What is... a supercharacter?

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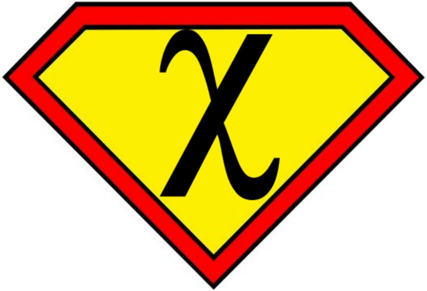

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Fix a basis for $V \cong \mathbb{C}^d$, then $X\colon G \to GL_d(\mathbb{C})$.

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$$X(\overline{1}) = i, X(\overline{2}) = -1, X(\overline{3}) = -i, X(\overline{0}) = 1.$$

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Manschke's theorem

U is also a subrepresentation.

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The number of irreducible representations is equal to the number of conjugacy classes of G

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$$\chi(\overline{0},\overline{0}) = 2, \quad \chi(\overline{1},\overline{0}) = 0, \quad \chi(\overline{0},\overline{1}) = -2, \quad \chi(\overline{1},\overline{1}) = 0.$$

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A character is *irreducible* if it is the trace of an irreducible representation.

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Irreducible characters are orthonormal w.r.t. this product: if χ_1, χ_2 are irreducible characters then

$$\langle \chi_1, \chi_2 \rangle = \delta_{\{\chi_1 = \chi_2\}}$$

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$\sim$: conjugation (we say $g \sim g'$ if $g' = h^{-1}gh$)

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$Irr(G)$ is an orthonormal basis for Cl_G.

Character table

	K_1	K_2	K_3	$\cdots$
χ_1	$\chi_1(K_1)$	$\chi_1(K_2)$	$\chi_1(K_3)$	
χ_2	$\chi_2(K_1)$	$\chi_2(K_2)$	$\chi_2(K_3)$	
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This table is orthonormal!

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χ_2	1	$e^{\frac{\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
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Classifying the irreducible representations of $U_n(\mathbb{F}_q)$ is a wild problem

OK, no irreducible characters. Now what?

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- if $\psi_1 \neq \psi_2$ then $I(\psi_1) \cap I(\psi_2) = \varnothing$

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   the superclasses are $\{\phi(A)([g_1]), \ldots, \phi(A)([g_r])\}$

   but A acts also on $Irr(G)$; call $\Omega_1, \ldots, \Omega_r$ the orbits, then the supercharacters are

   $$\sum_{\chi \in \Omega_i} \chi(1)\chi.$$

Brauer

This is a supercharacter theory

A nice supercharacter theory for $U_n(\mathbb{F}_q)$

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$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 4 & 0 & 3 \\ & 1 & 5 & 2 & 0 & 3 & 6 & 0 \\ & & 1 & 0 & 0 & 0 & 4 & 3 \\ & & & 1 & 0 & 0 & 3 & 0 \\ & & & & 1 & 0 & 4 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \cdot & 0 & \star & \cdot & \cdot & \cdot \\ & 1 & \star & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 1 & 0 & 0 & 0 & \cdot & \star \\ & & & 1 & 0 & 0 & \cdot & 0 \\ & & & & 1 & 0 & \star & \cdot \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix};$$

A nice supercharacter theory for $U_n(\mathbb{F}_q)$

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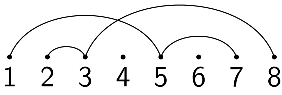
$\pi =$



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$$\chi^\pi(\sigma) = \begin{cases} \frac{q^{\dim(\pi)}}{q^{nst_\pi(\sigma)}} \left(\frac{q-1}{q}\right)^{d(\pi)} \left(\frac{1}{1-q}\right)^{d(\pi\cap\sigma)} & \text{if } D(\sigma) \subseteq Reg\pi; \\ 0 & \text{otherwise.} \end{cases}$$

Symmetric functions in noncommuting variables

$NCSym_n(x_1, x_2, \ldots)$ = symmetric functions in non commuting variables

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$$\begin{aligned} m_{13/24}(x_1, x_2, \ldots) &= x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + \\ &\qquad x_3x_1x_3x_1 + x_2x_3x_2x_3 + x_3x_2x_3x_2 + \ldots \\ &= \sum_{i \neq j} x_ix_jx_ix_j \end{aligned}$$

Superclass functions

$\mathcal{P}_n = \{$ set partitions of $n\}$

$SCl_n = \{f : \mathcal{P}_n \to \mathbb{C}\}$ is an algebra, and $\{\chi^\pi\}_{\pi \in \mathcal{P}_n}$ is an orthogonal basis

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SCl_n and $NCSym_n$ are isomorphic

Possible directions

- Properties of $U_n(\mathbb{F}_q)$ exposed by this supercharacter theory (random walks on $U_n(\mathbb{F}_q)$; random statistics for set partitions)

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- Properties of $U_n(\mathbb{F}_q)$ exposed by this supercharacter theory (random walks on $U_n(\mathbb{F}_q)$; random statistics for set partitions)
- Supercharacter theories in general (how many? Classification? Minimal integral?)
- Cool identities (involving Ramanujan sums) easily proved (???)

THANK YOU