0}^{\infty} \frac{(1+q^{7n+1})(1+q^{7n+2})(1+q^{7n+4})}{(1-dq^{7n+1})(1-dq^{7n+2})(1-dq^{7n+4})}. \end{aligned}$$

This is the generating function for overpartitions with parts congruent to 1, 2 or 4 modulo 7, which completes the proof of Theorem 1.2.

3.2 Proof of Theorem 1.3

Let $\pi_m(k,n)$ denote the number of overpartitions counted by $V(k,n)$ such that the largest part is $\leq m$ and overlined. Let $\phi_m(k,n)$ denote the number of overpartitions counted by $V(k,n)$ such that the largest part is $\leq m$ and non-overlined.

Lemma 3.2.

$$\pi_{7m+1}(k,n) = \pi_{7m}(k,n) + \phi_{7m-2}(k,n-7m-1) + \pi_{7m-9}(k,n-7m-1), \tag{3.22}$$

$$\pi_{7m+2}(k,n) = \pi_{7m+1}(k,n) + \phi_{7m-1}(k,n-7m-2) + \pi_{7m-8}(k,n-7m-2), \tag{3.23}$$

$$\pi_{7m+3}(k,n) = \pi_{7m+2}(k,n) + \phi_{7m+3}(k,n-7m-3) + \pi_{7m-4}(k,n-7m-3), \tag{3.24}$$

$$\pi_{7m+4}(k,n) = \pi_{7m+3}(k,n) + \phi_{7m-1}(k,n-7m-4) + \pi_{7m-8}(k,n-7m-4), \tag{3.25}$$

$$\pi_{7m+5}(k,n) = \pi_{7m+4}(k,n) + \phi_{7m+5}(k,n-7m-5) + \pi_{7m-2}(k,n-7m-5), \tag{3.26}$$

$$\pi_{7m+6}(k,n) = \pi_{7m+5}(k,n) + \phi_{7m+6}(k,n-7m-6) + \pi_{7m-1}(k,n-7m-6), \tag{3.27}$$

$$\pi_{7m+7}(k,n) = \pi_{7m+6}(k,n) + \phi_{7m-1}(k,n-7m-7) + \pi_{7m-8}(k,n-7m-7). \tag{3.28}$$

$$\phi_{7m+1}(k,n) = \phi_{7m}(k,n) + \phi_{7m-2}(k-1,n-7m-1) + \pi_{7m-9}(k-1,n-7m-1), \tag{3.29}$$

$$\phi_{7m+2}(k,n) = \phi_{7m+1}(k,n) + \phi_{7m-1}(k-1,n-7m-2) + \pi_{7m-8}(k-1,n-7m-2), \tag{3.30}$$

$$\phi_{7m+3}(k,n) = \phi_{7m+2}(k,n) + \phi_{7m+3}(k-1,n-7m-3) + \pi_{7m-4}(k-1,n-7m-3), \tag{3.31}$$

$$\phi_{7m+4}(k,n) = \phi_{7m+3}(k,n) + \phi_{7m-1}(k-1,n-7m-4) + \pi_{7m-8}(k-1,n-7m-4), \tag{3.32}$$

$$\phi_{7m+5}(k,n) = \phi_{7m+4}(k,n) + \phi_{7m+5}(k-1,n-7m-5) + \pi_{7m-2}(k-1,n-7m-5), \tag{3.33}$$

$$\phi_{7m+6}(k,n) = \phi_{7m+5}(k,n) + \phi_{7m+6}(k-1,n-7m-6) + \pi_{7m-1}(k-1,n-7m-6), \tag{3.34}$$

$$\phi_{7m+7}(k,n) = \phi_{7m+6}(k,n) + \phi_{7m-1}(k-1,n-7m-7) + \pi_{7m-8}(k-1,n-7m-7). \tag{3.35}$$

Proof: We give a proof of (3.22). The other equations can be proved in the same way. We break the set of overpartitions enumerated by $\pi_{7m+1}(k,n)$ into two sets, those with largest part less than $7m+1$ and those with largest part equal to $7m+1$. The first one is enumerated by $\pi_{7m}(k,n)$. The second is enumerated by $\phi_{7m-2}(k,n-7m-1) + \pi_{7m-9}(k,n-7m-1)$. To see this, we remove the largest part, so the number partitioned becomes $n-7m-1$. The largest part was overlined so the number of remaining non-overlined parts is still k. If the second part is overlined, it has to be $\leq 7m-9$ and we obtain an overpartition counted by $\pi_{7m-9}(k,n-7m-1)$. If it is not overlined, it has to be $\leq 7m-2$ and we obtain an overpartition counted by $\phi_{7m-2}(k,n-7m-1)$. □ □

For all m, n, k, let $\psi_m(k,n) = \pi_m(k,n) + \phi_m(k,n)$. By Lemma 3.2,

$$\psi_{7m+1}(k,n) = \psi_{7m}(k,n) + \psi_{7m-2}(k-1, n-7m-1) + \psi_{7m-9}(k, n-7m-1), \tag{3.36}$$

$$\psi_{7m+2}(k,n) = \psi_{7m+1}(k,n) + \psi_{7m-1}(k-1, n-7m-2) + \psi_{7m-8}(k, n-7m-2), \tag{3.37}$$

$$\psi_{7m+3}(k,n) = \psi_{7m+2}(k,n) + \psi_{7m+3}(k-1, n-7m-3) + \psi_{7m-4}(k, n-7m-3), \tag{3.38}$$

$$\psi_{7m+4}(k,n) = \psi_{7m+3}(k,n) + \psi_{7m-1}(k-1, n-7m-4) + \psi_{7m-8}(k, n-7m-4), \tag{3.39}$$

$$\psi_{7m+5}(k,n) = \psi_{7m+4}(k,n) + \psi_{7m+5}(k-1, n-7m-5) + \psi_{7m-2}(k, n-7m-5), \tag{3.40}$$

$$\psi_{7m+6}(k,n) = \psi_{7m+5}(k,n) + \psi_{7m+6}(k-1, n-7m-6) + \psi_{7m-1}(k, n-7m-6), \tag{3.41}$$

$$\psi_{7m+7}(k,n) = \psi_{7m+6}(k,n) + \psi_{7m-1}(k-1, n-7m-7) + \psi_{7m-8}(k, n-7m-7). \tag{3.42}$$

We define, for $m \geq 1$, $|q| < 1$, $|d| < 1$,

$$a_m(q,d) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \psi_m(k,n) q^n d^k,$$

and we set $a_{-m}(q,d) = 1$ for $0 \leq m \leq 7$, $a_{-m}(q,d) = 0$ for $m > 7$, and

$$a_1(q,d) = a_2(q,d) = 1,$$

$$a_3(q,d) = a_4(q,d) = \frac{1+q^3}{1-dq^3},$$

$$a_5(q,d) = \frac{1+q^3+q^5-dq^8}{(1-dq^3)(1-dq^5)},$$

$$a_6(q,d) = a_7(q,d) = \frac{1+q^3+q^5+q^6-dq^8-dq^9-dq^{11}+d^2q^{14}}{(1-dq^3)(1-dq^5)(1-dq^6)}.$$

This definition is consistent with the condition that $\lambda_s \neq 1, \overline{1}, 2, \overline{2}, 4, \overline{4}, 7, \overline{7}$.

As $m \to \infty$, $a_m(q,d) \to a(q,d)$ where $a(q,d)$ is the generating function for overpartitions counted by $V(k,n)$.

By equations (3.36)-(3.42), we have:

$$a_{7m-6}(q,d) = a_{7m-7}(q,d) + dq^{7m-6}a_{7m-9}(q,d) + q^{7m-6}a_{7m-16}(q,d), \tag{3.43}$$

$$a_{7m-5}(q,d) = a_{7m-6}(q,d) + dq^{7m-5}a_{7m-8}(q,d) + q^{7m-5}a_{7m-15}(q,d), \tag{3.44}$$

$$a_{7m-4}(q,d) = a_{7m-5}(q,d) + dq^{7m-4}a_{7m-4}(q,d) + q^{7m-4}a_{7m-11}(q,d), \tag{3.45}$$

$$a_{7m-3}(q,d) = a_{7m-4}(q,d) + dq^{7m-3}a_{7m-8}(q,d) + q^{7m-3}a_{7m-15}(q,d), \tag{3.46}$$

$$a_{7m-2}(q,d) = a_{7m-3}(q,d) + dq^{7m-2}a_{7m-2}(q,d) + q^{7m-2}a_{7m-9}(q,d), \tag{3.47}$$

$$a_{7m-1}(q,d) = a_{7m-2}(q,d) + dq^{7m-1}a_{7m-1}(q,d) + q^{7m-1}a_{7m-8}(q,d), \tag{3.48}$$

$$a_{7m}(q,d) = a_{7m-1}(q,d) + dq^{7m}a_{7m-8}(q,d) + q^{7m}a_{7m-15}(q,d). \tag{3.49}$$

By (3.48),

$$(1-dq^{7m-1})a_{7m-1}(q,d) = a_{7m-2}(q,d) + q^{7m-1}a_{7m-8}(q,d). \tag{3.50}$$

Replacing m by $m-1$ in (3.48), multiplying this by $-q^5$ and adding it to (3.46), we get

$$a_{7m-3}(q,d) = a_{7m-4}(q,d) + q^5a_{7m-8}(q,d) - q^5a_{7m-9}(q,d). \tag{3.51}$$

Substituting (3.51) into (3.47) leads to

$$(1 - dq^{7m-2})a_{7m-2}(q,d) = a_{7m-4}(q,d) + q^5 a_{7m-8}(q,d) + (-q^5 + q^{7m-2})a_{7m-9}(q,d). \tag{3.52}$$

Let (A) denote the equation obtained by replacing m by $m-1$ in (3.49), and adding it together with (3.43) and (3.44). Let (B) denote the equation obtained by adding (3.46), (3.47) and (3.48) and replacing m by $m-1$. Then $(A) - q^3(B)$ yields

$$a_{7m-5}(q,d) = (1 + q^3)a_{7m-8}(q,d) - q^3 a_{7m-11}(q,d). \tag{3.53}$$

Substituting (3.53) into (3.45), we obtain

$$(1 - dq^{7m-4})a_{7m-4}(q,d) = (1 + q^3)a_{7m-8}(q,d) + (-q^3 + q^{7m-4})a_{7m-11}(q,d). \tag{3.54}$$

By (3.50), (3.52) and (3.54), we obtain:

$$\begin{aligned}
&(1 - dq^{7m-1})(1 - dq^{7m-2})(1 - dq^{7m-4})a_{7m-1}(q,d) = \\
&[1 + q^{7m-1} + q^{7m-2} + q^{7m-4} + dq^{7m-3} + dq^{7m-5} + dq^{7m-6} \\
&- dq^{14m-3} - dq^{14m-5} - dq^{14m-6} - dq^{14m-10} - dq^{14m-12} - dq^{14m-13} \\
&- d^2q^{14m-7} - d^2q^{14m-14} + d^2q^{21m-7} + d^2q^{21m-14} + d^2q^{21m-21}]a_{7m-8}(q,d) \\
&+ (1 - q^{7m-7})[q^{7m-3} + q^{7m-5} + q^{7m-6} + dq^{7m-7} \\
&- dq^{14m-7} - dq^{14m-14} - dq^{14m-21}]a_{7m-15}(q,d) \\
&+ (1 - q^{7m-7})(1 - q^{7m-14})q^{7m-7}a_{7m-22}(q,d).
\end{aligned}$$

Let $\alpha_m(q,d) = a_{7m-1}(q,d)$ and :

$$\beta_m(q,d) = \alpha_m(q,d)\prod_{k=0}^{m-1}\frac{(1 - dq^{7k+3})}{(1 - q^{7k+7})}.$$

Thus $\beta_0(q,d) = 1$, $\beta_{-1}(q,d) = \beta_{-2}(q,d) = 0$ and:

$$\begin{aligned}
&(1 - dq^{7m-1})(1 - dq^{7m-2})(1 - q^{7m})\beta_m(q,d) = \\
&[1 + q^{7m-1} + q^{7m-2} + q^{7m-4} + dq^{7m-3} + dq^{7m-5} + dq^{7m-6} \\
&- dq^{14m-3} - dq^{14m-5} - dq^{14m-6} - dq^{14m-10} - dq^{14m-12} - dq^{14m-13} \\
&- d^2q^{14m-7} - d^2q^{14m-14} + d^2q^{21m-7} + d^2q^{21m-14} + d^2q^{21m-21}]\beta_{m-1}(q,d) \\
&+ (1 - dq^{7m-11})[q^{7m-3} + q^{7m-5} + q^{7m-6} + dq^{7m-7} \\
&- dq^{14m-7} - dq^{14m-14} - dq^{14m-21}]\beta_{m-2}(q,d) \\
&+ (1 - dq^{7m-11})(1 - dq^{7m-18})q^{7m-7}\beta_{m-3}(q,d).
\end{aligned} \tag{3.55}$$

For $|x| < 1$, let

$$f(x) = \sum_{m=0}^{\infty}\beta_m(q,d)x^m.$$

From (3.55) we deduce

$$\begin{aligned}
&(1 - x)f(x) = \\
&(1 + xq^3)[dq^{-2} + dq^{-1} + 1 + xq^5 + xq^6 + dxq^4 + x^2q^{11}]f(xq^7) \\
&- (1 + xq^3)(1 + xq^{10})[dq^{-1} + dq^{-2} + d^2q^{-3} + dxq^4 + dxq^{11}]f(xq^{14}) \\
&+ (1 + xq^3)(1 + xq^{10})(1 + xq^{17})d^2q^{-3}f(xq^{21}).
\end{aligned}$$

Let

$$f(x)=F(x)\prod_{k=0}^{\infty}(1+xq^{7k+3}).$$

Thus from (3.55) we deduce

$$\begin{aligned}(1-x)F(x)=&[dq^{-2}+dq^{-1}+1+xq^{5}+xq^{6}+dxq^{4}+x^{2}q^{11}]F(xq^{7})\\&-[dq^{-1}+dq^{-2}+d^{2}q^{-3}+dxq^{4}+dxq^{11}]F(xq^{14})\\&+d^{2}q^{-3}F(xq^{21}).\end{aligned}$$

Let

$$F(x)=\sum_{n=0}^{\infty}s_{n}x^{n}.$$

Then $s_0=F(0)=1$ and

$$\begin{aligned}&(1-dq^{7n-1})(1-dq^{7n-2})(1-q^{7n})s_{n}=\\&(1+q^{7n-2}+q^{7n-1}+dq^{7n-3}-dq^{14n-10}-dq^{14n-3})s_{n-1}\\&+q^{7n-3}s_{n-2}.\end{aligned}$$

Let

$$\mu_{n}=s_{n}\prod_{k=0}^{n-1}(1-q^{7n+7}).$$

Thus

$$\begin{aligned}&(1-dq^{7n-1})(1-dq^{7n-2})\mu_{n}=\\&(1+q^{7n-2}+q^{7n-1}+dq^{7n-3}-dq^{14n-10}-dq^{14n-3})\mu_{n-1}\\&+q^{7n-3}(1-q^{7n-7})\mu_{n-2},\end{aligned}\tag{3.56}$$

and $\mu_0=s_0=1$, $\mu_1=\frac{(1+q^6+q^5-dq^{11})}{(1-dq^6)(1-dq^5)}$.

To solve (3.56), we apply Lemma 2.4 with $N=7$. We obtain that

$$\mu_{\infty}=\prod_{k=0}^{\infty}\frac{(1+q^{7k+5})(1+q^{7k+6})}{(1-dq^{7k+5})(1-dq^{7k+6})}.$$

So

$$s_{\infty}=\prod_{k=0}^{\infty}\frac{(1+q^{7k+5})(1+q^{7k+6})}{(1-dq^{7k+5})(1-dq^{7k+6})(1-q^{7k+7})}.$$

We have

$$\sum_{n=0}^{\infty}\beta_{n}(q,d)x^{n}=f(x)=F(x)\prod_{k=0}^{\infty}(1+xq^{7k+3})=\prod_{k=0}^{\infty}(1+xq^{7k+3})\sum_{n=0}^{\infty}s_{n}x^{n}.\tag{3.57}$$

We multiply both sides of (3.57) by $(1-x)$ and we apply Appell's Comparison Theorem. We obtain

$$\beta_{\infty}(q,d)=s_{\infty}\prod_{k=0}^{\infty}(1+q^{7k+3})=\prod_{k=0}^{\infty}\frac{(1+q^{7k+3})(1+q^{7k+5})(1+q^{7k+6})}{(1-dq^{7k+5})(1-dq^{7k+6})(1-q^{7k+7})}.$$

Thus

$$\begin{aligned}\alpha_{\infty}(q,d)&=\prod_{k=0}^{\infty}\frac{(1-q^{7k+7})}{(1-dq^{7k+3})}\beta_{m}(q,d)\\&=\prod_{k=0}^{\infty}\frac{(1+q^{7k+3})(1+q^{7k+5})(1+q^{7k+6})}{(1-dq^{7k+3})(1-dq^{7k+5})(1-dq^{7k+6})}.\end{aligned}$$

This is the generating function for overpartitions with parts congruent to 3, 5 or 6 modulo 7. This finishes the proof of Theorem 1.3.

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