The free energy of the two-dimensional dilute Bose gas

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The two-dimensional homogeneous Bose gas in experiments

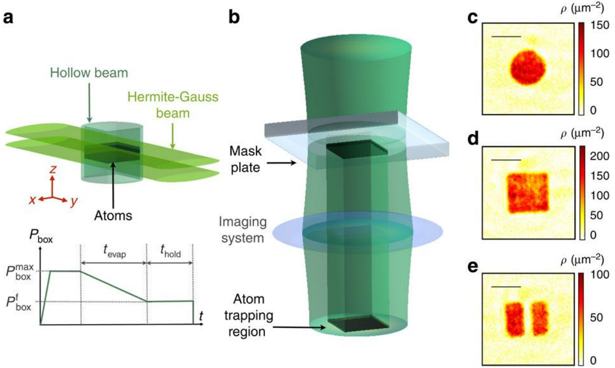

Figure: L. Chomaz, L. Corman, T. Bienaime, R. Desbuquois, C. Weitenberg, S. Nascimbéne, J. Beugnon and J. Dalibard, Phys. Rev. Lett. **110**, 200406 (2013)

Also possible in boxes: 3d Bose gas, 2d and 3d Fermi gases.

The model in finite volume

Hamiltonian of the system describing N particles in box $[0, L]^d$:

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} v\left(x_i - x_j\right).$$

Free energy at inverse temperature β:

$$F(\beta, N, L) = -\frac{1}{\beta} \ln\left(\mathrm{Tr}_{\mathrm{sym}}\left[e^{-\beta H_N}\right]\right),$$

where the trace is taken over permutation-symmetric functions.

Ground state energy:

$$E(N, L) = \inf_{\|\Psi\|=1} \langle \Psi, H_N \Psi \rangle = \lim_{\beta \to \infty} F(\beta, N, L).$$

The thermodynamic limit

Free energy and ground state energy in **thermodynamic limit**:

$$f(\beta,\varrho) = \lim_{\substack{N,L\to\infty \\ \varrho = N/L^d}} \frac{F(\beta,N,L)}{L^d},$$

$$e(\varrho) = \lim_{\substack{N,L\to\infty \\ \varrho = N/L^d}} \frac{E(N,L)}{L^d}.$$

For existence and independence of boundary conditions see e.g. books by Robinson '71 and Ruelle '69.

The scattering length

Assume that $v \geq 0$ is a radial and measurable function with range R_0 and let for $R > R_0$

$$\mathcal{E}(\phi) = \int_{B(R)} \left(|\nabla \phi(x)|^2 + \frac{1}{2} v(x) |\phi(x)|^2 \right) \mathrm{d}x,$$

with $\phi(x) = 1$ for $|x| = R$. Then

$$\inf_{\phi \in H^1(B(R))} \mathcal{E}(\phi) = \begin{cases} \frac{2\pi}{\ln(R/a)} & \text{if d=2,} \\ \frac{4\pi a}{1 - \frac{a}{R}} & \text{if d=3.} \end{cases}$$

The quantity a is called the scattering length of v.

The dilute limit

The dilute limit is defined by $\varrho a^d \ll 1$ and $\beta \varrho^{2/d} \gtrsim 1$.

Assume that v has scattering length 1 then

$$v_a(x) = \frac{1}{a^2} v(x/a)$$

has scattering length a (by scaling).

Known results I: 3-dim. Bose gas at $T = 0$

Ground state energy asymptotics:

$$e(\varrho) = 4\pi a \varrho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\varrho a^3} + o\left(\sqrt{\varrho a^3}\right)\right).$$

Leading order:

- Upper bound: Dyson '57 (in case of hard spheres)
- Lower bound: Lieb, Yngvason '98 (general interactions)
- Upper bound: Lieb, Seiringer, Yngvason '00 (general interactions)

Second order (Lee–Huang–Yang formula)

- Upper bound: Yau, Yin '09 (potentials excluding hard cores)
- Lower bound: Fournais, Solovej '19 (potentials excluding hard cores)
- Conjecture: Lee, Huang, Yang '57

Known results II: 2-dim. Bose gas at $T = 0$

Ground state energy asymptotics:

$$e(\varrho) = \frac{4\pi\varrho^2}{|\ln \varrho a^2|}\,(1 + o(1))\,.$$

Conjectured correction:

$$-\frac{4\pi\varrho^2 \ln\left|\ln \varrho a^2\right|}{\left|\ln \varrho a^2\right|^2}.$$

References:

- Upper and lower bound: Lieb, Yngvason '01 (general interactions)
- Conjecture leading order: Schick '71
- Conjecture second order: e.g. Andersen '05; Pilati, Boronat, Casulleras, Giorgini, '05; Mora, Castin '09.

The ideal Bose gas at $T > 0$

The free energy $f_0(\beta, \varrho)$ of the ideal Bose gas:

$$f_0(\beta, \varrho) = \sup_{\mu \leq 0} \left\{ \mu\varrho + \frac{1}{\beta(2\pi)^d} \int_{\mathbb{R}^d} \ln\left(1 - e^{-\beta\left(p^2 - \mu\right)}\right) \right\}.$$

- $d = 2$: Maximum attained at $\mu = \mu_0(\beta, \varrho) < 0$ for all $\beta > 0$.
- $d = 3$: Maximum attained at

$$\mu = \begin{cases} \mu_0(\beta, \varrho) < 0 & \text{if } \beta < \beta_{\mathrm{c}}, \\ 0 & \text{if } \beta \geq \beta_{\mathrm{c}}. \end{cases}$$

Inverse critical temperature for Bose–Einstein condensation:

$$\beta_{\mathrm{c}} = \frac{1}{4\pi} \left(\frac{\zeta(3/2)}{\varrho} \right)^{2/3}.$$

Known results III: 3-dim. Bose gas at $T > 0$

Free energy asymptotics:

$$f(\beta,\varrho) = f_0(\beta,\varrho) + 4\pi a\varrho^2 \left(2 - \left[1 - \left(\frac{\beta_c}{\beta} \right)^{3/2} \right]_+^2 \right) (1 + o(1)),$$

with

$$\beta_c = \frac{1}{4\pi} \left(\frac{\zeta(3/2)}{\varrho} \right)^{2/3}.$$

References:

- Lower bound: Seiringer '08 (general interactions)
- Upper bound: Yin '10 (potentials excluding hard cores)
- Conjectured corrections: see e.g. Napiórkowski, Reuvers, Solovej '17

Known results IV: Gross-Pitaevskii limit and Fermi gases

- **Dilute Fermi gas in thermodynamic limit**: ($T = 0$) Lieb, Seiringer, Solovej '05; ($T > 0$) Seiringer '06
- **Ground state asymptotics in GP limit**: Lieb, Seiringer, Yngvason '00; Lieb, Seiringer '02; Boccato, Brennecke, Cenatiempo, Schlein '17, '18
- **GP limit of rotating Bose gas**: Lieb, Seiringer '06; Nam, Rougerie, Seiringer '16
- **Bogoliubov theory in GP scaling**: Boccato, Brennecke, Cenatiempo, Schlein '18
- **Dynamics of BEC in GP limit**: Erdös, Schlein, Yau '09 and '10; Pickl '15; Benedikter, de Oliveira, Schlein '15
- **GP limit at positive temperature**: Deuchert, Seiringer, Yngvason '19, Deuchert, Seiringer '19

Main Result: 2-dim. Bose gas at $T > 0$

Theorem (Free energy asymptotics)

$$f(\beta, \varrho) = f_0(\beta, \varrho) + \frac{4\pi\varrho^2}{|\ln \varrho a^2|} \left(2 - \left[1 - \frac{\beta_{\mathrm{BKT}}(\varrho, a)}{\beta} \right]_+^2 \right) (1 + o(1)),$$

with the **inverse BKT critical temperature for superfluidity**

$$\beta_{\mathrm{BKT}}(\varrho, a) = \frac{\ln \left| \ln \varrho a^2 \right|}{4\pi\varrho}.$$

References:

- Lower bound: Deuchert, Mayer, Seiringer '19 (general interactions)
- Upper bound: Mayer, Seiringer (in preparation, general interactions)
- Conjecture: Popov '77; Hohenberg, Fisher '88

Remarks on Theorem

- **Explicit rate for remainder (lower bound)**:
  $|o(1)| \lesssim C(\beta\varrho)\frac{\ln\ln\left|\ln \varrho a^2\right|}{\ln\left|\ln \varrho a^2\right|}$ with $C(x)$ uniformly bounded for $x \geq c > 0$.
- The **error rate is improving** if one stays away from $\beta_{\mathrm{BKT}}(\varrho, a)$.
- Origin of the **temperature dependence in the interaction term** follows from

$$\inf_{0\leq\rho_0\leq\rho}\left\{f_0(\beta,\rho-\rho_0)+\frac{4\pi}{|\ln a^2\rho|}\left(2\rho^2-\rho_0^2\right)\right\}$$
$$=f_0(\beta,\rho)+\frac{4\pi}{|\ln a^2\rho|}\left(2\rho^2-\rho_{\mathrm{s}}^2\right)(1+o(1))$$

as $a^2\rho \to 0$. Optimal choice of ρ_0 (to leading order) is

$$\rho_{\mathrm{s}} = \rho[1-\beta_{\mathrm{BKT}}(\varrho, a)/\beta]_+.$$

Remarks on proof of the lower bound

- As in the case of $T = 0$, we adjust the **proof of the 3-dim. result**.
- The main idea is to **treat the thermal cloud perturbatively** and to use coherent states to obtain the **non-pertubative effects** related to low momentum modes.
- The **new temperature scale** complicates the analysis w.r.t. the proof in $d = 3$.

And now some more ideas of the proof!