Gross-Pitaevskii limit of a homogeneous Bose gas at positive temperature

Andreas Deuchert

Institute of Science and Technology Austria (IST Austria)

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The homogeneous Bose gas in experiments

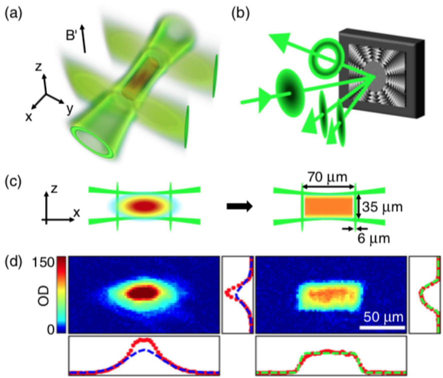

Figure: A. L. Gaunt, T. F. Schmidutz, I. Gotlibovych, R. P. Smith, Z. Hadzibabic, Phys. Rev. Lett. **110**, 200406 (2013)

Also possible: 2d Bose gas, 2d and 3d Fermi gases.

The ideal Bose gas

Consider the ideal Bose gas on $[0, L]^3$ with periodic boundary conditions. The **expected number of particles** in the grand canonical ensemble is given by

$$N = \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} \frac{1}{\exp\left(\left(p^2 - \mu\right)/T\right) - 1}.$$

Here $\mu(T, N, L)$ and T denote the chemical potential and the temperature.

The expected number of particles in the Bose-Einstein condensate (BEC) $N_0(T, N, L) = [e^{-\mu/T} - 1]^{-1}$ is, as $N \to \infty$, to leading order given by

$$\frac{N_0(T, N, L)}{N} \simeq \left[1 - \left(\frac{T}{T_\mathrm{c}}\right)^{3/2}\right]_+ \quad \text{with} \quad T_\mathrm{c} = 4\pi \left(\frac{N/L^3}{\zeta(3/2)}\right)^{2/3}.$$

The Hamiltonian of the interacting model

Hamiltonian with Gross-Pitaevskii scaling:

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} L^{-2} N^2 v\left(N|x_i - x_j|/L\right).$$

Here Δ is the Laplacian on $[0, L]^3$ with periodic boundary conditions and $v \geq 0$ such that scattering length is finite.

The **scattering length** a_N of $L^{-2} N^2 v\left(N|x|/L\right)$ behaves as

$$a_N \sim L N^{-1} \quad \Rightarrow \quad a_N \ll \varrho^{-1/3}.$$

Free energy and Gibbs variational principle

The **free energy** of the gas is given by

$$F(T,N,L) = -T \ln\left(\mathrm{Tr}\left[e^{-H_N/T}\right]\right),$$

where the trace is taken over functions that are symmetric under an exchange of the coordinates.

Gibbs variation principle: Let

$$\mathcal{S}_N = \left\{\Gamma \in \mathcal{L}\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{3N}\right)\right) \,\middle|\, 0 \leq \Gamma \leq 1 \text{ and } \mathrm{Tr}\,\Gamma = 1\right\},$$

then

$$F(T,N,L) = \inf_{\Gamma \in \mathcal{S}_N} \underbrace{\{\mathrm{Tr}\,[H_N \Gamma] - TS(\Gamma)\}}_{=\mathcal{F}(\Gamma)} \quad \text{with} \quad S(\Gamma) = -\,\mathrm{Tr}\,[\Gamma \ln(\Gamma)]\,.$$

1-pdm and Bose-Einstein condensation

The **one-particle reduced density matrix (1-pdm)** of a state $\Gamma \in \mathcal{S}_N$ can be defined via its integral kernel by

$$\gamma(x, y) = \mathrm{Tr}\left[a_y^* a_x \Gamma\right].$$

Here a_x^* and a_x denote the usual creation and annihilation operators.
Equivalently, this kernel can be defined by

$$\gamma(x, y) = N \int_{\mathbb{R}^{3(N-1)}} \Gamma(x, q_1, ..., q_{N-1}; y, q_1, ..., q_{N-1}) \mathrm{d}(q_1, ..., q_{N-1}).$$

A sequence of states $\Gamma_N \in \mathcal{S}_N$ with 1-pdms γ_N is said to show **Bose-Einstein condensation (BEC)** if

$$\liminf_{N \to \infty} \sup_{\|\phi\|=1} \frac{\langle \phi, \gamma_N \phi \rangle}{N} > 0.$$

Mathematical literature on dilute Bose gases, $T = 0$

- **Ground state asymptotics of dilute Bose gas in thermodynamic limit**: Dyson '57 (Upper bound hard spheres), Lieb, Yngvason '98 (Lower bound), Lieb, Seiringer, Yngvason '00 (General upper bound)
- **Ground state asymptotics in GP limit**: Lieb, Seiringer, Yngvason '00, Lieb, Seiringer '02, Boccato, Brennecke, Cenatiempo, Schlein '17, '18
- **GP limit of rotating Bose gas**: Lieb, Seiringer '06, Nam, Rougerie, Seiringer '16
- **Bogoliubov theory in GP scaling**: Boccato, Brennecke, Cenatiempo, Schlein '18
- **Dynamics of BEC in GP limit**: Erdös, Schlein, Yau '09 and '10, Pickl '15, Benedikter, de Oliveira, Schlein '15

Mathematical literature on dilute Bose gases, $T > 0$

Results in thermodynamic limit:

- Free energy asymptotics of dilute Bose gas in thermodynamic limit: Seiringer '08 (Lower bound)
- Free energy asymptotics of dilute Bose gas in thermodynamic limit: Yin '10 (Upper bound)

Results in GP limit:

- Free energy asymptotics and prove of BEC for trapped gas in GP limit: Deuchert, Seiringer, Yngvason '18

Theorem: Part 1 (Asymptotics of free energy)

Assumptions:

- v a nonnegative, radial and measurable function which is integrable outside some finite ball ($\Leftrightarrow a_N < \infty$)
- Limit: $N \to \infty$, $T \lesssim T_{\mathrm{c}}$ and $a_N \sim LN^{-1}$

Notation:

- $F_0(T, N, L) \sim L^3 T^{5/2} \sim L^{-2} N^{5/3}$ the free energy of the ideal gas
- $\varrho_0(T, N, L) = N_0(T, N, L)/L^3$ expected density of particles in condensate of ideal Bose gas

We have

$$F(T, N, L) = F_0(T, N, L) + 4\pi a_N L^3 \left(2\varrho^2 - \varrho_0\,(T, N, L)^2\right)(1 + o(1)).$$

Note that $4\pi a_N L^3 \varrho^2 \sim L^{-2} N$.

Theorem: Part 2 (Asymptotics of 1-pdm)

Notation:

- State Γ_N with 1-pdm γ_N and free energy $\mathcal{F}(\Gamma_N)$
- $\gamma_{N,0}$ denotes 1-pdm of the non-interacting canonical Gibbs state

For any sequence of approximate minimizers Γ_N of the free energy in the sense

$$\mathcal{F}_N(\Gamma_N) = F_0(T,N,L) + 4\pi a_N L^3 \left(2\varrho^2 - \varrho_0(T,N,L)^2\right)(1+o(1))$$

we have

$$\|\gamma_N - \gamma_{N,0}\|_1 = o(N).$$

Remarks

- **Result for 1-pdm implies BEC** in the form

$$\lim_{N\to\infty} \sup_{\|\phi\|=1} \frac{\langle \phi, \gamma_N \phi \rangle}{N} = \left[1 - \left(\frac{T}{T_{\mathrm{c}}}\right)^{3/2}\right]_+$$

  with critical temperature T_{c} of the ideal gas.

- **Quantities related to the ideal gas**, that is, $N_0(T, N, L)$ and $F_0(T, N, L)$, can be replaced by their **grand canonical versions**.

- **Uniformity in temperature** as long as $T \leq C T_{\mathrm{c}}$ for some $C > 0$.

- Treatment of **Dirichlet boundary conditions** possible.

Ingredients of the proof

- **Upper bound:** Much simpler proof than in thermodynamic limit (Yin '10) possible because the system in the GP scaling is much more dilute (5 vs. 55 pages).
- **Lower bound:** Adaption of the proof of the lower bound in the thermodynamic limit (Seiringer '08) with an error of the same size.
- **Asymptotics of 1-pdm and BEC:** C-number substitution with general state instead of interacting Gibbs state, novel bound for bosonic relative entropy, Griffith argument to detect condensate.

Thank you for your attention!