Lecture Notes for the courses

- Advanced topics in Analysis
- Variational methods in Analysis

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Disclaimer:
- these handwritten notes are not self-contained, and are certainly full of minor mistakes;
- the file will be updated in the course of the semester, with the topics that will be discussed in class.

0 Recap of previous results

We will assume everyone is familiar with Analysis I, II, III, in particular Lebesgue integration and measurability theory.
We will also assume familiarity with the definition

Definition 0.1 (Banach space)

A normed vector space $(X, \|\cdot\|)$ is called Banach space if it is complete, i.e., every Cauchy sequence in X has a limit in X.

Particularly important examples of Banach spaces are the L^p spaces.

Definition 0.2 (L^p spaces)

Let $1 \leq p < +\infty$. We define

$$L^p(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C} \text{ such that } f \text{ is measurable and } |f|^p \text{ is summable} \}.$$

In fact, one rather considers $f \in L^p(\mathbb{R}^d)$ up to a set of measure zero. We should redefine $L^p(\mathbb{R}^d)$ as the set of equivalence classes of functions that coincide up to a set of measure zero. We will not do this though.
Defining the norm

$$\|f\|_{L^p(\mathbb{R}^d)} \equiv \|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

it turns out that $(L^p(\mathbb{R}^d), \|\cdot\|_p)$ is a Banach space.

Unless explicitly mentioned we will always work with complex-valued functions. It is by far better for a number of applications, and there is basically no trade-off. Just remember that $|\cdot|$ denotes the modulus of a complex number. Notice also that we use $\mathbb{R}^d$ by simplicity, but all notions in this section can be rephrased for $L^p(\Omega, d\mu)$, where Ω is a measure space with positive measure μ.

Definition 0.3 (L^∞)

We define

$L^\infty(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C}$ such that f is measurable and $\exists K$ finite such that $|f(x)| \leq K$ for almost every $x \in \mathbb{R}^d \}$.

Again here we should actually consider equivalence classes up to a set of measure zero.
The L^∞ norm is defined as

$$\|f\|_\infty := \inf \{ K \text{ such that } |f(x)| \leq K \text{ for almost every } x \in \mathbb{R}^d \}.$$

We state a few important results.

Theorem 0.4 (Hölder's inequality)

Let p, p' be dual indices, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, with $1 \leq p \leq \infty$ (and therefore $1 \leq p' \leq \infty$). Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$. Then the pointwise product $fg \in L^1(\mathbb{R}^d)$ and

$$\left| \int_{\mathbb{R}^d} f(x)\, g(x)\, dx \right| \leq \int_{\mathbb{R}^d} |f(x)|\, |g(x)|\, dx \leq \|f\|_p \, \|g\|_{q'}.$$

The relation between L^p and $L^{p'}$ is actually deeper than this already very important inequality. Let us recall that the dual of a (complex) Banach space is the space of continuous linear functionals, i.e.

$$X^* = \{ L : X \to \mathbb{C} \mid L \text{ linear and continuous} \}.$$

It turns out that L^p spaces are dual to each other through dual indices. For all $1 \leq p < +\infty$, the dual of $L^p(\mathbb{R}^d)$ is isomorphic to $L^{p'}(\mathbb{R}^d)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, i.e.

$$\left(L^p(\mathbb{R}^d)\right)^* = \{ L : L^p(\mathbb{R}^d) \to \mathbb{C} \mid L \text{ linear continuous} \} \simeq L^{p'}(\mathbb{R}^d),$$

in the sense that, for every $L \in \left(L^p(\mathbb{R}^d)\right)^*$ there exists a unique $v \in L^{p'}(\mathbb{R}^d)$ such that

$$L(f) = \int_{\mathbb{R}^d} v f \quad \forall f \in L^p(\mathbb{R}^d).$$

This is <u>not</u> true for $p = \infty$. The dual of $L^\infty(\mathbb{R}^d)$ is a less nice space. Depending on the presence or absence of the axiom of choice it might be larger than $L^1(\mathbb{R}^d)$.

A property of the dual of a Banach space X is that

$$\|f\| = \sup_{\substack{L \in X^* \\ \|L\| \leq 1}} |L(f)| \quad \forall f \in X.$$

By the above property of duality,

$$\|f\|_{L^p} = \sup_{\substack{g \in L^{p'} \\ \|g\|_{L^{p'}} \leq 1}} \left| \int_{\mathbb{R}^d} g f \right|, \quad \forall f \in L^p(\mathbb{R}^d), \ \frac{1}{p} + \frac{1}{p'} = 1$$

or, in other words, the L^p norm can be obtained, by duality, testing against $L^{p'}$ functions.

Notice that, for $1 < p < +\infty$, $L^{p'}(\mathbb{R}^d)$ is the dual of $L^p(\mathbb{R}^d)$, again if $\frac{1}{p} + \frac{1}{p'} = 1$. But this means that $L^p(\mathbb{R}^d)$ is the dual of $L^{p'}(\mathbb{R}^d)$, since $1 < p' < +\infty$ if $1 < p < +\infty$. This shows that $L^p(\mathbb{R}^d)$ coincides with its bidual $(L^p(\mathbb{R}^d)^*)^*$, a property that is called <u>reflexivity</u>. This is closely linked with the property of a space to have weak limits for bounded sequences. It will play an important role later in the course. $L^1(\mathbb{R}^d)$, in turn, is not reflexive.

We end this recap section with two approximation results which we will refer to quite frequently during the course.

Theorem 0.5 (Approximation by simple functions).

Let $1 \le p < \infty$, and $f \in L^p(\mathbb{R}^d)$. Then there exists a sequence of simple functions ϕ_n of the form

$$\phi_n(x) = \sum_{J=1}^{K^{(n)}} C_J^{(n)} \chi_J^{(n)}(x)$$

where $C_J^{(n)} \in \mathbb{C}$ and $\chi_J^{(n)}$ is the characteristic function of some measurable set $A_J^{(n)}$ such that

$$\lim_{n \to \infty} \| \phi_n - f \|_{L^p} = 0 .$$

Theorem 0.6 (Approximation by C^∞-functions, mollification)

Let $J \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} J = 1$, and for $\varepsilon > 0$ define $J_\varepsilon(x) := \frac{1}{\varepsilon^d} J\left(\frac{x}{\varepsilon}\right)$ so that $\int_{\mathbb{R}^d} J_\varepsilon = 1$, and $\| J_\varepsilon \|_1 = \| J \|_1$. Let $f \in L^p(\mathbb{R}^d)$ for some $1 \le p < \infty$ and define the convolution

$$f_\varepsilon(x) := (J_\varepsilon * f)(x) = \int_{\mathbb{R}^d} J_\varepsilon(x-y) f(y)\, dy .$$

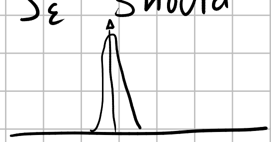
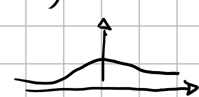
Then

$$f_\varepsilon \in L^p(\mathbb{R}^d) \quad , \quad \| f_\varepsilon \|_{L^p} \le \| J \|_{L^1} \cdot \| f \|_{L^p}$$

and $\quad f_\varepsilon \to f$ strongly in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$, that is,

$$\lim_{\varepsilon \to 0} \| f_\varepsilon - f \|_{L^p} = 0 .$$

Finally, if $J \in C_c^\infty(\mathbb{R}^n)$, then $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.

Intuitively, the function J_ε should be thought of, as $\varepsilon \to 0$, as something of the type  if J is of the type 

In the limit $\varepsilon \to 0$, in the convolution $\int f(x-y) J_\varepsilon(y)\, dy$, only $y = 0$ should matter, which is why $f * J_\varepsilon \to f$ at least pointwise.

The approximating sequence can in fact even be compactly supported.

Theorem 0.7 (Approximation by C_c^∞-functions)

The space $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \le p < \infty$, i.e., for every $f \in L^p(\mathbb{R}^d)$ there exists a sequence $f_n \in C_c^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \| f_n - f \|_{L^p} = 0 .$$

The main message here is that, even though L^p functions can be quite rough (discontinuous, non smooth, ..) or singular, they can be approximated strongly (i.e. in norm) by very regular functions. This will be extremely useful in a number of proofs..

1 Fourier transform

The Fourier transform is a crucial tool in countless areas of mathematics, science, statistics, ... It will play an important role in this course, and we discuss here one very simple possible application to the solution of Partial Differential Equations.

Suppose you want to solve the Heat Equation

$$\begin{cases} \partial_t u - \Delta u = 0 & x \in \mathbb{R}^n,\ t \in (0,\infty) \\ u = g & x \in \mathbb{R}^n,\ t = 0 \end{cases}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is fixed and $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is the unknown.

A formal expression of the type $u(x,t) = (e^{t\Delta} u)(x)$ has a chance of being a solution, but the operator $e^{t\Delta}$ is not necessarily easy to digest. The Fourier transform provides a nice way of writing the solution.

Assume you have an operator F mapping $u(x,t)$ to some $\hat{u}(k,t)$ with the properties

$$\begin{cases} \partial_{x_j} u(k,t) \longmapsto i k_j \hat{u}(k,t) \\ \partial_t u(k,t) \longmapsto \partial_t \hat{u}(k,t). \end{cases}$$

Then the Heat equation is rewritten as

$$\begin{cases} \partial_t \hat{u}(k,t) + k^2 \hat{u}(k,t) = 0 & k \in \mathbb{R}^n,\ t \in (0,\infty) \\ \hat{u}(k,0) = \hat{g}(k) & k \in \mathbb{R}^n,\ t = 0. \end{cases}$$

For any k this is an ODE with solution

$$\hat{u}(k,t) = e^{-t|k|^2} \hat{g}(k).$$

If we now have a way of inverting the transform (non necessarily explicitly), we have proven the existence of a solution.

Definition 1.1 (Fourier transform in L^1)

Let $f \in L^1(\mathbb{R}^d)$. We define its Fourier transform $\hat{f}: \mathbb{R}^d \to \mathbb{C}$ by

$$\hat{f}(k) = \int_{\mathbb{R}^d} e^{-2\pi i\, k \cdot x} f(x)\,dx \qquad k \in \mathbb{R}^d$$

where $k \cdot x = \sum_{i=1}^{d} k_i x_i$.

Remark: the integral converges absolutely for every $k \in \mathbb{R}^d$ since

$$\int \left| e^{-2\pi i k \cdot x} f(x) \right| dx \le \int |f(x)| < +\infty \quad \text{since } f \in L^1(\mathbb{R}^d).$$

This definition has the following straightforward but important properties.

Proposition 1.2

1) The map $f \mapsto \hat{f}$ is linear.
2) Translations are mapped to complex phases: given $(\tau_h f)(x) := f(x-h)$ for $x, h \in \mathbb{R}^d$, $f \in L^1(\mathbb{R}^d)$, we have

$$\widehat{(\tau_h f)}(k) = e^{-2\pi i\, k \cdot h} \hat{f}(k)$$

3) Scaling property: given $(\delta_\lambda f)(x) := f\left(\frac{x}{\lambda}\right)$, $\lambda > 0$, $x \in \mathbb{R}^d$, $f \in L^1$, we have

$$\widehat{(\delta_\lambda f)}(k) = \lambda^d \hat{f}(\lambda k)$$

Proof

We prove 2) only, and leave the others as exercise.

We have

$$\widehat{(\tau_h f)}(k) = \int e^{-2\pi i k \cdot x} (\tau_h f)(x)\,dx = \int e^{-2\pi i k \cdot x} f(x-h)\,dx$$

$$\underset{x-h=y}{=} \int e^{-2\pi i k \cdot (y+h)} f(y)\,dy = e^{-2\pi i k \cdot y} \hat{f}(k)$$

∎

Other properties are:

Proposition 1.3

1) $\hat{f} \in C^0(\mathbb{R}^d)$

2) $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

3) (Riemann-Lebesgue Lemma) $\hat{f}(k) \to 0$ as $k \to \infty$

Proof

3) will be proven in exercise classes.

To prove 1), let $k_n \to k$ in $\mathbb{R}^d$. Then

$$\left|\hat{f}(k_n) - \hat{f}(k)\right| \leq \int \left|\left(e^{-2\pi i (k_n - k)\cdot x} - 1\right) f(x)\right| dx$$

We can apply dominated convergence if $\exists g \in L^1(\mathbb{R}^d)$ such that

$$\left|\left(e^{-2\pi i (k_n - k)\cdot x} - 1\right) f(x)\right| \leq g(x)$$

since the integrand converge pointwise to 0.

But such a function is $g(x) = 2|f(x)|$. Hence $\hat{f}(k_n) \to \hat{f}(k)$.

To prove 2), since $\hat{f} \in C^0$ it is measurable. Then

$$\|\hat{f}\|_{L^\infty} = \sup_{k \in \mathbb{R}^d} \left| \int e^{-2\pi i k\cdot x} f(x)\, dx \right| \leq \sup_{k \in \mathbb{R}^d} \int |f(x)|\, dx = \|f\|_{L^1} < +\infty.$$

∎

The definition of Fourier transform makes certainly sense, as shown, for $f \in L^1(\mathbb{R}^d)$. For many applications however, it would be very convenient to have a way of considering $\hat{f}$ for $f \in L^2(\mathbb{R}^d)$. And Definition 1.1 is not enough.

What is usually done is to define $\hat{f}$ through a suitable approximation argument.

Theorem 1.4 (Plancherel's theorem)

Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then $\hat{f} \in L^2(\mathbb{R}^d)$ and

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

The linear map

$$L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \ni f \longmapsto \hat{f} \in L^2(\mathbb{R}^d)$$

has a unique extension from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ which is an isometry (so the formula above holds $\forall f \in L^2(\mathbb{R}^d)$).

Before proving Theorem 1.4 we state an important consequence.

Corollary 1.5 (Parseval's formula)

For any $f, g \in L^2(\mathbb{R}^d)$, we have

$$\langle f, g \rangle := \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, dx = \int_{\mathbb{R}^d} \overline{\hat{f}(K)}\, \hat{g}(K)\, dK = \langle \hat{f}, \hat{g} \rangle.$$

Proof

The result is a consequence of the polarization identity $\forall f, g \in L^2(\mathbb{R}^d)$

$$\langle f, g \rangle = \frac{1}{2}\Big[\|f+g\|_2^2 - i\|f+ig\|_2^2 - (1-i)\|f\|_2^2 - (1-i)\|g\|_2^2\Big].$$

It is very important for this to hold that we are working in a complex valued linear space (complex-valued L^p functions). The proof of the polarization identity is a straightforward calculation. Once that is accomplished, Theorem 1.4 immediately concludes the proof. ∎

Proof of Theorem 1.4

For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\hat{f}(K)$ is bounded, and therefore

$$Q_\varepsilon := \int |\hat{f}(K)|^2 e^{-\varepsilon \pi K^2}\, dK$$

is well defined.

Moreover, the function $\overline{f(x)}\, f(y)\, e^{-\varepsilon \pi K^2}$ belongs to $L^1(\mathbb{R}^{3d})$.
We can rewrite the integral above as

$$
\begin{aligned}
Q_\varepsilon &= \int \overline{\hat{f}(K)}\, \hat{f}(K)\, e^{-\varepsilon \pi K^2}\, dK \\
&= \int e^{i 2\pi K\cdot x}\, \overline{f(x)}\, e^{-i2\pi K\cdot y} f(y)\, e^{-\varepsilon \pi K^2}\, dx dy dK . \\
&= \int \overline{f(x)}\, f(y)\, e^{2\pi i K\cdot(x-y)}\, e^{-\varepsilon \pi K^2}\, dx dy dK .
\end{aligned}
$$

We now focus on the dK-integral (using Fubini).
We have the following formula for the Fourier transform of of a gaussian (exercise class)

$$\int e^{2\pi i K\cdot(x-y)}\, e^{-\varepsilon\pi K^2}\, dK = \varepsilon^{-\frac{n}{2}}\, e^{-\pi \frac{(x-y)^2}{\varepsilon}} .$$

This yields to

$$Q_\varepsilon = \int \overline{f(x)}\, f(y)\, \varepsilon^{-\frac{n}{2}}\, e^{-\pi\frac{(x-y)^2}{\varepsilon}}\, dy dx$$

We now use the fact that (Lemma 1.5)

$$\varepsilon^{-\frac{n}{2}} \int e^{-\pi\frac{(x-y)^2}{\varepsilon}} f(y)\, dy \underset{\varepsilon\to 0}{\longrightarrow} f(x)$$

in $L^2(\mathbb{R}^d)$.

Hence $Q_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \int |f(x)|^2 dx$, which means that Q_ε is uniformly bounded in ε.

Since $|\hat{f}(K)|^2 e^{-\varepsilon\pi K}$ is an increasing sequence with respect to ε we have

$$\|f\|_{L^2}^2 = \int |f(x)|^2 dx = \lim_{\varepsilon\to 0} \int |\hat{f}(K)|^2 e^{-\varepsilon\pi K^2}\, dK = \int |\hat{f}(K)|^2 dK = \|\hat{f}\|_{L^2}^2$$

where the third equality follows by monotone convergence.

Now, let $f \in L^2(\mathbb{R}^d)$ but $f \notin L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.
By density of $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$,

$$\exists f_J \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \text{ such that } \|f - f_J\|_{L^2} \to 0.$$

But then, by the equality we just proved,

$$\|\hat{f}_J - \hat{f}_K\|_{L^2} = \|f_J - f_K\|_{L^2},$$

which means that $\hat{f}_J$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$.
Since $L^2(\mathbb{R}^d)$ is a Banach space, $\hat{f}_J$ has a limit in $L^2(\mathbb{R}^d)$, which we call $\hat{f}$.

This limit does not depend on the choice of the approximating sequence, since, if f_J' was another approximating sequence, with another limit $\hat{f}'$, then

$$\|\hat{f}' - \hat{f}\|_{L^2} = \lim_{J\to\infty} \|\hat{f}_J' - \hat{f}_J\|_{L^2} = \lim_{J\to\infty} \|f_J' - f_J\|_{L^2} = \|f - f\|_{L^2} = 0.$$

Moreover,

$$\|\hat{f}\|_{L^2} = \lim_{J\to\infty} \|\hat{f}_J\|_{L^2} = \lim_{J\to\infty} \|f_J\|_{L^2} = \|f\|_{L^2}.$$

∎

The function $\hat{f}$ that Theorem 1.4 associates to $f \in L^2(\mathbb{R}^d)$ will be by definition the Fourier tranform in $L^2(\mathbb{R}^d)$.
The operation of computing Fourier transforms in $L^2(\mathbb{R}^d)$ has particularly nice properties. Not only it is an isometry, as per the above result, but it is an invertible isometry (unitary operator) on the Hilbert space $L^2(\mathbb{R}^d)$.

Theorem 1.6 (Inversion formula)

For $f \in L^2(\mathbb{R}^d)$, define its inverse Fourier transform as

$$\check{f}(k) := \hat{f}(-k).$$

Then

$$f(x) = (\check{\hat{f}})(x) \quad \text{almost everywhere.}$$

Proof

Define the gaussian $g_\lambda(k) = e^{-\lambda\pi|k|^2}$, whose Fourier transform is

$$\hat{g}_\lambda(x) = \lambda^{-\frac{n}{2}} e^{-\pi|x|^2/\lambda}.$$

We claim that the following formula holds for any $f \in L^2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^n} \hat{g}_\lambda(y-x)\, f(y)\, dy = \int_{\mathbb{R}^n} g_\lambda(k)\, \hat{f}(k)\, e^{2\pi i k\cdot x}\, dk. \quad (*)$$

We will show at the end that this is true.
As $\lambda \to 0$, the left hand side converges to $f(x)$ in $L^2(\mathbb{R}^d)$ by Theorem 0.6. Moreover, by dominated convergence, in the right hand side we have

$$g_\lambda \cdot \hat{f} \longrightarrow \hat{f} \quad \text{in } L^2 \quad \text{as } \lambda \to 0.$$

Indeed,

$$|g_\lambda(k)\hat{f}(k) - \hat{f}(k)|^2 \leq |g_\lambda(k) - 1|^2 |\hat{f}(k)|^2 \leq 4|\hat{f}(k)|^2 \in L^1(\mathbb{R}^d),$$

which justifies the application of dominated convergence.
Now, since $g_\lambda \hat{f} \to \hat{f}$ as $\lambda \to 0$ in $L^2(\mathbb{R}^d)$, we can use the fact that, by Theorem 1.4, the Fourier transform is an isometry on $L^2(\mathbb{R}^d)$, and thus it is in particular continuous. Thus, the right and side of $(*)$ converges in L^2 to $\hat{\hat{f}}(-x) \equiv \check{\hat{f}}(x)$. Comparing the limits of the two sides of $(*)$ we get

$$f(x) = (\check{\hat{f}})(x)$$

if ⊛ holds, which is precisely what we wanted.

To prove ⊛ we approximate f through $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ functions $f_n \to f$ in $L^2(\mathbb{R}^d)$. Then ⊛ holds for f replaced by f_n since

$$\int_{\mathbb{R}^d} \hat{g}_\lambda(y-x)\, f_n(y)\, dy = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-2\pi i K\cdot(y-x)} g_\lambda(K)\, dK \right) f_n(y)\, dy$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-2\pi i K\cdot(y-x)} f_n(y)\, dy \right) g_\lambda(K)\, dK$$

$$= \int_{\mathbb{R}^d} e^{2\pi i K x}\, \hat{f}_n(K)\, g_\lambda(K)\, dK$$

where the second equality follows by Fubini.

Moreover, by Plancherel, $\hat{f}_n \to \hat{f}$ in $L^2(\mathbb{R}^d)$. Since g_λ and $\hat{g}_\lambda$ are $L^2(\mathbb{R}^d)$ as well, the result follows by a density argument as in

$$\int \hat{g}_\lambda(x-y) f(y)\, dy - \int g_\lambda(K)\, \hat{f}(K) e^{2\pi i K\cdot x}\, dK = \int \hat{g}_\lambda(x-y)\big(f(y) - f_n(y)\big)\, dy - \int g_\lambda(K)\big(\hat{f}(K) - \hat{f}_n(K)\big) e^{2\pi i K\cdot x}\, dK \xrightarrow{n\to\infty} 0. \qquad \blacksquare$$

We defined $\hat{f}$ for $f \in L^2(\mathbb{R}^d)$ through an approximation argument with a sequence $f_J \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. What about $\hat{f}$ for $f \in L^p(\mathbb{R}^d)$ with some $1 < p < \infty$ and $p \neq 2$?

We can replicate the approximation argument, using however $f_J \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

What about the norm bounds? We proved

- $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ (and the transform is invertible)
- $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ (but no invertibility, e.g. $f(x) = 1$ is $L^\infty(\mathbb{R}^d)$, but $\hat{f} \notin L^1(\mathbb{R}^d)$)

Is there a bound of the type

- $\|\hat{f}\|_{L^q} \leq C_{p,q} \|f\|_{L^p}$?

Yes, but:
- $p \leq 2$
- q must be $q = p'$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.7 (Hausdorff-Young inequality)

Let $1<p<2$ and $f \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then

$$\|\hat{f}\|_{L^{p'}} \leq C_p \|f\|_{L^p}$$

with p' such that $\frac{1}{p}+\frac{1}{p'}=1$.

This extends to the same inequality for $f \in L^p(\mathbb{R}^d)$, but the map $f \longmapsto \hat{f}$ between $L^p(\mathbb{R}^d)$ and $L^{p'}(\mathbb{R}^d)$ is not invertible.

The last topic of this chapter is the relation between Fourier transform and convolutions.

We define the convolution between f and g as

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

It is an interesting question in itself to which L^p space $f*g$ belongs depending on the properties of f and g.

We state, without proof, the following result.

Theorem 1.8 (Young's inequality)

Let $p,q,r \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$.
Then $f*g \in L^r(\mathbb{R}^d)$ and

$$\|f*g\|_{L^r} \leq C_{r,p,q} \|f\|_{L^p} \cdot \|g\|_{L^q}.$$

Notable examples:

- The convolution of two L^2 functions is L^∞ $\left(1+\frac{1}{\infty}=\frac{1}{2}+\frac{1}{2}\right)$.
- Similarly, the convolution of $f \in L^p$, $g \in L^{p'}$ with $\frac{1}{p}+\frac{1}{p'}=1$ is L^∞ $\left(1+\frac{1}{\infty}=\frac{1}{p}+\frac{1}{p'}=1\right)$.
- The convolution of two L^1 functions is L^1 $(1+1=1+1)$.

What is the Fourier transform of a convolution?

Theorem 1.9

Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ for $1 \leq p, q \leq 2$, and suppose $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for some $1 \leq r \leq 2$. Then

$$\widehat{(f * g)}(k) = \hat{f}(k)\, \hat{g}(k).$$

The key message of this result is that the Fourier transform maps convolutions to products (and viceversa).

Proof

By Young's inequality, $f * g \in L^r(\mathbb{R}^d)$. Moreover, by Theorem 1.7, $\hat{f} \in L^{p'}$ and $\hat{g} \in L^{q'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

By Hölder's inequality we thus have $\hat{f}\hat{g} \in L^{r'}(\mathbb{R}^d)$ with $\frac{1}{r} + \frac{1}{r'} = 1$

Since $h = f * g \in L^r(\mathbb{R}^d)$, it follows again by Theorem 1.8 that $\hat{h} \in L^{r'}(\mathbb{R}^d)$.

Now, in the special case $p = q = 1$, the above discussion proves $f * g \in L^1(\mathbb{R}^d)$. We can thus use the L^1 definition of Fourier transform to write (using Fubini)

$$\widehat{(f * g)}(k) = \int_{\mathbb{R}^d} (f * g)(x)\, e^{-2\pi i k \cdot x}\, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)\, g(y)\, e^{-2\pi i k \cdot x}\, dx\, dy$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)\, g(y)\, e^{-2\pi i k \cdot (x-y)}\, e^{-2\pi i k \cdot y}\, dx\, dy$$

$$= \hat{f}(k)\, \hat{g}(k).$$

For the general case of p or $q \neq 1$ one needs an approximation argument similar to the previous ones.

We pick two sequences $f_J \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $g_J \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ with $f_J \to f$ in L^p and $g_J \to g$ in L^q. Then $f_J * g_J \in L^1(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$, and we can therefore write

$$\widehat{(f_J * g_J)}(k) = \hat{f}_J(k)\, \hat{g}_J(k)$$

by the formula proven above. Notice moreover that $\widehat{f_J * g_J} \in L^{r'}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

We now consider

$$\|\widehat{f*g} - \hat{f}\hat{g}\|_{L^{r'}} = \|\widehat{f*g} - \widehat{f_J * g_J} + \hat{f}_J \hat{g}_J - \hat{f}\hat{g}\|_{L^{r'}}$$

$$\leq \|\widehat{f*g} - \widehat{f_J * g_J}\|_{L^{r'}} + \|\hat{f}_J \hat{g}_J - \hat{f}\hat{g}\|_{L^{r'}}.$$

We estimate the first norm using Hausdorff-Young and then Young

$$\|\widehat{f*g} - \widehat{f_J * g_J}\|_{L^{r'}} \leq C\|f*g - f_J * g_J\|_{L^r} = C\|(f-f_J)*g + f_J*(g-g_J)\|_{L^r}$$

$$\leq C\|f-f_J\|_{L^p}\|g\|_{L^q} + C\|f_J\|_{L^p}\|g-g_J\|_{L^q} \to 0.$$

For the second norm we argue similarly. ■

The scheme we used is a very very common proof technique. Instead of showing a property for any f in some Banach space, one proves it for f in a suitable dense subspace. If the subspace is smartly chosen, one can (often) ensure that

- proving the property for f in the subspace is much simpler;
- density allows to extend to the whole Banach space.