WEIGHTED DEPENDENCY GRAPHS AND THE ISING MODEL

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ABSTRACT. We consider the Ising model on the d-dimensional square lattice at either very high temperature, very low temperature or in a strong magnetic field. In each of these three regimes, we give a simple proof that the joint cumulants of spins decay exponentially fast with respect to the tree-length of the set of spins.

With the recent terminology of weighted dependency graphs introduced by the second author, this means that we have a weighted dependency graph structure on the spins. We use this structure to reprove Newman's central limit theorem for the magnetization in a growing box, and to extend it to the number of occurrences of a given spin pattern.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **The Ising model.** The Ising model is a mathematical model of ferro-magnetism in statistical physics. It was introduced in 1920 by Wilhelm Lenz who gave it as a problem to his Ph.D. student Ernst Ising [13]. It can be defined on any finite graph, but we restrict ourselves to finite subsets Λ of $\mathbb{Z}^d$. For any lattice site $i \in \Lambda$, there is a random variable σ_i which is equal to either 1 or -1 and represents the *spin* at site i. The distribution of spins depends on the magnetic field h and the inverse temperature β in a way that will be detailed later (see Section 2.1). In particular, spins corresponding to neighbour sites i and j are more likely to be aligned (i.e. both equal to 1 or both equal to -1); the bigger β is (i.e. the lower the temperature is), the more important is this phenomenon.

In his Ph.D. thesis [13], Ising solved the model for the one-dimensional case $d = 1$, and showed that there is no phase transition. But in 1936, Peierls [25] showed that, in dimensions 2 and 3, when $h = 0$, the Ising model undergoes a phase transition at a critical inverse temperature β_c. He used a combinatorial argument now known as Peierls' argument. The two-dimensional model for $h = 0$ was then exactly solved by Onsager [24] in 1944, using analytic techniques and the transfer matrix method. It turns out that in higher dimensions, there is also a phase transition for $h = 0$ (see [25] or [8] for a more modern treatment) and no phase transition when there is a magnetic field $h \neq 0$ [16, 28].

The Ising model is *a priori* defined on a finite subset $\Lambda \subset \mathbb{Z}^d$, but it is well-known that we can take the *thermodynamic limit* $\Lambda \uparrow \mathbb{Z}^d$ (see eg. [8]). This defines, for each pair of parameters (β, h), a measure $\mu_{\beta,h}$ on the set $\{-1, 1\}^{\mathbb{Z}^d}$ of spin configurations on the whole d-dimensional lattice $\mathbb{Z}^d$. (In low temperature without magnetic field, i.e. β large and $h = 0$, this measure is not unique; we will consider the one corresponding to $+$ boundary conditions, see Section 2.1 for details.)

The Ising model has since been studied in thousands of research articles. Among many others, a subject of interest has been the estimation of the decay of covariances. Writing $\langle \cdot \rangle_{\beta,h}$ for the expectation with magnetic field h and inverse temperature β, the covariance of the spins σ_i and σ_j is defined as

$$\langle \sigma_i ; \sigma_j \rangle_{\beta,h} := \langle \sigma_i \sigma_j \rangle_{\beta,h} - \langle \sigma_i \rangle_{\beta,h} \langle \sigma_j \rangle_{\beta,h}.$$

In 1974, Duneau, Iagolnitzer and Souillard [4] showed the exponential decay of the covariance $\langle \sigma_i ; \sigma_j \rangle_{\beta,h}$ in the distance $\|i - j\|_1$ between the sites i and j in two different regimes.

Theorem 1.1 (Duneau-Iagolnitzer-Souillard, 1974)**.** *For $h \neq 0$ and all $\beta < \infty$ or for $h = 0$ and β small enough, there exists some positive constant c such that*

$$\langle \sigma_i; \sigma_j \rangle_{\beta,h} \leq e^{-c\|i-j\|_1},$$

for all $i, j \in \mathbb{Z}^d$.

It has later been showed by Aizenman, Barsky and Fernandez [1] that the theorem also holds when $h = 0$ and $\beta < \beta_c$ (with $+$ boundary condition). Alternate proofs using the random current representation for the non-zero magnetic field and the high temperature regimes have recently been given by Ioffe [12] and Duminil-Copin, Tassion [2] respectively. The same result is believed to be true for $\beta > \beta_c$, but to our knowledge it has only been proved for $d = 2$ [21] and β very large (see Schor [26] or Malyshev, Minlos [17]). However the decay is not exponential any more when β is equal to the critical inverse temperature β_c [20, 21].

The decay of *joint cumulants* (also called *truncated k-point functions* or *Ursell functions* in the physics literature) has also been a subject of interest. As the covariance, cumulants behave nicely with independence: namely, if a set of variables can be split in two mutually independent sets, the corresponding joint cumulant vanishes. Therefore a bound on joint cumulants may be seen as a quantification of the dependency in a set of random variables. Moreover, joint moments can be computed from joint cumulants (and vice-versa) so that they can be used to prove central limit theorems, as we shall see in the next section.

The joint cumulant of order r of spins $\sigma_{i_1}, \ldots, \sigma_{i_r}$ is defined as

$$\kappa_{\beta,h}(\sigma_{i_1}, \ldots, \sigma_{i_r}) = [t_1 \ldots t_r] \log \langle \exp(t_1 \sigma_{i_1} + \cdots + t_r \sigma_{i_r}) \rangle_{\beta,h}.$$

The covariance is the joint cumulant of order 2 (more detail in Section 2.2).

Bounds on cumulants in the physics literature are often called *cluster properties*. There is in fact a hierarchy of cluster properties (corresponding to sharper or weaker bounds on cumulants); we refer to [3] or [17, Chapter 6, §1] for definitions of various kinds of cluster properties.

In the case of the Ising model, a first bound on cumulants was obtained by Martin-Löf [18, Eq. (20)] — see also [15, Section 1] —: he proved that cumulants decrease exponentially in $\mathrm{diam}(A)/r$, where $\mathrm{diam}(A)$ is the the diameter of A and r the order of the cumulants. In [4], Duneau, Iagolnitzer and Souillard sharpened this bound in the case of a strong magnetic field h, or for $h = 0$ and very high temperature: the joint cumulant $\kappa_{\beta,h}(\sigma_i; i \in A)$ decays exponentially in $\ell_T(A)$, the minimum length of a tree connecting vertices of A (see Section 2.3 for a formal definition of the tree-length). In [17], Malyshev and Minlos have a similar result in the case $h = 0$ and very low temperature. Both their approaches use cluster expansion, a powerful tool introduced by Mayer and Montroll [19] which consists in viewing our model in terms of macroscopic geometrical objects instead of considering its original microscopic components. Both proofs use additional ingredients of different nature: Duneau, Iagolnitzer and Souillard use Lee-Yang circle theorem and complex analysis arguments, while the second relies on combinatorial developments and bounds on joint cumulants for *contours* as an intermediate step (contours will be defined in Section 3.2).

Here we give a simpler and more unified approach of the decays of joint cumulants in all of these regimes. Indeed we only use elementary reasoning on the cluster expansion in all three regimes, with additionally in the very low temperature case a lemma from [7] which gives an equivalence between bounds on certain quotients and bounds on joint cumulants. Our result is stated as follows.

Theorem 1.2. *We consider the Ising model on $\mathbb{Z}^d$ with parameters (β, h). There exist positive constants $\varepsilon(d) < 1$, $\beta_1(d)$, $\beta_2(d)$ and $h_1(d)$ depending on the dimension d with the following property. Assume we are in one of three following regimes:*

very low temperature: *$\beta > \beta_2(d)$ and $h = 0$;*
very high temperature: *$\beta < \beta_1(d)$ and $h = 0$;*
strong magnetic field: *$h > h_1(d)$.*

Then for any $r \geq 1$, there exists a constant D_r such that for all $A = \{i_1, \ldots, i_r\} \subset \mathbb{Z}^d$, we have

$$\left| \kappa_{\beta,h}(\sigma_{i_1}, \ldots, \sigma_{i_r}) \right| \leq D_r \varepsilon(d)^{\ell_T(A)}.$$

That the tree-length appears as exponent is important to make the connection with weighted dependency graphs, as we shall explain now.

1.2. Weighted dependency graphs.

The theory of weighted dependency graphs, introduced in [7], is a toolbox to prove central limit theorems. Here, a weighted graph is a graph such that to each edge e is associated a weight w_e in $[0, 1]$, where a weight 0 is the same as no edge. Informally, that a family of random variables $\{Y_a, a \in A\}$ admits a weighted graph G as weighted dependency graph means the following:

- G has vertex-set A, i.e. we have one vertex in G per variable in $\{Y_a, a \in A\}$;
- the smaller the weight of an edge $\{a, b\}$ is, the *closer to independent* Y_a and Y_b should be. In particular, an edge of weight 0, or equivalently no edge between a and b, means that Y_a and Y_b are independent.

Formally, this closeness to independence is not only measured by a bound on the covariance (as could be expected), but also involves bounds on higher order cumulants (see Definition 2.3 for more precision).

Suppose now that, for each n, the family of random variables $\{Y_{a,n}, a \in A_n\}$ has a weighted dependency graph G_n. Consider the renormalised sum $\widetilde{X_n} = \frac{1}{a_n} \left(\sum_{a \in A_n} Y_{a,n} \right)$. Under some easy-to-check conditions on the renormalising factor a_n, the variance of $\widetilde{X_n}$ and the maximal weighted degree of G_n, $\widetilde{X_n}$ tends in distribution towards a Gaussian law (see Theorem 2.4). In short, weighted dependency graphs yield central limit theorems.

A nice feature of weighted dependency graphs is the following stability property: a weighted dependency graph for a family $\{Y_a, a \in A\}$ automatically gives a weighted dependency graph for monomials $Y_I = \prod_{a \in I} Y_a$ in the Y_a's with a fixed bound on the degree (here, I is a multiset of elements of A). As a consequence, we can potentially prove central limit theorems for sums of such monomials. We refer the reader to [7] for a detailed presentation of the theory of weighted dependency graphs.

Let us come back to the Ising model. The bounds on joint cumulants of Theorem 1.2 can be naturally translated in terms of weighted dependency graphs for the random variables $\{\sigma_i : i \in \mathbb{Z}^d\}$.

Theorem 1.3. *Let $\{\sigma_i : i \in \mathbb{Z}^d\}$ be a spin configuration distributed according to $\mu_{\beta,h}$, where either $h > h_1(d)$ or $(h = 0; \beta < \beta_1(d))$ or $(h = 0; \beta > \beta_2(d))$. Let G be the complete weighted graph with vertex set $\mathbb{Z}^d$, such that every edge $e = (i, j)$ has weight $w_e = \varepsilon(d)^{\frac{\|i-j\|_1}{2}}$, where $\varepsilon(d)$ comes from Theorem 1.2.*

Then G is a $\mathbf{C}$-weighted dependency graph for the family $\{\sigma_i : i \in \mathbb{Z}^d\}$, for some sequence $\mathbf{C} = (C_r)_{r \geq 1}$.

Theorem 1.3 is proved in Section 4.1.1. The proof uses Theorem 1.2, some general results of [7] and elementary considerations. As explained above, this automatically yields a weighted dependency graph for products of a finite number of spins, which will be presented in Theorem 4.6 below.

1.3. Central limit theorems.

Central limit theorems play a key role in probability theory and have also been a subject of interest in the study of the Ising model. We define $\Lambda_n := [-n, n]^d$ the d-dimensional cube centred at 0 of side $2n$. Central limit theorems for the magnetization $S_n := \sum_{i \in \Lambda_n} \sigma_i$ and the energy $\sum_{i,j \in \Lambda} \sigma_i \sigma_j$, when $\Lambda \uparrow \mathbb{Z}^d$, have first been proved by Newman [22, 23], using the FKG inequalities (see also an earlier work of Martin-Löf [18], for the very low temperature regime). This is still a subject of interest: central limit theorems for the magnetization in the Ising model on random graphs have recently been given, either in the quenched setting [10] or in the annealed one [9].

Using the weighted dependency graph for spins, we can first reprove Newman's result for the magnetization (in the regimes where Theorem 1.2 and Theorem 1.3 hold; Newman's result holds in more generality).

Theorem 1.4. *Consider the Ising model on $\mathbb{Z}^d$, with inverse temperature β and magnetic field h, such that either $h > h_1(d)$ or $(h = 0;\ \beta < \beta_1(d))$ or $(h = 0;\ \beta > \beta_2(d))$. Then*

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{|\Lambda_n|}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, v^2).$$

Moreover $v^2 > 0$ so the Gaussian law is non-degenerate.

As said above, we not only have a weighted dependency graph for spins, but also for products of spins. This makes it easy and natural to also obtain a central limit theorem for the number of occurrences $S_{n,\mathcal{P}}$ of a fixed *pattern* $\mathcal{P}$ in Λ_n; see Section 4.2 for a formal definition. For example, for an appropriate $\mathcal{P}$, the quantity $S_{n,\mathcal{P}}$ is the number of isolated positive spins, i.e. spins $+1$ whose nearest neighbours are all -1's.

Theorem 1.5. *Consider the Ising model on $\mathbb{Z}^d$, with inverse temperature β and magnetic field h, such that either $h > h_1(d)$ or $(h = 0;\ \beta < \beta_1(d))$ or $(h = 0;\ \beta > \beta_2(d))$. Then*

$$\frac{S_{n,\mathcal{P}} - \mathbb{E}(S_{n,\mathcal{P}})}{\sqrt{|\Lambda_n|}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, v_{\mathcal{P}}^2).$$

It should also be possible to adapt our techniques to reprove Newman's result on the energy. On the other hand, Newman's techniques based on FKG inequalities may also be used to prove this central limit theorem for occurrences of patterns, though it might be quite technical.

We believe however that our approach is of independent interest for two reasons. First it fits in the general framework of weighted dependency graphs developed in [7]. Second it is a direct formalization of the following intuition: since covariances and cumulants decay exponentially in the distance, one may think as if a given spin would depend only on a finite number of its neighbours.

1.4. **Outline of the paper.** The remainder of the paper is organised as follows. In Section 2, we give some preliminary definitions and basic results about the Ising model, the theory of joint cumulants and weighted dependency graphs. In Section 3, we discuss the cluster expansion for the Ising model in the three different regimes we consider (high magnetic field, very high temperature, very low temperature) and deduce bounds on joint cumulants. In Section 4, we use the theory of weighted dependency graphs to prove our central limit theorems.

Note: all constants throughout the paper depend on the dimension of the space and we shall not make it explicit from now on.

2. Preliminaries

2.1. **The Ising model.** We consider the Ising model on a finite subset Λ of $\mathbb{Z}^d$. For any lattice site $i \in \Lambda$, we define a random variable σ_i which is equal to either 1 or -1 and represents the spin at site i. A *spin configuration* $\omega = (\sigma_i(\omega))_{i \in \Lambda}$ is an assignment of spins to every site of Λ. We use the notation of [8], that we define now.

Let $\mathcal{E}_\Lambda := \{\{i, j\} \subset \Lambda : \|i - j\|_1 = 1\}$ be the set of nearest neighbour pairs in Λ. (Here, and throughout the paper, $\|i - j\|_1$ denotes the graph distance in $\mathbb{Z}^d$ between two points i and j.) To each spin configuration ω, we associate its Hamiltonian

$$H_{\Lambda;\beta,h}(\omega) := -\beta \sum_{\{i,j\}\in\mathcal{E}_\Lambda} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i\in\Lambda} \sigma_i(\omega),$$

where $\beta \geq 0$ and h are two real parameters, respectively called *inverse temperature* and *magnetic field*.

The probability of a spin configuration ω is given by the Gibbs distribution

$$\mu_{\Lambda;\beta,h}(\omega) := \frac{1}{Z_{\Lambda;\beta,h}} e^{-H_{\Lambda;\beta,h}(\omega)},$$

where

$$Z_{\Lambda;\beta,h} := \sum_{\omega\in\{-1,1\}^\Lambda} e^{-H_{\Lambda;\beta,h}(\omega)}$$

is called the partition function.

The quantities defined so far are with "free boundary conditions", which means that the value of the spins outside of Λ is not taken into consideration. We can also define the same quantities with boundary condition, by considering the Ising model on the full lattice $\mathbb{Z}^d$, but where the values of the spins outside of Λ are fixed. Fixing a spin configuration $\eta \in \{-1, 1\}^{\mathbb{Z}^d}$, we define a spin configuration in Λ with boundary condition η as an element of the set

$$\Omega_\Lambda^\eta := \left\{ \omega \in \{-1, 1\}^{\mathbb{Z}^d} : \omega_i = \eta_i, \forall i \notin \Lambda \right\}.$$

We now define the Hamiltonian as

$$H_{\Lambda;\beta,h}^\eta(\omega) := -\beta \sum_{\{i,j\} \in \mathcal{E}_\Lambda^b} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in \Lambda} \sigma_i(\omega),$$

where $\mathcal{E}_\Lambda^b := \left\{ \{i, j\} \subset \mathbb{Z}^d : \|i - j\|_1 = 1 \text{ and } \{i, j\} \cap \Lambda \neq \emptyset \right\}.$

The Gibbs distribution of the Ising model in Λ with boundary condition η and parameters β and h is the probability distribution defined on Ω_Λ^η by

$$\mu_{\Lambda;\beta,h}^\eta(\omega) := \frac{1}{Z_{\Lambda;\beta,h}^\eta} e^{-H_{\Lambda;\beta,h}^\eta(\omega)},$$

where

$$Z_{\Lambda;\beta,h}^\eta := \sum_{\omega \in \Omega_\Lambda^\eta} e^{-H_{\Lambda;\beta,h}^\eta(\omega)}$$

is the partition function with boundary condition η.

The most classical boundary conditions are the $+$ boundary condition, where $\eta_i = +1$ for all $i \in \mathbb{Z}^d$, and the $-$ boundary condition, where $\eta_i = -1$ for all $i \in \mathbb{Z}^d$. When considering quantities with $+$ (resp. $-$) boundary condition, we write them with superscript $+$ (resp. $-$), e.g. $\mu_{\Lambda;\beta,h}^+(\omega)$.

We now take an increasing sequence Λ_n of finite subsets of $\mathbb{Z}^d$ with $\bigcup_{n \geq 1} \Lambda_n = \mathbb{Z}^d$. It is a well-known fact (see, *e.g.*, [8, Chapter 3]) that the sequence of measures $\mu_{\Lambda_n;\beta,h}^+$ converges in the weak sense towards a measure denoted $\mu_{\beta,h}^+$ as $n \to \infty$. In the high temperature case ($\beta < \beta_c(d)$, $h = 0$) or in the presence of a magnetic field ($h \neq 0$), the limiting measure is independent of the choice of boundary conditions. At low temperature ($\beta > \beta_c(d)$, $h = 0$), the limiting measure depends on the boundary conditions; in this article, we restrict ourselves to $+$ boundary conditions to have a well-defined limiting measure in all cases. Also, we drop the superscript $+$ and denote the limiting measure by $\mu_{\beta,h}$.

In this article, we work with this limiting measure $\mu_{\beta,h}$ and prove our central limit theorem under this measure. In comparison with the measure $\mu_{\Lambda_n;\beta,h}^+$, it has the advantage to be translation invariant, which simplifies in particular the variance estimates.

2.2. Joint cumulants.
As usual in the physics literature (and as done in the introduction), we use the notation $\langle f \rangle$ for the expectation of $f(X)$ (adding parameters of the measure as indices if necessary). For random variables $X_1, \ldots, X_r$ on the same probability space with finite moments, we define their *joint cumulant* (or *mixed cumulant*) as

$$(2.1) \qquad \kappa(X_1, \ldots, X_r) = [t_1 \ldots t_r] \log \left\langle \exp(t_1 X_1 + \cdots + t_r X_r) \right\rangle.$$

The notation $[t_1 \ldots t_r]F$ stands here for the coefficient of $t_1 \ldots t_r$ in the series expansion of F in positive powers of $t_1, \ldots, t_r$. The finite moments assumption ensures that this series expansion exists. If all random variables $X_1, \cdots, X_r$ are equal to the same variable X, we denote $\kappa_r(X) = \kappa(X, \ldots, X)$ and this is the usual *cumulant* of a single random variable.

Joint cumulants have a long history in statistics and theoretical physics, see *e.g.* [27]. In the case where the X_i's are indicator functions of the presence of particles (or $+$ spins for examples), they are often referred to in the statistical physics literature as *truncated correlation functions* or *Ursell functions*.

2.3. **Spanning trees of maximal weight and tree lengths.**

Definition 2.1. A spanning tree of a graph $L = (V, E)$ is a subset E' of E such that (V, E') is a tree.

If G is an edge-weighted graph, we define the weight $w(T)$ of a spanning tree of G as the *product* of the weights of the edges in T. The maximum weight of a spanning tree of G is denoted $\mathcal{M}(G)$.

We will be mainly interested in the case where V is a finite subset A of $\mathbb{Z}^d$, E consists of all pairs of vertices and the weights are of the form $w(i, j) = \varepsilon^{\|i-j\|_1}$ for some positive constant $\varepsilon < 1$. We denote this weighted graph by G_A. Then, for a spanning tree T of G_A

$$w(T) = \varepsilon^{\sum_{(i,j)\in T} \|i-j\|_1}$$

and the maximal such weight $\mathcal{M}(G_A)$ is obtained by *minimizing* the quantity $\sum_{(i,j)\in T} \|i - j\|_1$. Therefore we define

$$\ell'_T(A) = \min_T \sum_{(i,j)\in T} \|i - j\|_1,$$

where the minimum is taken over all spanning trees T of G_A, *i.e.* of the complete graph on A. Then we have $\mathcal{M}(G_A) = \varepsilon^{\ell'_T(A)}$.

The quantity $\ell'_T(A)$ is sometimes referred to as the *tree-length* of A. There is another closely related notion of tree-length defined as $\ell_T(A) = \min_B \ell'_T(A \cup B)$, where the minimum is taken over all finite subsets B of $\mathbb{Z}^d$. In other words, this is the minimum length of a tree connecting vertices of A and possibly other vertices of $\mathbb{Z}^d$. Equivalently, this is the minimal size of a connected set of edges of the lattice $\mathbb{Z}^d$ such that each vertex of A is incident to at least one edge in the set. These two notions of tree-length are illustrated on Fig. 1.

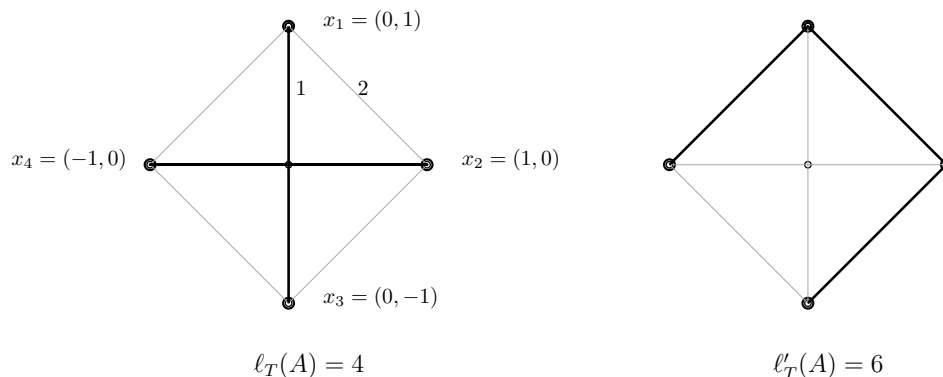$$\ell_T(A) = 4 \qquad\qquad\qquad \ell'_T(A) = 6$$

FIGURE 1. The two tree lengths on an example

In [3, page 197], Duneau, Iagolnitzer and Souillard proved the following bound, which will be useful later in our computations.

Proposition 2.2. *For all $A = \{x_1, \ldots, x_n\}$ finite subset of $\mathbb{Z}^d$, we have*

$$\ell_T(A) \le \ell'_T(A) \le 2\ell_T(A).$$

2.4. **Weighted dependency graphs.** Weighted dependency graphs have been introduced by the second author in [7]. The following is a simplified definition, sufficient for the purpose of this paper (it corresponds to the case $\Psi \equiv 1$ of the general definition, given in [7]).

Definition 2.3. Let $\{Y_\alpha, \alpha \in A\}$ be a family of random variables with finite moments, living in the same probability space; and let $\boldsymbol{C} = (C_1, C_2, \cdots)$ be a sequence of positive real numbers.

A weighted graph G is a $\boldsymbol{C}$-*weighted dependency graph* for $\{Y_\alpha, \alpha \in A\}$ if, for any multiset $B = \{\alpha_1, \ldots, \alpha_r\}$ of elements of A, one has

$$(2.2) \qquad\qquad \left| \kappa(Y_\alpha; \alpha \in B) \right| \le C_r \, \mathcal{M}(G[B]).$$

Here $G[B]$ denotes the graph induced by G on the vertex set B.

Weighted dependency graphs are a toolbox to prove central limit theorems. Here is a normality criterion, which is a slightly modified version of the main theorem in [7]

Theorem 2.4. *Suppose that, for each n, $\{Y_{n,i}, 1 \leq j \leq N_n\}$ is a family of random variables with finite moments defined on the same probability space. Let $\mathbf{C} = (C_r)_{r \geq 1}$ be a fixed sequence that does not depend on n.*

Assume that, for each n, one has a $\mathbf{C}$-weighted dependency graph G_n for $\{Y_{n,i}, 1 \leq j \leq N_n\}$ and denote $\Delta_n - 1$ its maximal weighted degree.

Let $X_n = \sum_{i=1}^{N_n} Y_{n,i}$ and $v_n^2 = Var(X_n)$. Assume that there exists a sequence (a_n), an integer $s \geq 3$ and a real number v such that

(1)
$$\frac{v_n^2}{a_n^2} \xrightarrow[n \to \infty]{} v^2,$$

(2)
$$\text{for all } n, \ a_n^2 \leq N_n \Delta_n,$$

(3)
$$\left(\frac{N_n}{\Delta_n}\right)^{\frac{1}{s}} \frac{\Delta_n}{a_n} \xrightarrow[n \to \infty]{} 0.$$

Then in distribution,
$$\frac{X_n - \mathbb{E}(X_n)}{a_n} \xrightarrow[n \to \infty]{d} \mathcal{N}(0, v^2).$$

Proof: The proof is almost identical to the proof of the normality criterion in [7, Section 4.3] replacing σ_n by a_n. Indeed, as noticed in [7, Section 4.3], in the special case $\Psi \equiv 1$ (to which we restrict ourselves in this article), the quantities R_n and Q_n defined there can be replaced respectively by N_n (the number of vertices) and Δ_n (the maximal weighted degree plus one). $\square$

3. Cluster expansions and bounds on joint cumulants

The cluster expansion is a powerful tool in statistical mechanics, which consists in studying a system in terms of macroscopic geometrical objects instead of considering its original microscopic components. It was introduced in a work of Mayer and Montroll [19] studying molecular distribution and has since been used in several other topics; for the Ising model, see for example [6] or Chapter 5 of [8]. In this section, we will use the cluster expansion in three different regimes of the Ising model to prove the bounds on joint cumulants of Theorem 1.2. This will later be useful to apply the theory of weighted dependency graphs. Theorem 1.2 is proved in Sections 3.1.2, 3.2.2 and 3.3, depending on the considered regime.

Remark 3.1. As already mentioned, cluster expansion is a key step to obtain bounds for cumulants in each of the three regimes. Therefore we will use each time classical notation for cluster expansion, such as Ξ, wt, ... Note however that these quantities may have different meanings in different regimes. Since they are only used for the proof of Theorem 1.2 and since the proofs in the different regimes are independent from each other, this should not create any difficulty.

3.1. **At very high temperature, without magnetic field.**

3.1.1. *The cluster expansion of the (multivariate) moment generating function.* Let us start with the regime where $h = 0$ and β is sufficiently small (very high temperature).

Fix a finite domain $\Lambda \subset \mathbb{Z}^d$ and let $A = \{x_1, \ldots, x_r\}$ be a set of points in Λ. We consider the (multivariate) moment generating function

$$\left\langle \exp\left(\sum_{j=1}^{r} t_j \sigma_{x_j}\right) \right\rangle_{\Lambda;\beta,0} = \frac{\sum_{\omega \in \Omega_\Lambda} \exp\left(\sum_{j=1}^{r} t_j \sigma_{x_j}(\omega)\right) e^{-H_{\Lambda;\beta,0}(\omega)}}{Z_{\Lambda;\beta,0}}.$$

Let us call $Z^A_{\Lambda;\beta,0}$ the numerator of the right-hand side. The denominator $Z_{\Lambda;\beta,0}$ is then exactly $Z^\emptyset_{\Lambda;\beta,0}$. Let $\mathcal{E}^{\mathrm{even}}_A$ (resp. $\mathcal{E}^{\mathrm{even}}_{\Lambda,A}$) be the set of pairs (E,B), where $E \subset \mathcal{E}_{\mathbb{Z}^d}$ (resp. $E \subset \mathcal{E}_\Lambda$) and $B \subseteq A$ are such that a vertex of Λ is incident to an odd number of edges in E if and only if it is in B. For such a pair (E,B), we denote $\mathrm{wt}(E,B) = (\tanh \beta)^{|E|} \prod_{j \in B}(\tanh t_j)$.

Lemma 3.2 (high temperature representation)**.** *We have*

$$(3.1) \qquad Z^A_{\Lambda;\beta,0} = 2^{|\Lambda|}(\cosh \beta)^{|\mathcal{E}_\Lambda|} \left(\prod_{j=1}^r \cosh(t_j) \right) \Xi^A_{\Lambda;\beta,0},$$

where $\Xi^A_{\Lambda;\beta,0} = \sum_{(E,B) \in \mathcal{E}^{even}_{\Lambda,A}} \mathrm{wt}(E,B)$.

Proof: This proof is a straight-forward extension of the case $A = \emptyset$, see e.g. [8, Eq. (5.40)]. We write in short σ_i for $\sigma_i(\omega)$. Since σ_{x_j} is in $\{-1,+1\}$, we can write

$$\exp(t_j \sigma_{x_j}) = \cosh(t_j) + \sigma_{x_j} \sinh(t_j) = \cosh(t_j)\big(1 + \sigma_{x_j} \tanh(t_j)\big);$$
$$\exp(\beta \sigma_i \sigma_j) = \cosh(\beta)\big(1 + \sigma_i \sigma_j \tanh(\beta)\big).$$

This gives the following expression for $Z^A_{\Lambda;\beta,0}$:

$$Z^A_{\Lambda;\beta,0} = (\cosh \beta)^{|\mathcal{E}_\Lambda|} \left(\prod_{j=1}^r \cosh(t_j) \right) \sum_{\omega \in \Omega_\Lambda} \left[\sum_{E \subseteq \mathcal{E}_\Lambda} \prod_{\{i,j\} \in E} (\sigma_i \sigma_j \tanh(\beta)) \right] \cdot \left[\sum_{B \subseteq A} \prod_{x_j \in B} (\sigma_{x_j} \tanh(t_j)) \right].$$

Changing the order of summation we should evaluate, for $E \subseteq \mathcal{E}_\Lambda$ and $B \subseteq A$, the quantity:

$$\sum_{\omega \in \Omega_\Lambda} \prod_{\{i,j\} \in E} (\sigma_i \sigma_j) \prod_{x_j \in B} (\sigma_{x_j}).$$

By an easy symmetry argument, this sum is zero unless all σ_i's appear an even number of time, which corresponds to the condition $(E,B) \in \mathcal{E}^{\mathrm{even}}_{\Lambda,A}$. In this latter case, the sum is the number of spin configurations $|\Omega_\Lambda| = 2^{|\Lambda|}$. This ends the proof of the high temperature expansion. $\qquad\square$

Pairs (E,B) in $\mathcal{E}^{\mathrm{even}}_{\Lambda,A}$ can be considered as subgraphs of Λ, where the vertices are all vertices incident to an edge of E and the edge-set is precisely E (vertices in B must be adjacent to at least one edge in E). This graph has a unique decomposition (up to reordering) into r connected components, each again being the graph of some (E_i, B_i) (for $1 \le i \le r$). Notice that the weight function is multiplicative with respect to connected components, *i.e.* $\mathrm{wt}(E,B) = \prod_i \mathrm{wt}(E_i, B_i)$. Therefore using notation of [8],

$$\Xi^A_{\Lambda;\beta,0} = 1 + \sum_{r \ge 1} \frac{1}{r!} \sum_{\substack{(E_1,B_1),\ldots,(E_r,B_r) \\ \mathrm{connected}}} \prod_{i=1}^r \mathrm{wt}(E_i, B_i) \prod_{1 \le i < j \le r} \delta\big[(E_i,B_i),(E_j,B_j)\big],$$

where $\delta\big[(E_i,B_i),(E_j,B_j)\big] = 1$

if the graphs corresponding to (E_i, B_i) and (E_j, B_j) do not share a vertex, and 0 otherwise (this factor encodes the fact that connected components should not intersect). We set $\zeta((E,B),(E',B')) = \delta\big[(E,B),(E',B')\big] - 1$ as it is usually done in the field.

To compute cumulants, we need an expansion of $\log\left(\left\langle \exp\left(i \sum_{j=1}^r t_j \sigma_{x_j}\right) \right\rangle_{\Lambda;\beta,0} \right)$ and thus of $\log(\Xi^A_{\Lambda;\beta,0})$. Such an expansion will be given by cluster expansion, but we should first check some conditions ensuring convergence, *e.g.* the ones given in [8, Section 5.4]. We let $\overline{\mathrm{wt}}(E,B) = (\tanh \beta)^{|E|}$, which dominates all functions $\mathrm{wt}(E,B)$ when the t_j's are complex parameters of moduli at most $\tanh^{-1}(1)$.

Lemma 3.3. *For* $(E,B) \in \mathcal{E}^{even}_{\Lambda,A}$, *let* $a(E;B) = |V(E,B)|$, *that is the number of vertices in the graph associated to* (E,B). *Then, there exists a constant* $\beta^{ht}_{ce}(d)$ *such that the following holds for* $\beta < \beta^{ht}_{ce}(d)$:

(1) *for any $\Lambda \subset \mathbb{Z}$, we have* $\qquad \sum\limits_{(E,B)\in\mathcal{E}^{even}_{\Lambda,A}} |\overline{\mathrm{wt}}(E,B)| e^{a(E,B)} < \infty$;

(2) *for each fixed pair $(E_\star, B_\star)$ where $E_\star$ is a finite subset of $\mathcal{E}_{\mathbb{Z}^d}$ and $B^\star \subseteq A$, one has*

$$S_{(E_\star,B_\star)} := \sum_{\substack{(E,B)\in\mathcal{E}^{even}_A \\ (E,B)\ connected}} |\overline{\mathrm{wt}}(E,B)| e^{a(E,B)} |\zeta[(E,B),(E_\star,B_\star)]| \leq a(E_\star,B_\star).$$

Remark 3.4. To prove the convergence of cluster expansion, it is actually enough to prove a weaker version of Item 2, where $(E_\star, B_\star)$ is necessarily in $\mathcal{E}^{even}_{\Lambda,A}$ and the sum only runs on $(E,B) \in \mathcal{E}^{even}_{\Lambda,A}$. The stronger version stated here will nevertheless be useful in the proof of Lemma 3.6 below and is proved in the same way, which explains our choice.

Proof: The first condition is trivial since the set $\mathcal{E}^{even}_{\Lambda,A}$ is finite. Let us consider the second one. By definition, $\zeta[(E,B),(E_\star,B_\star)] = -1$ if (E,B) and $(E_\star,B_\star)$ share a vertex and 0 otherwise. Thus

$$S_{(E_\star,B_\star)} \leq \sum_{v\in V(E_\star,B_\star)} \left[\sum_{\substack{E\subset\mathcal{E}_{\mathbb{Z}^d},B\subset\mathbb{Z}^d,(E,B)\ connected \\ v\in V(E,B)}} |\overline{\mathrm{wt}}(E,B)|\, e^{|V(E,B)|} \right].$$

A simple translation argument shows that the quantity between brackets is independent of v so that

$$S_{(E_\star,B_\star)} \leq |V(E_\star,B_\star)| \left[\sum_{\substack{E\subset\mathcal{E}_{\mathbb{Z}^d},B\subset\mathbb{Z}^d,(E,B)\ connected \\ 0\in V(E,B)}} |\overline{\mathrm{wt}}(E,B)|\, e^{|V(E,B)|} \right].$$

Note that (E,B) connected implies in particular that B is included in the vertex set of the graph associated to E. Therefore the sum can be simplified as a sum only over E by paying a factor $2^{|V(E,B)|}$. Moreover connectedness implies $|V(E,B)| \leq |E| + 1$. Thus we get:

$$S_{(E_\star,B_\star)} \leq |V(E_\star,B_\star)| \left[\sum_{\substack{E\subset\mathcal{E}_{\mathbb{Z}^d},\ connected \\ 0\in V(E)}} 2^{|E|+1} (\tanh\beta)^{|E|} e^{|E|+1} \right].$$

The summand depends only on the size k of E. From [8, Lemma 3.59], the number of connected sets $E \subset \mathcal{E}_{\mathbb{Z}^d}$ containing 0 of size k is bounded from above by $(2d)^{2k}$, so that

$$S_{(E_\star,B_\star)} \leq |V(E_\star,B_\star)| \left[\sum_{k\geq 1} (2d)^{2k}\, 2^{k+1} (\tanh\beta)^k\, e^{k+1} \right].$$

For β small enough, say $\beta < \beta^{ht}_{ce}(d)$, the sum is smaller than 1 and the second inequality is fulfilled. $\qquad\square$

We can now state the *cluster expansion* of $\log(\Xi^A_{\Lambda;\beta,0})$. In this context, a cluster is a multiset $X = \{(E_1, B_1), \ldots, (E_r, B_r)\}$ of elements of $\mathcal{E}^{even}_{\Lambda,A}$. The multiplicity of (E,B) in the multiset X is denoted $n_X(E,B)$. The support $\overline{X}$ is the union of the vertex-sets $V(E_i,B_i)$. We say that two clusters intersect if $\zeta[(E,B),(E_\star,B_\star)] = -1$ i.e. if they share a *vertex*.

Proposition 3.5. *For $\beta < \beta^{ht}_{ce}(d)$, We have the following expansion:*

(3.2)
$$\log(\Xi^A_{\Lambda;\beta,0}) = \sum_{X:\overline{X}\subset\Lambda} \Psi_\beta(X),$$

where for a cluster $X = \{(E_1,B_1),\ldots,(E_r,B_r)\}$,

$$\Psi_\beta(X) := \left(\prod_{(E,B)\in\mathcal{E}^{even}_{\Lambda,A}} \frac{1}{n_X(E,B)!} \right) \phi((E_1,B_1),\ldots,(E_r,B_r)) \left(\prod_{i=1}^r \mathrm{wt}(E_i,B_i) \right),$$

where

$$\phi((E_1, B_1), \ldots, (E_r, B_r)) = \sum_{G \subset G_r,\, connected} \prod_{\{i,j\} \in G} \zeta\big[(E_i, B_i), (E_j, B_j)\big]$$

and G_r denotes the complete graph on r vertices. The convergence of the series in Eq. (3.2) holds in the sense of locally uniform convergence of analytic functions in the complex parameters $t_1, \ldots, t_r$ for $|t_1|, \ldots, |t_r| \leq \tanh^{-1}(1)$.

Proof: This follows from the general theory of cluster expansions, see e.g. [8, Chapter 5]. For the analyticity in the parameters, see specifically [8, Section 5.5]. $\qquad\square$

Notice that the functional $\phi((E_1, B_1), \ldots, (E_r, B_r))$ depends only on which pairs (E_i, E_j) intersect and vanishes if X can be split into two mutually non-intersecting subsets.

3.1.2. *Bounds on joint cumulants.* Recall that $A = \{x_1, \ldots, x_r\}$ is a set of points in the finite domain Λ. The joint cumulant $\kappa_{\Lambda;\beta,0}(\sigma_{x_1}, \ldots, \sigma_{x_r})$ is the coefficient of $t_1 \ldots t_r$ in

$$\log \left\langle \exp\left(\sum_{j=1}^{r} t_j \sigma_{x_j}\right) \right\rangle_{\Lambda,\beta,0} = \log Z_{\Lambda;\beta,0}^{A} - \log Z_{\Lambda;\beta,0}^{\emptyset}$$

$$= |\Lambda| \log 2 + |\mathcal{E}_\Lambda| \log(\cosh \beta) + \sum_{j=1}^{r} \log(\cos t_j) + \log \Xi_{\Lambda;\beta,0}^{A} - \log Z_{\Lambda;\beta,0}^{\emptyset}.$$

Only the summand $\log \Xi_{\Lambda;\beta,0}^{A}$ contributes to the coefficient of $t_1 \ldots t_r$. Therefore, using Proposition 3.5, we have

$$(3.3) \qquad \kappa_{\Lambda;\beta,0}(\sigma_{x_1}, \ldots, \sigma_{x_r}) = [t_1 \ldots t_r] \sum_{X:\overline{X} \subset \Lambda} \Psi_\beta(X) = \sum_{X:\overline{X} \subset \Lambda} [t_1 \ldots t_r] \Psi_\beta(X).$$

The exchange of infinite sum and coefficient extraction is valid since we have uniform convergence of analytic functions on a neighborhood of 0. A cluster $X = \{(E_1, B_1), \ldots, (E_r, B_r)\}$ contributes to the coefficient of $t_1 \ldots t_r$ only if $A = B_1 \uplus \cdots \uplus B_r \subseteq \overline{X}$. Then

$$[t_1 \ldots t_r] \Psi_\beta(X) = \left(\prod_{(E,B) \in \mathcal{E}_{\Lambda,A}^{\text{even}}} \frac{1}{n_X(E,B)!}\right) \varphi((E_1, B_1), \ldots, (E_r, B_r)) (\tanh \beta)^{e(X)}$$

$$\leq C_r \, \mathbf{1}[X \text{ is connected}] (\tanh \beta)^{e(X)},$$

where $e(X) = |E_1| + \cdots + |E_r|$ and $\mathbf{1}[\text{event}]$ is the indicator function of the corresponding event. Back to Eq. (3.3), we get

$$|\kappa_{\Lambda;\beta,0}(\sigma_{x_1}, \ldots, \sigma_{x_r})| \leq \sum_{\substack{X:\overline{X} \subset \Lambda,\, X \text{ connected} \\ A \subseteq \overline{X}}} C_r (\tanh \beta)^{e(X)}.$$

Taking the limit $\Lambda \uparrow \mathbb{Z}^d$, we get a similar upper bound for the cumulant $\kappa_{\beta,0}(\sigma_{x_1}, \ldots, \sigma_{x_r})$ under the probability measure $\mu_{\beta,0}$ corresponding to the whole lattice $\mathbb{Z}^d$:

$$(3.4) \qquad |\kappa_{\beta,0}(\sigma_{x_1}, \ldots, \sigma_{x_r})| \leq \sum_{\substack{X \text{ connected} \\ A \subseteq \overline{X}}} C_r (\tanh \beta)^{e(X)}.$$

The key point in the above formula is that any connected cluster X with $A \subseteq \overline{X}$ fulfills $e(X) \geq \ell_T(A)$. We now need the following lemma, whose proof is inspired by the end of the proof of Theorem 5.27 in [8].

Lemma 3.6. *There exist constants $\beta_{jc}^{ht}(d) > 0$ and $\varepsilon > 0$ such that, for $\beta \leq \beta_{jc}^{ht}(d)$, we have the following inequality:*

$$\sum_{\substack{X \text{ connected} \\ x_1 \in \overline{X},\ e(X) \geq R}} (\tanh \beta)^{e(X)} \leq \varepsilon^R.$$

Proof: The proof involves different values of the inverse temperature β so that we will here make explicit the dependency of the weight in β: we write $\overline{\mathrm{wt}}_\beta(E, B)$ instead of $\overline{\mathrm{wt}}(E, B)$. We first prove that for $\beta' \leq \beta_{\mathrm{jc}}^{\mathrm{ht}}(d)$, we have

$$(3.5) \qquad \sum_{\substack{X \text{ connected} \\ x_1 \in \overline{X}}} (\tanh \beta')^{e(X)} = \sum_{\substack{X \text{ connected} \\ x_1 \in \overline{X}}} \prod_{i=1}^r \overline{\mathrm{wt}}_{\beta'}(E_i, B_i) \leq 1.$$

This uses the same argument as in [8, Eq. (5.31)]:

$$\sum_{\substack{X \text{ connected} \\ x_1 \in \overline{X}}} \prod_{i=1}^r \overline{\mathrm{wt}}_{\beta'}(E_i, B_i) \leq \sum_{r \geq 1} r \sum_{\substack{(E_1, B_1) \text{ connected} \\ x_1 \in V(E_1, B_1)}} \sum_{(E_2, B_2), \ldots, (E_r, B_r)} \prod_{i=1}^r \overline{\mathrm{wt}}_{\beta'}(E_i, B_i)$$

$$\leq \sum_{\substack{(E_1, B_1) \text{ connected} \\ x_1 \in V(E_1, B_1)}} \overline{\mathrm{wt}}_{\beta'}(E_1, B_1) e^{|a(E_1, B_1)|} \leq a(\emptyset, \{x_1\}) = 1,$$

where we used Lemma 3.3 and [8, Theorem 5.4]. This proves (3.5).

Let us fix a value β' as above. There exists a constant $\varepsilon < 1$ such that for β small enough, we have $\tanh \beta < \varepsilon \tanh \beta'$. We can now write

$$\sum_{\substack{X \text{ connected} \\ x_1 \in \overline{X}, \ e(X) \geq R}} (\tanh \beta)^{e(X)} \leq \varepsilon^R \sum_{\substack{X \text{ connected} \\ x_1 \in \overline{X}, \ e(X) \geq R}} (\tanh \beta')^{e(X)} \leq \varepsilon^R,$$

where the last inequality uses (3.5). This ends the proof of the lemma. $\qquad\square$

Combining Eq. (3.4) and Lemma 3.6, we get the desired bound: for $\beta \leq \beta_{\mathrm{jc}}^{\mathrm{ht}}(d)$,

$$|\kappa_{\beta,0}(\sigma_{x_1}, \ldots, \sigma_{x_r})| \leq C_r \varepsilon^{\ell_T(A)}.$$

3.2. **At very low temperature, without magnetic field.**

3.2.1. *The cluster expansion of the partition function.* We now turn to the regime without magnetic field ($h = 0$) and very low temperature (β large). Intuitively, in that case, the spin configurations with fewer pairs of neighbours having opposite spins appear with higher probability. To emphasize the role of these pairs, we rewrite the Hamiltonian as follows:

$$H^\eta_{\Lambda;\beta,h}(\omega) = -\beta |E^\eta_\Lambda| - \beta \sum_{\{i,j\} \in \mathcal{E}^\eta_\Lambda} (\sigma_i(\omega)\sigma_j(\omega) - 1).$$

The only non-zero terms in the sum are those where two neighbours i and j have opposite spins. Let us consider a finite subset $\Lambda \subset \mathbb{Z}^d$ with $+$ boundary condition. A typical spin configuration will then look as a sea of $+$'s with some islands of $-$'s. Therefore the interesting macroscopic components for the cluster expansion in that case are the frontiers between the areas of $+$'s and those of $-$'s, which are called *contours*. Let us define them more rigorously.

Given $\omega \in \Omega^+_\Lambda$, let $\Lambda^-(\omega)$ denote the set of lattice points i where $\sigma_i(\omega) = -1$. For each $i \in \mathbb{Z}^d$ we define $\mathcal{S}_i := i + [\frac{-1}{2}, \frac{1}{2}]^d$ to be the unit cube of $\mathbb{R}^d$ centred at i. Now let

$$\mathcal{M}(\omega) := \bigcup_{i \in \Lambda^-(\omega)} \mathcal{S}_i,$$

and consider the set of maximal connected components of the boundary of $\mathcal{M}(\omega)$, which we denote

$$\Gamma'(\omega) = \{\gamma_1, \ldots, \gamma_r\}.$$

Each of the γ_i's is a *contour* of ω. Contours are connected sets of $(d-1)$-dimensional faces of the cubes $\mathcal{S}_i$. We denote by $|\gamma_i|$ the number of such faces in γ_i. Let $\Gamma_\Lambda := \{\gamma \in \Gamma'(\omega) : \omega \in \Omega^+_\Lambda\}$ denote the set of all possible contours in Λ. Finally, a collection of contours $\Gamma' \subset \Gamma_\Lambda$ is said to be *admissible* if there exists a spin configuration $\omega \in \Omega^+_\Lambda$ such that $\Gamma'(\omega) = \Gamma'$.

Thus when Λ is simply connected (which we will assume from now on in this paper), the partition function can be rewritten as

$$Z^+_{\Lambda;\beta,0} = e^{\beta|\mathcal{E}^+_\Lambda|}\Xi^+_{\Lambda;\beta,0},$$

where

$$\Xi^+_{\Lambda;\beta,0} := \sum_{\Gamma' \subset \Gamma_\Lambda \, admissible} \prod_{\gamma \in \Gamma'} e^{-2\beta|\gamma|}.$$

The cluster expansion is an expression for $\log \Xi^+_{\Lambda;\beta,0}$ as an absolutely convergent series. In this case, a cluster is a collection $X = \{\gamma_1, \ldots, \gamma_r\}$ of contours such that for every two contours γ_i and γ_j, there is a "path" of faces of contours of X connecting γ_i and γ_j. Note that X is actually a multiset, and denote by $n_X(\gamma)$ the number of copies of γ appearing in X and by $\overline{X}$ the support of X, ie $\overline{X} = \cup_{\gamma \in X}\gamma$. In the following we write $\overline{X} \subseteq \Lambda$ to say that $\overline{X} \subseteq \cup_{i \in \Lambda}\mathcal{S}_i$ as subsets of $\mathbb{R}^d$.

In can be shown (see e.g. [8, Chapter 5]) that the cluster expansion converges for β large enough.

Proposition 3.7. *There exists $\beta^{lt}_{ce}(d)$ such that for all $\beta > \beta^{lt}_{ce}(d)$,*

$$\log \Xi^+_{\Lambda;\beta,0} = \sum_{X:\overline{X}\subseteq\Lambda} \Psi_\beta(X),$$

where for a cluster $X = \{\gamma_1, \ldots, \gamma_r\}$,

$$\Psi_\beta(X) := \left(\prod_{\gamma\in\Gamma_\Lambda}\frac{1}{n_X(\gamma)!}\right)\phi(\gamma_1,\ldots,\gamma_r)e^{-2\beta\sum_{i=1}^r|\gamma_i|},$$

where

$$\phi(\gamma_1,\ldots,\gamma_r) = \sum_{G\subseteq G_r\, connected} \prod_{\{i,j\}\in G}\zeta(\gamma_i,\gamma_j),$$

$$\zeta(\gamma_i,\gamma_j) := \begin{cases} 0 \; if \; \gamma_i \cap \gamma_j = \emptyset, \\ -1 \; otherwise, \end{cases}$$

and G_r denotes the complete graph on r vertices.

3.2.2. *Bounds on joint cumulants.* This cluster expansion can be used to compute expectations and therefore deduce some bounds on joint cumulants.

Let $A \subseteq \Lambda$ and let us define $\sigma_A := \prod_{i\in A}\sigma_i$. Its expectation is given by

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \sum_{\omega\in\Omega^+_\Lambda}\sigma_A(\omega)\frac{e^{-H_{\Lambda;\beta,0}(\omega)}}{Z^+_{\Lambda;\beta,0}}.$$

For any spin configuration $\omega \in \Omega^+_\Lambda$ and any contour $\gamma \in \Gamma'(\omega)$, let us define the *interior* of γ (written $\mathrm{Int}(\gamma)$) as the set of points of Λ which would have spin -1 if γ was the only contour of ω. We also write $\mathrm{Int}(X) := \bigcup_{\gamma\in X}\mathrm{Int}(\gamma)$. Thus for any $\omega \in \Omega^+_\Lambda$ and any $i \in \Lambda$,

$$\sigma_i(\omega) = (-1)^{|\{\gamma\in\Gamma'(\omega):i\in\mathrm{Int}(\gamma)\}|},$$

and thus

$$\sigma_A(\omega) = (-1)^{\sum_{i\in A}|\{\gamma\in\Gamma'(\omega):i\in\mathrm{Int}(\gamma)\}|}$$
$$= (-1)^{\sum_{\gamma\in\Gamma'(\omega)}|\{i\in A:i\in\mathrm{Int}(\gamma)\}|}.$$

Therefore one can write

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \frac{\Xi^{+,A}_{\Lambda;\beta,0}}{\Xi^+_{\Lambda;\beta,0}},$$

where

$$\Xi^{+,A}_{\Lambda;\beta,0} := \sum_{\Gamma'\subseteq\Gamma_\Lambda\, admissible}\prod_{\gamma\in\Gamma'}(-1)^{|\{i\in A:i\in\mathrm{Int}(\gamma)\}|}e^{-2\beta|\gamma|}.$$

The cluster expansion converges, which means that we have an analogue of Proposition 3.7 for $\Xi^{+,A}_{\Lambda;\beta,0}$, and thus $\langle\sigma_A\rangle^+_{\Lambda;\beta,0}$ can be expressed as

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \exp\left(\sum_{X:\overline{X}\subseteq\Lambda}\Psi^A_\beta(X) - \sum_{X:\overline{X}\subseteq\Lambda}\Psi_\beta(X)\right),$$

where for a cluster $X = \{\gamma_1,\ldots,\gamma_r\}$,

$$\Psi^A_\beta(X) := \left(\prod_{\gamma\in\Gamma_\Lambda}\frac{1}{n_X(\gamma)!}\right)\left(\sum_{G\subseteq G_r connected}\zeta(\gamma_i,\gamma_j)\right)$$
$$\times (-1)^{\sum_{j=1}^r |\{i\in A: i\in\mathrm{Int}(\gamma_j)\}|}e^{-2\beta\sum_{j=1}^r |\gamma_j|}.$$

For all clusters X such that no vertex of A is in the interior of any of its contours, the exponent of (-1) in the definition of $\Psi^A_\beta(X)$ is always 0 and thus $\Psi^A_\beta(X) = \Psi_\beta(X)$.

Therefore we obtain

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \exp\left(\sum_{X\sim A:\overline{X}\subseteq\Lambda}(\Psi^A_\beta(X) - \Psi_\beta(X))\right),$$

where $X \sim A$ means that X contains at least one contour γ such that a point of A is in the interior of γ. The series is absolutely convergent and we can let $\Lambda\uparrow\mathbb{Z}^d$, obtaining the following proposition.

Proposition 3.8 (Equation (5.49) in [8]). *For β large enough,*

$$\langle\sigma_A\rangle^+_{\beta,0} = \exp\left(\sum_{X\sim A}(\Psi^A_\beta(X) - \Psi_\beta(X))\right).$$

We now want to find estimates on the joint cumulants of the variables $\{\sigma_i : i\in A\}$, for all $A\subset\mathbb{Z}^d$ finite. But in this case, it is easier to estimate first another quantity related to cumulants. We define, for some set B and random variables $(Y_i)_{i\in B}$ defined on the same probability space,

$$Q\left(Y_j; j\in B\right) := \prod_{\substack{\delta\subseteq B\\ \delta\neq\emptyset}}\left\langle\prod_{j\in\delta}Yj\right\rangle^{(-1)^{|\delta|}}.$$

For example,

$$Q_{\beta,0}(\sigma_1,\sigma_2) = \frac{\langle\sigma_1\sigma_2\rangle^+_{\beta,0}}{\langle\sigma_1\rangle^+_{\beta,0}\langle\sigma_2\rangle^+_{\beta,0}}.$$

We show a bound on the quantities $Q_{\beta,0}\left(\sigma_j; j\in A\right)$ for all finite $A\subset\mathbb{Z}^d$.

Lemma 3.9. *Let A be a finite subset of $\mathbb{Z}^d$ of size r. Then for β large enough,*

$$|Q_{\beta,0}\left(\sigma_j; j\in A\right) - 1| \leq C_r e^{-c\beta\ell_T(A)},$$

where $c = c(d)$ and C_r are positive constants depending respectively on d and r.

Proof: Using Proposition 3.8, we have

$$\log Q\left(\sigma_j; j\in A\right) = \sum_{\delta\subseteq A,\delta\neq\emptyset}(-1)^{|\delta|}\sum_{X\sim\delta}(\Psi^\delta_\beta(X) - \Psi_\beta(X)).$$

Recall that $X\sim\delta$ means that at least one point of δ is in the interior of a contour in X. We split the second sum depending on the exact subset $I\subseteq\delta$ of points that are in the interior of a contour in X.

By definition of Ψ observe, if I is as above, $\Psi_\beta^\delta(X) = \Psi_\beta^I(X)$. Therefore

$$\log Q\left(\sigma_j; j \in A\right) = \sum_{\delta \subseteq A, \delta \neq \emptyset} (-1)^{|\delta|} \sum_{\substack{I \subseteq \delta: \\ I \neq \emptyset}} \sum_{\substack{X: \\ \mathrm{Int}(X) \cap \delta = I}} (\Psi_\beta^I(X) - \Psi_\beta(X))$$

$$= \sum_{\substack{I \subseteq A: \\ I \neq \emptyset}} \sum_{\substack{X: \\ I \subseteq \mathrm{Int}(X)}} (\Psi_\beta^I(X) - \Psi_\beta(X)) \sum_{\substack{\delta: \\ I \subseteq \delta \subseteq (I \cup (A \setminus \mathrm{Int}(X)))}} (-1)^{|\delta|}.$$

But the last sum is equal to 0 unless A is contained in $\mathrm{Int}(X)$. Therefore we obtain

$$\log Q\left(\sigma_j; j \in A\right) = \sum_{\substack{X: \\ A \subseteq \mathrm{Int}(X)}} \sum_{\substack{I \subseteq A: \\ I \neq \emptyset}} (-1)^{|I|}(\Psi_\beta^I(X) - \Psi_\beta(X)).$$

Finally, there are $2^r - 1$ non-empty subsets of A, and $|\Psi_\beta^I(X)| \leq |\Psi_\beta(X)|$ for all X and I, thus

$$(3.6) \qquad |\log Q\left(\sigma_j; j \in A\right)| \leq \sum_{\substack{X: \\ A \subseteq \mathrm{Int}(X)}} 2(2^r - 1)|\Psi_\beta(X)|.$$

We conclude by using a trick similar to Lemma 3.6. By [8, Equation (5.31)], if $\beta \geq \beta_{\mathrm{ce}}^{\mathrm{lt}}(d)$, for any vertex $v \in \mathbb{Z}^d$, we have the bound

$$\sum_{X: \overline{X} \ni v} |\Psi_\beta(X)| \leq 1.$$

Thus if $\beta \geq 2\beta_{\mathrm{ce}}^{\mathrm{lt}}(d)$,

$$\sum_{X: \overline{X} \ni v} \left|\Psi_\beta(X)e^{\beta|\overline{X}|}\right| \leq \sum_{X: \overline{X} \ni v} \left|\Psi_{\frac{\beta}{2}}(X)\right| \leq 1.$$

So for any positive integer R,

$$(3.7) \qquad \sum_{\substack{X: \overline{X} \ni v \\ |\overline{X}| \geq R}} |\Psi_\beta(X)| \leq e^{-\beta R} \sum_{X: \overline{X} \ni v} \left|\Psi_\beta(X)e^{\beta|\overline{X}|}\right| \leq e^{-\beta R}.$$

Let us now turn back to Eq. (3.6). Every cluster X such that $A \subseteq \mathrm{Int}(X)$ satisfies $|\overline{X}| \geq 2\ell_T(A)$. Moreover, if a cluster of size R has $j_1 \in A$ in its interior, then it contains at least a point v which is at distance at most R of j_1. There are at most CR^d such points, therefore

$$\sum_{\substack{X: \\ A \subseteq \mathrm{Int}(X)}} |\Psi_\beta(X)| \leq \sum_{R \geq 2\ell_T(A)} CR^d \left[\sum_{\substack{X: \overline{X} \ni v \\ |\overline{X}| = R}} |\Psi_\beta(X)|\right]$$

$$\leq C \sum_{R \geq 2\ell_T(A)} R^d e^{-\beta R}$$

$$\leq C' e^{-c\beta \ell_T(A)},$$

for β large enough, where C' and c are some positive constants (and c depends on the dimension d of the ambient space). Thus by (3.6),

$$|\log Q\left(\sigma_j; j \in A\right)| \leq C_r' e^{-c\beta \ell_T(A)},$$

for some positive constant C_r' depending on r. Exponentiating completes the proof. $\qquad \square$

Now we can convert this estimate into the desired bound on joint cumulants.

Proposition 3.10. *Let A be a finite subset of $\mathbb{Z}^d$ of size r. Then, for β large enough,*

$$|\kappa_{\beta,0}\left(\sigma_j; j \in A\right)| \leq D_r e^{\frac{-c\beta \ell_T(A)}{2}},$$

where $c = c(d)$ is given by Lemma 3.9 and D_r is a positive constant depending on r.

Proof: By Lemma 3.9,

$$Q_{\beta,0}\left(\sigma_j; j \in A\right) = 1 + \mathcal{O}(\mathcal{M}\left(G[A]\right)),$$

where G is the weighted graph defined on $\mathbb{Z}^d$ such that for each $e = (i, j)$, $w_e = e^{\frac{-c\beta\|i-j\|_1}{2}}$. Indeed in that case $\mathcal{M}\left(G[A]\right) = e^{\frac{-c\beta\ell'_T(A)}{2}} \geq e^{-c\beta\ell_T(A)}$; see the discussion in Section 2.3.

Then using Proposition 5.8 of [7], we deduce that

$$\left|\kappa_{\beta,0}\left(\sigma_j; j \in A\right)\right| = \prod_{j \in A} \langle\sigma_j\rangle^+_{\beta,0} \times \mathcal{O}(\mathcal{M}\left(G[A]\right)).$$

But for all $j \in A$, we have $\langle\sigma_j\rangle^+_{\beta,0} \leq 1$, and for β large enough, $\langle\sigma_j\rangle^+_{\beta,0} > 0$.

Hence

$$0 < \prod_{j \in A} \langle\sigma_j\rangle^+_{\beta,0} \leq 1.$$

Moreover by Proposition 2.2,

$$\mathcal{M}\left(G[A]\right) = e^{\frac{-c_d\beta\ell'_T(A)}{2}} \leq e^{\frac{-c_d\beta\ell_T(A)}{2}}.$$

Thus $\left|\kappa_{\beta,0}\left(\sigma_j; j \in A\right)\right| \leq D_r e^{\frac{-c_d\beta\ell_T(A)}{2}}$, as claimed. $\qquad\square$

3.3. With a strong magnetic field.

The last regime we consider is the Ising model with a strong magnetic field, *i.e.* h is bigger than some value $h_1 > 0$ (h_1 is to be determined later). The case of negative h (smaller than $-h_1 < 0$) is obviously symmetric.

In this regime, there is also a well-known cluster expansion for the partition function [8, Section 5.7]. Let us present it briefly.

Fix $\Lambda \subset \mathbb{Z}^d$ and consider the Ising model on Λ with $+$ boundary conditions. We first write its partition function in a suitable form. For a subset Λ^- of Λ, we denote

$$\delta_e\Lambda^- = \{\{i, j\} \in \mathcal{E}^b_\Lambda, \ i \in \Lambda^-, \ j \notin \Lambda^-\}.$$

Let also be $\mathrm{wt}(\Lambda^-) = \exp(-2\beta\left|\delta_e\Lambda^-\right| - 2h|\Lambda^-|)$. Then we have:

Lemma 3.11 (strong magnetic field representation). *With the above notation,*

$$Z^+_{\Lambda,\beta,h} = \exp\left(\beta|\mathcal{E}^b_\Lambda| + h|\Lambda|\right)\left(\sum_{\Lambda^- \subseteq \Lambda} \mathrm{wt}(\Lambda^-)\right).$$

Proof: The proof is not difficult and can be found, *e.g.*, in [8, Section 5.7]. It is important to note that the sum over $\Lambda^- \subseteq \Lambda$ corresponds to the sum over spin configurations in the definition of the partition function: the correspondence simply associates with a spin configuration the set Λ^- of positions of its minus spins. $\qquad\square$

Let A be a subset of Λ. It is straightforward to modify the argument to find a similar expression for the numerators of $\langle\sigma_A\rangle$ (as in Section 3.2) or of $\left\langle\exp\left(\sum_{i \in A} t_i\sigma_i\right)\right\rangle$ (as in Section 3.1).

$$(3.8) \quad \sum_{\omega \in \Omega_\Lambda} \sigma_A(\omega)e^{-H^+_{\Lambda;\beta,h}(\omega)} = \exp\left(\beta|\mathcal{E}^b_\Lambda| + h|\Lambda|\right)\left(\sum_{\Lambda^- \subseteq \Lambda} (-1)^{|A \cap \Lambda^-|}\mathrm{wt}(\Lambda^-)\right).$$

$$(3.9) \quad \sum_{\omega \in \Omega_\Lambda} \exp\left(\sum_{i \in A} t_i\sigma_i\right) = e\exp\left(\beta|\mathcal{E}^b_\Lambda| + h|\Lambda| + \sum_{i \in A} t_i\right)\left(\sum_{\Lambda^- \subseteq \Lambda}\left[\prod_{i \in A \cap \Lambda^-} \exp(-2t_i)\right]\mathrm{wt}(\Lambda^-)\right).$$

A set $\Lambda^- \subseteq \Lambda$ can be seen as a subgraph of the lattice $\mathbb{Z}^d$. As such, it admits a unique decomposition as disjoint union of its connected components $\Lambda^- = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_r$. The weight wt behaves multiplicatively with respect to this decomposition $\mathrm{wt}(\Lambda^-) = \prod_{i=1}^r \mathrm{wt}(S_i)$; the same is true for the modified weight $(-1)^{|A \cap \Lambda^-|}\mathrm{wt}(\Lambda^-)$ and $\left[\prod_{i \in A \cap \Lambda^-} \exp(-2t_i)\right]\mathrm{wt}(\Lambda^-)$ which appear in Eqs. (3.8) and (3.9) above. This enables to use the technique of cluster expansion.

The convergence of this cluster expansion is proved for the partition function in [8, Section 5.7]. The argument can be directly adapted to get a cluster expansion of the expression in Eqs. (3.8) and (3.9) above. The same reasoning as in Section 3.1 or in Section 3.2 leads to similar bounds on joint cumulants, which proves Theorem 1.2 in the strong magnetic field regime.

4. WEIGHTED DEPENDENCY GRAPHS AND CENTRAL LIMIT THEOREMS

We will now use the bounds on cumulants obtained in the previous section to show that the family of random variables $\{\sigma_i : i \in \mathbb{Z}^d\}$ has a weighted dependency graph, and we will use this fact to deduce central limit theorems. We consider any of the regimes studied in the previous section: very high temperature, or very low temperature with $+$ boundary condition, or strong magnetic field with any boundary condition. To have uniform notation, we omit from now on the notation of the boundary condition in low temperature.

4.1. Weighted dependency graph for the σ_i's and central limit theorem for the magnetization.

4.1.1. *The weighted dependency graph.* We start by proving Theorem 1.3, which gives a weighted dependency graph for $\{\sigma_i : i \in \mathbb{Z}^d\}$.

Proof of Theorem 1.3: Let $B = \{i_1, ..., i_r\}$ be a multiset of elements of $\mathbb{Z}^d$ and consider the induced subgraph $G[B]$. Then the maximum weight $\mathcal{M}(G[B])$ of a spanning tree in $G[B]$ satisfies

$$\mathcal{M}(G[B]) = \varepsilon^{\frac{\ell'_T(B)}{2}}.$$

Thus by Proposition 2.2,

$$(4.1) \qquad\qquad \varepsilon^{\ell_T(B)} \le \mathcal{M}(G[B]) \le \varepsilon^{\frac{\ell_T(B)}{2}}.$$

By Proposition 5.2 of [7], it is sufficient to show that

$$\left| \kappa_{\beta,h} \left(\prod_{\alpha \in B_1} \sigma_\alpha, \ldots, \prod_{\alpha \in B_k} \sigma_\alpha \right) \right| \le D_r \mathcal{M}(G[B]),$$

for some sequence $\mathbf{D} = (D_r)_{r \ge 1}$, where $B_1, \ldots, B_k$ are the vertex-set of the connected components of $G_1[B]$, which is the graph induced by edges of weight 1 of G on B.

The vertices i and j are connected in G_1 if and only if $i = j$, because of the definition of the weights w_e in G. Moreover the random variables σ_i are equal to $+1$ or -1, thus

$$\sigma_i^j = \begin{cases} \sigma_i \text{ if } j \text{ odd}, \\ 1 \text{ if } j \text{ even}. \end{cases}$$

Therefore it is sufficient to prove that for any set B' of distinct $i_1, \ldots, i_r$,

$$(4.2) \qquad\qquad |\kappa_{\beta,h}(\sigma_{i_1}, \ldots, \sigma_{i_r})| \le D_r \mathcal{M}(G[B']).$$

But by Theorem 1.2,

$$|\kappa_{\beta,h}(\sigma_{i_1}, \ldots, \sigma_{i_r})| \le D_r \varepsilon^{\ell_T(B')},$$

for some sequence $\mathbf{D}$ depending only on r. Thus using (4.1), Equation (4.2) is proved, which completes the proof of the theorem. $\qquad\square$

Remark 4.1. For all $i \in \mathbb{Z}^d$, let us define $X_i = \frac{\sigma_i + 1}{2}$. Thus $X_i = 1$ (resp. 0) if and only if $\sigma_i = 1$ (resp. -1). In the remainder of this paper, it will sometimes be more convenient to consider the X_i's rather than the σ_i's. Since $\kappa_{\beta,h}(X_{i_1}, \ldots, X_{i_r}) = \frac{1}{2^r} \kappa_{\beta,h}(\sigma_1, \ldots, \sigma_r)$ for $r \ge 2$, the weighted graph G defined in Theorem 1.3 is also a $\mathbf{C}$-weighted dependency graph for the family $\{X_i : i \in \mathbb{Z}^d\}$, for some sequence $\mathbf{C} = (C_r)_{r \ge 1}$.

4.1.2. *The central limit theorem.* We now use the weighted dependency graph from last section to obtain the central limit theorem for the magnetization.

We consider the Ising model on $\mathbb{Z}^d$, with inverse temperature β and magnetic field h. For any positive integer n, we define $\Lambda_n := [-n, n]^d$ the d-dimensional cube centred at 0 of side $2n$. We define the *magnetization*

$$S_n := \sum_{i \in \Lambda_n} \sigma_i,$$

and let v_n^2 denote the variance of S_n. Let us further define the covariance

$$\langle \sigma_i; \sigma_j \rangle_{\beta,h} := \langle \sigma_i \sigma_j \rangle_{\beta,h} - \langle \sigma_i \rangle_{\beta,h} \langle \sigma_j \rangle_{\beta,h}.$$

Let us first state a lemma of [5] which will be useful to prove our central limit theorem.

Lemma 4.2. [5, Lemma V.7.1] *The limit*

$$v^2 := \lim_{n \to \infty} \frac{v_n^2}{|\Lambda_n|}$$

exists as an extended real valued number and

$$v^2 = \sum_{i \in \mathbb{Z}^d} \langle \sigma_0; \sigma_i \rangle_{\beta,h}.$$

But by Theorem 1.2, in the regimes we consider, the cumulants (so in particular the covariance) are exponentially small, so the sum is absolutely convergent and we actually have the stronger statement:

Corollary 4.3. *Suppose that either $h > h_1$ or ($h = 0$ and $\beta < \beta_1$) or ($h = 0$ and $\beta > \beta_2$). The limit*

$$v^2 := \lim_{n \to \infty} \frac{v_n^2}{|\Lambda_n|} = \sum_{i \in \mathbb{Z}^d} \langle \sigma_0; \sigma_i \rangle_{\beta,h}$$

is finite.

We are now ready to prove Theorem 1.4, the central limit theorem for the magnetization.

Proof of Theorem 1.4: We will use Theorem 2.4. Let G be the weighted dependency graph defined in Theorem 1.3. Then for all n, $G[\Lambda_n]$ is a **C**-weighted dependency graph for $\{\sigma_i; i \in \Lambda_n\}$. The number of vertices of $G[\Lambda_n]$ is

$$N_n = |\Lambda_n| = (2n+1)^d,$$

and its maximal weighted degree is

$$\Delta_n - 1 = \max_{i \in \Lambda_n} \sum_{j \in \Lambda_n} \varepsilon^{\frac{\|i-j\|_1}{2}}.$$

There are $2^d \binom{d+y-1}{d-1}$ points at distance y of 0 in $\mathbb{Z}^d$. Indeed such a point has coordinates $(y_1, \ldots, y_d)$ such that $|y_1| + \cdots + |y_d| = y$. There are $\binom{d+y-1}{d-1}$ choices for the values of $|y_1|, \ldots, |y_d|$, and each y_i can be either positive or negative, which multiplies the number of choices by 2^d. Thus there are at most $2^d \binom{d+y-1}{d-1}$ points at distance y of any point x in Λ_n, and

$$\Delta_n - 1 \leq \sum_{y=0}^{2dn} \varepsilon^{\frac{y}{2}} 2^d \binom{d+y-1}{d-1} \leq C,$$

for some constant C because the infinite series is absolutely convergent.

We now have to find a sequence (a_n) and integers s and v such that conditions (1)-(3) of Theorem 2.4 are satisfied. We set for all n, $a_n = \sqrt{|\Lambda_n|} = (2n+1)^{\frac{d}{2}}$, $v = \sqrt{\sum_{i \in \mathbb{Z}^d} \langle \sigma_0; \sigma_i \rangle_{\beta,h}}$ as in Lemma 4.2 and we can choose s to be any integer ≥ 3.

Now condition (1) is satisfied because of Lemma 4.2, as

$$\frac{v_n^2}{|\Lambda_n|} = \frac{v_n^2}{a_n^2} \xrightarrow[n \to \infty]{} v^2.$$

Condition (2) is also satisfied as $a_n^2 = (2n+1)^d = N_n$.

Finally, for some constant C',

$$\left(\frac{N_n}{\Delta_n}\right)^{\frac{1}{s}} \frac{\Delta_n}{a_n} \le C' \frac{(2n+1)^{\frac{d}{s}}}{(2n+1)^{\frac{d}{2}}},$$

and the right-hand side tends to 0 as n tends to infinity for $s \ge 3$. So (3) is satisfied too.

The central limit theorem is proved.

Moreover, whatever the values of β and h are, the spin at 0 is not constant, thus $\langle \sigma_0, \sigma_0 \rangle_{\beta,h} > 0$. On the other hand, because of the GKS inequalities [11, 14], for all $i \in \mathbb{Z}^d$, $\langle \sigma_0; \sigma_i \rangle_{\beta,h} \ge 0$ (see eg. [8]). Therefore,

$$v^2 = \langle \sigma_0, \sigma_0 \rangle_{\beta,h} + \sum_{i \in Z^d \setminus \{0\}} \langle \sigma_0; \sigma_i \rangle_{\beta,h} > 0,$$

which ends the proof of the theorem.   □

4.2. **Central limit theorem for occurrences of a given pattern.**

4.2.1. *Power of weighted dependency graphs.* A major advantage of the theory of weighted dependency graphs is that this structure is stable by taking *powers*.

Definition 4.4. Let G be an edge-weighted graph with vertex set A and weight function w; we also consider a positive integer m. We denote by $\mathrm{MSet}_{\le m}(A)$ the subset of multisets of elements of A with cardinality at most m. Then the m-th power G^m of G is by definition the graph with vertex-set $\mathrm{MSet}_{\le m}(A)$ and where the weight between I and J is given by $w_m(I,J) = \max_{i \in I, j \in J} w(i,j)$.

(Edges not in the graph should be seen as edges of weight 0.)

This definition is justified by the following property, proved in [7, Section 5.3].

Proposition 4.5. *Let $\{Y_a, a \in A\}$ be a family of random variables with a weighted dependency graph G. Then G^m is a weighted dependency graph for the family $\{Y_I, I \in \mathrm{MSet}_{\le m}(A)\}$, where $Y_I = \prod_{a \in I} Y_a$.*

Instead of applying this to the variables σ_i, we will rather work with the variables $X_{(i,+)} := X_i = \frac{1+\sigma_i}{2}$ and $X_{(i,-)} = 1 - X_i$. Start with the following observation. We have, for all $A = \{i_1, \ldots, i_r\} \subseteq \mathbb{Z}^d$,

$$|\kappa_{\beta,h}(X_{i_1}, \ldots, X_{i_r})| = |\kappa_{\beta,h}(Y_{i_1}, \ldots, Y_{i_r})|,$$

where on some subset B of A, $Y_i = X_{(i,+)}$, and on $A \setminus B$, $Y_i = X_{(i,-)}$.

Combining this observation with the remark after Theorem 1.3, we obtain the following. Let G_s be the complete weighted graph with vertex set $\mathbb{Z}^d \times \{+, -\}$, such that for all $i, j \in \mathbb{Z}^d$,

$$w'((i,+),(j,+)) = w'((i,+),(j,-)) = \varepsilon^{\frac{1}{2}\|i-j\|_1}.$$

In other words, we ignore the sign and use the weight function w from the previous section. Then G_s is a **C**-weighted dependency graph for the family $\{X_{(i,s)}; i \in \mathbb{Z}^d, s \in \{+, -\}\}$, for some sequence $\mathbf{C} = (C_r)_{r \ge 1}$.

By considering the powers of G_s and using Proposition 4.5, we obtain weighted dependency graphs for the products of $X_{(i,+)}$'s and $X_{(i,-)}$'s with a bounded number of terms.

Theorem 4.6. *Consider the Ising model on $\mathbb{Z}^d$, with inverse temperature β and magnetic field h, either for $h > h_1$ or ($h = 0$ and $\beta < \beta_1$) or ($h = 0$ and $\beta > \beta_2$). Let m be a fixed positive integer; for multisets I of elements of $\mathbb{Z}^d \times \{+, -\}$, we define $X_I := \prod_{i \in I} X_i$. Then G_s^m is a $\mathbf{D}_m$-weighted dependency graph for the family of random variables $\{X_I; I \in \mathrm{MSet}_{\le m}(\mathbb{Z}^d \times \{+, -\})\}$, for some sequence $\mathbf{D}_m$ depending only on m.*

4.2.2. *The central limit theorem.* In this section, we establish a central limit theorem for the number of occurrences of a given pattern of spins (for example a $+$ surrounded by $-$'s). Formally, we define a *pattern* $\mathcal{P}$ to be a pair $(\mathcal{D}, \mathfrak{s})$, where $\mathcal{D}$ is a finite subset of $\mathbb{Z}^d$ containing 0 and $\mathfrak{s}$ is a function $\mathcal{D} \longrightarrow \{+, -\}$. The cardinality of $\mathcal{D}$ is called the *size* of the pattern $\mathcal{P}$. In our example, the subset is $\mathcal{D} = \{j \in \mathbb{Z}^d : \|j\|_1 \leq 1\}$, while the sign function is given by $\mathfrak{s}(j) = -$ for all $j \in \mathcal{D} \setminus \{0\}$, $\mathfrak{s}(0) = +$; this pattern has size $2d + 1$. An *occurrence* of a pattern $\mathcal{P} = (\mathcal{D}, \mathfrak{s})$ is a set $\{(i + j, s(j)) : j \in \mathcal{D}\}$, where $i \in \mathbb{Z}^d$ is the *position* of the occurrence.

To find a weighted dependency graph for the potential occurrences of a pattern $\mathcal{P}$ of size m, we consider $G_{\mathcal{P}}$ the restriction of G_s^m to the Z_I's of the form

$$Z_i^{\mathcal{P}} = \prod_{j \in \mathcal{D}} X_{(i+j, \mathfrak{s}(j))}.$$

Note that vertices of $G_{\mathcal{P}}$ are canonically indexed by $i \in \mathbb{Z}^d$ so that we will think of $G_{\mathcal{P}}$ as a graph with vertex set $\mathbb{Z}^d$. The weight of the edge between i_1 and i_2 is then

$$w_{\mathcal{P}}(i_1, i_2) = \max_{j_1 \in \mathcal{D}, j_2 \in \mathcal{D}} w(i_1 + j_1, i_2 + j_2) \leq \varepsilon^{\frac{1}{2}(\|i-j\|_1 - \max_{\alpha, \beta \in \mathcal{D}} \|\alpha - \beta\|_1)}.$$

The graph $G_{\mathcal{P}}$ is a **D**-weighted dependency graph for $\{Z_i, i \in \mathbb{Z}^d\}$, for some sequence **D** depending only on $\mathcal{P}$. Indeed, it is a restriction of the weighted dependency graph given in Theorem 4.6.

We define

$$S_{n, \mathcal{P}} := \sum_{i \in \Lambda_n} Z_i^{\mathcal{P}},$$

the number of occurrences of $\mathcal{P}$ whose position is in Λ_n. In our example,

$$S_{n, \mathcal{P}} := \sum_{i \in \Lambda_n} \left(X_i \prod_{j : \|i-j\|_1 = 1} (1 - X_j) \right).$$

This counts the number of isolated $+$ spins in the box Λ_n. It is also easy to encode in this framework the number of $+$ connected components of any given shape.

Let $v_{n, \mathcal{P}}^2$ denote the variance of $S_{n, \mathcal{P}}$. We have a lemma analogous to Lemma 4.2.

Lemma 4.7. *As n tends to infinity, the quantity $\frac{v_{n, \mathcal{P}}^2}{|\Lambda_n|}$ tends to $v_{\mathcal{P}}^2 := \sum_{k \in \mathbb{Z}^d} \langle Z_0^{\mathcal{P}} ; Z_k^{\mathcal{P}} \rangle_{\beta, h} < \infty$.*

Proof: That $G_{\mathcal{P}}$ is a weighted dependency graph for the family $Z_i^{\mathcal{P}}$ implies that

$$\langle Z_0^{\mathcal{P}} ; Z_k^{\mathcal{P}} \rangle_{\beta, h} \leq D_2 \varepsilon^{\frac{1}{2}(k - \max_{\alpha, \beta \in \mathcal{D}} \|\alpha - \beta\|_1)}.$$

This proves that $v_{\mathcal{P}}^2$ is finite as claimed.

Let $\varepsilon > 0$ be fixed. We want to show that for n large enough,

$$\left| \frac{v_{n, \mathcal{P}}^2}{|\Lambda_n|} - \sum_{k \in \mathbb{Z}^d} \langle Z_0^{\mathcal{P}} ; Z_k^{\mathcal{P}} \rangle_{\beta, h} \right| \leq \varepsilon.$$

We have

$$\frac{v_{n, \mathcal{P}}^2}{|\Lambda_n|} = \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \sum_{j \in \Lambda_n} \langle Z_i^{\mathcal{P}} ; Z_j^{\mathcal{P}} \rangle_{\beta, h}$$

$$= \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \sum_{j \in \mathbb{Z}^d} \langle Z_i^{\mathcal{P}} ; Z_j^{\mathcal{P}} \rangle_{\beta, h} - \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \sum_{j \in \mathbb{Z}^d \setminus \Lambda_n} \langle Z_i^{\mathcal{P}} ; Z_j^{\mathcal{P}} \rangle_{\beta, h}.$$

By translation invariance of $\langle Z_i^{\mathcal{P}} ; Z_j^{\mathcal{P}} \rangle_{\beta, h}$, the first sum equals $\sum_{k \in \mathbb{Z}^d} \langle Z_0^{\mathcal{P}} ; Z_k^{\mathcal{P}} \rangle_{\beta, h}$. Thus the only thing left to do is to show that for n large enough, the absolute value of the second term is bounded by ε. We cut the sum on i into two parts : the points that are far from the boundary of Λ_n and those which are not. Recall that the boundary $\partial \Lambda_n$ consists of points j not in Λ, which have a neighbour in Λ_n. We denote $\delta(i, \partial \Lambda_n) = \min_{j \in \partial \Lambda_n} \| i - j \|_1$, which is the distance between i and $\partial \Lambda_n$ For R a positive

integer, let us consider the points $i \in \Lambda_n$ at distance more than R from the boundary of Λ_n. We have, again by translation invariance

$$\frac{1}{|\Lambda_n|} \sum_{\substack{i \in \Lambda_n \\ \delta(i, \partial \Lambda_n) > R}} \sum_{j \in \mathbb{Z}^d \setminus \Lambda_n} |\langle Z_i^{\mathcal{P}}; Z_j^{\mathcal{P}} \rangle_{\beta, h}| \leq \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > R}} |\langle Z_0^{\mathcal{P}}; Z_k^{\mathcal{P}} \rangle_{\beta, h}|.$$

But the series $\sum_{k \in \mathbb{Z}^d} |\langle Z_0^{\mathcal{P}}; Z_k^{\mathcal{P}} \rangle_{\beta, h}|$ is absolutely convergent so the sum above tends to 0 as R tends to infinity. Therefore, there exists some integer R_0 such that

$$(4.3) \qquad \frac{1}{|\Lambda_n|} \sum_{\substack{i \in \Lambda_n \\ \delta(i, \partial \Lambda_n) > R_0}} \sum_{j \in \mathbb{Z}^d \setminus \Lambda_n} |\langle Z_i^{\mathcal{P}}; Z_j^{\mathcal{P}} \rangle_{\beta, h}| \leq \frac{\varepsilon}{2}.$$

Now let us consider the points of Λ_n that are at distance at most R_0 of $\partial \Lambda_n$. There are at most $C|\partial \Lambda_n| R_0^d$ such points. Therefore

$$\frac{1}{|\Lambda_n|} \sum_{\substack{i \in \Lambda_n \\ \delta(i, \partial \Lambda_n) \leq R_0}} \sum_{j \in \mathbb{Z}^d \setminus \Lambda_n} |\langle Z_i^{\mathcal{P}}; Z_j^{\mathcal{P}} \rangle_{\beta, h}| \leq \frac{1}{|\Lambda_n|} \sum_{\substack{i \in \Lambda_n \\ \delta(i, \partial \Lambda_n) \leq R_0}} \sum_{k \in \mathbb{Z}^d} |\langle Z_0^{\mathcal{P}}; Z_k^{\mathcal{P}} \rangle_{\beta, h}| \leq C' \frac{|\partial \Lambda_n|}{|\Lambda_n|} R_0^d.$$

But as n tends to ∞, $\frac{|\partial \Lambda_n|}{|\Lambda_n|}$ tends to 0. Therefore for n large enough,

$$(4.4) \qquad \frac{1}{|\Lambda_n|} \sum_{\substack{i \in \Lambda_n \\ \delta(i, \partial \Lambda_n) \leq R_0}} \sum_{j \in \mathbb{Z}^d \setminus \Lambda_n} |\langle Z_i^{\mathcal{P}}; Z_j^{\mathcal{P}} \rangle_{\beta, h}| \leq \frac{\varepsilon}{2}.$$

Adding (4.3) and (4.4) completes the proof.   □

We are now ready to prove the central limit theorem.

Proof of Theorem 1.5: We proceed as in the proof of Theorem 1.4. We consider $G_{\mathcal{P}}[\Lambda_n]$.

The number of vertices is $N_n = |\Lambda_n'| = |\Lambda_n|$ and, from the discussion above, its maximal weighted degree $\Delta_n - 1$ is bounded as follows:

$$\Delta_n - 1 \leq \max_{i \in \Lambda_n} \sum_{j \in \Lambda_n} \varepsilon^{\frac{1}{2}(\|i-j\|_1 - \max_{\alpha, \beta \in \mathcal{D}} \|\alpha - \beta\|_1)} \leq \max_{i \in \Lambda_n} \sum_{j \in \Lambda_n} C_{\mathcal{P}} \varepsilon^{\frac{1}{2} \|i-j\|_1},$$

where $C_{\mathcal{P}}$ is a positive constant depending only on the pattern $\mathcal{P}$. Thus by the same argument as in the proof of Theorem 1.4, $\Delta_n - 1 \leq C_{\mathcal{P}}'$, for some other constant $C_{\mathcal{P}}'$.

Again we set for all n, $a_n = \sqrt{|\Lambda_n|}$. We also set $v = v_{\mathcal{P}}$ as in Lemma 4.7 and we can choose s to be any integer ≥ 3.

Conditions (1) to (3) of Theorem 2.4 are satisfied again and the theorem is proved.   □

Remark 4.8. The variance $v_{\mathcal{P}}$ appearing in Theorem 1.5 might be equal to 0 for some patterns $\mathcal{P}$, in which case the central limit theorem is degenerate. If the pattern has only plus signs, the same proof as before gives $v_{\mathcal{P}} > 0$.

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