Sum of matrix entries of representations of the symmetric group and its asymptotics

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Partitions

A partition $\lambda \vdash n$ is a non increasing sequence of positive integers

$$\lambda = (\lambda_1, \ldots, \lambda_l)$$

such that $\sum \lambda_i = n$

Example

$$\lambda = (3, 2) \vdash 5$$

$\lambda =$ 

Representations

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where V is finite dimensional $\mathbb{C}$ vector space

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$$\pi^\lambda, \quad \dim\lambda := \dim V^\lambda$$

$$\chi^\lambda(\sigma) = tr(\pi^\lambda(\sigma)), \qquad \hat{\chi}^\lambda(\sigma) = \frac{tr(\pi^\lambda(\sigma))}{\dim\lambda}$$

Standard Young tableaux

1	2	8	9	12
3	5	10	13	
4	7			
6				
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$$\lambda = (3,2) \Rightarrow \dim \lambda = 5$$

1	2	3
4	5	

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Plancherel measure

$$\sum_{\lambda \vdash n} (\dim \lambda)^2 = n!$$

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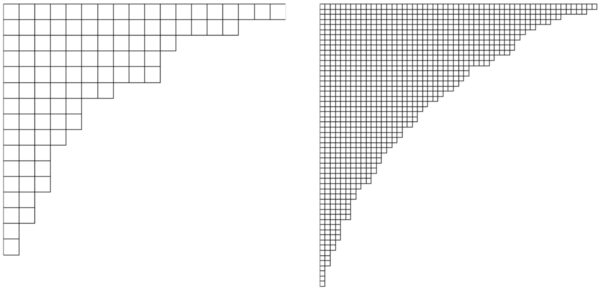
Probability on the set $\mathbb{Y}_n$ of partitions of n

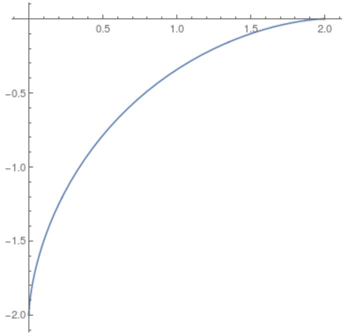
Limit shape

λ distributed with the Plancherel measure and renormalized, then



(a) (b)

Image from D. Romik "The Surprising Mathematics of Longest Increasing Subsequences"



$$\omega_x(\theta) = \left(1 + \frac{2\theta}{\pi}\right) \sin\theta + \frac{2}{\pi}\cos\theta$$

$$\omega_y(\theta) = \left(1 - \frac{2\theta}{\pi}\right) \sin\theta - \frac{2}{\pi}\cos\theta$$

Theorem (Kerov 1999)

$$n^{\frac{wt(\rho)}{2}} \hat{\chi}^{\lambda}_{\rho} \to \prod_{k\geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k)$$

Relations with random matrices

Rows $\lambda_1, \lambda_2, \lambda_3, \ldots$ of a random Young diagram

First, second, third, ... biggest eigenvalues of a Gaussian random Hermitian matrix

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First, second, third, ... biggest eigenvalues of a Gaussian random Hermitian matrix

Same first order asymptotics
Same joint fluctuation (Tracy-Widom law)

Similar tools: *moment method*, link with *free probability theory*

Signed distance

$$d_k(T) \;=\; \begin{array}{rl} & \text{length of northeast path from } k \text{ to } k+1 \\ \text{or} & -\ \text{length of southwest path from } k \text{ to } k+1 \end{array}$$

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$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \Rightarrow d_3(T) = -3$$

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$$(3,4)\ \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \;=\; \begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

Young seminormal representation

$$\pi^{\lambda}((k,k+1))_{T,\tilde{T}} = \begin{cases} 1/d_k(T) & \text{if } T = \tilde{T} \\ \sqrt{1 - \frac{1}{d_k(T)^2}} & \text{if } (k,k+1)T = \tilde{T} \\ 0 & \text{else} \end{cases}$$

Example

$$\lambda = (3,2)$$

$$\pi^\lambda((2,4,3)) = \pi^\lambda((3,4)(2,3)) = \pi^\lambda((3,4))\pi^\lambda((2,3))$$

$$= \begin{bmatrix} -1/3 & \sqrt{8/9} & 0 & 0 & 0 \\ \sqrt{8/9} & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & \sqrt{3/4} & 0 & 0 \\ 0 & \sqrt{3/4} & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3/4} \\ 0 & 0 & 0 & \sqrt{3/4} & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & -\sqrt{2/9} & \sqrt{2/3} & 0 & 0 \\ \sqrt{8/9} & -1/6 & \sqrt{1/12} & 0 & 0 \\ 0 & \sqrt{3/4} & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3/4} \\ 0 & 0 & 0 & -\sqrt{3/4} & -1/2 \end{bmatrix}$$

$$0 \leq u \leq 1$$

Partial trace

$$PT_u^\lambda(\sigma) := \sum_{i \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda}$$

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- We would like to refine Kerov's result

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Partial trace

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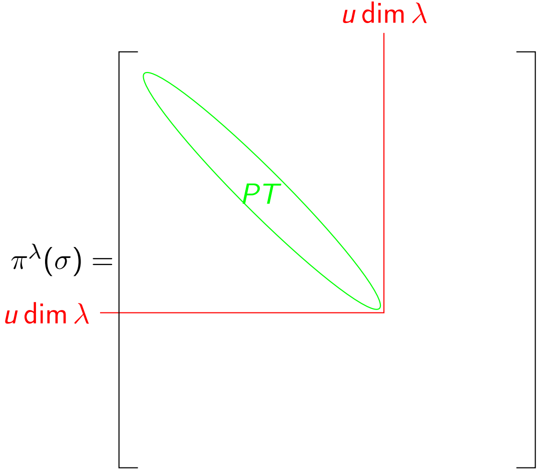
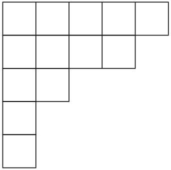
- We would like to refine Kerov's result
- The partial trace has been studied in random matrix theory, *e.g.* for orthogonal random matrices

Visually



Decomposition of PT

$\lambda =$ 

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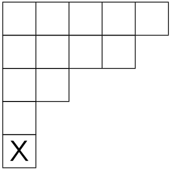
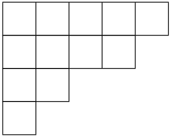
$\lambda =$ 

$\mu_1 =$ 

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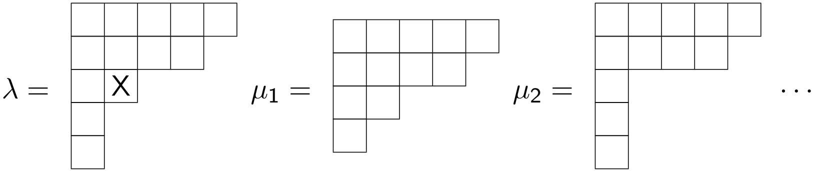

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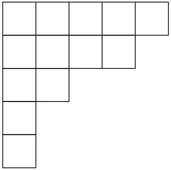
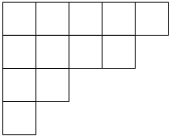
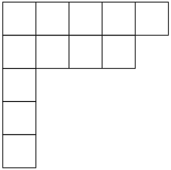
$\lambda =$  $\mu_1 =$  $\mu_2 =$  $\cdots$

Proposition (DS)

$$PT_u^{\lambda}(\sigma) = \sum_{i<\bar{k}} \frac{\chi^{\mu_i}(\sigma)}{\dim \lambda} + \text{Rem}$$

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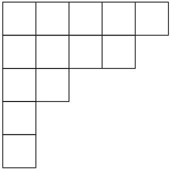
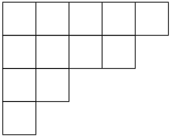
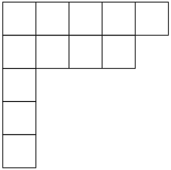
$\lambda =$  $\mu_1 =$  $\mu_2 =$  $\cdots$

Proposition (DS)

$$PT_u^{\lambda}(\sigma) = \sum_{i<\bar{k}} \frac{\chi^{\mu_i}(\sigma)}{\dim \lambda} + \text{Rem}$$

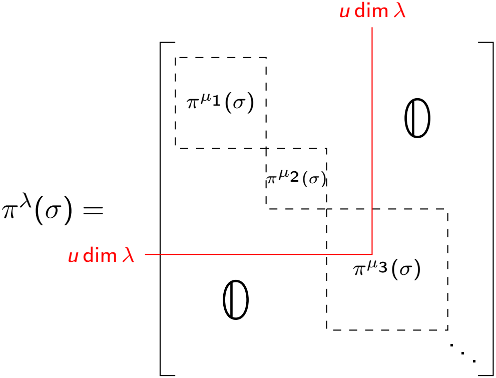
$$\text{Rem} = \sum_{i \leq \tilde{u} \dim \mu_{\bar{k}}} \frac{\pi^{\mu_{\bar{k}}}(\sigma)_{i,i}}{\dim \lambda} = \frac{\dim \mu_{\bar{k}}}{\dim \lambda} PT_{\tilde{u}}^{\mu_{\bar{k}}}(\sigma)$$

Proof

$$\pi^{\lambda}(\sigma) = \begin{bmatrix} \pi^{\mu_1}(\sigma) & & & \mathbb{0} \\ & \pi^{\mu_2}(\sigma) & & \\ & & \pi^{\mu_3}(\sigma) & \\ \mathbb{0} & & & \ddots \end{bmatrix}$$

Proof

$$PT^{\lambda}(\sigma) = \sum_{i<\bar{k}} \frac{\chi^{\mu_j}(\sigma)}{\dim \lambda} + \text{Rem}$$



Asymptotics

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$$F_{sc}(c) n^{-\frac{wt(\sigma)}{2}} \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k)$$

Asymptotics

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$$\downarrow$$

$$A \cdot n^{-\frac{wt(\sigma)}{2}} B$$

Theorem (Kerov 1993)

$$\sum_{j<\bar{k}} \frac{\dim \mu_j}{\dim \lambda} \to A \text{ (deterministic)}$$

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$$n^{\frac{wt(\sigma)}{2}} \hat{\chi}^{\lambda}(\sigma) \to B \text{ (random)}$$

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The two objects are *asymptotically independent*

First, a definition

Contents

$$c(\square) := col(\square) - row(\square)$$

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0	1	2	3	4
-1	0	1	2	
-2	-1			
-3				
-4				

Jucys-Murphy elements

$$J_k := (1,k) + (2,k) + \ldots + (k-1,k) \in Z(\mathbb{C}[S_n])$$

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$$\pi^{\lambda}(J_k) = \begin{bmatrix} c_{T_1}(\boxed{k}) & & \mathbb{0} \\ & c_{T_2}(\boxed{k}) & \\ \mathbb{0} & & \ddots \end{bmatrix}$$

$$\chi^{\lambda}(J_2 + \ldots + J_n)$$

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$$\binom{n}{2}\chi^\lambda(\tau) = \chi^\lambda(J_2 + \ldots + J_n) = \sum_{i=2}^{n} \chi^\lambda(J_i) = \sum_{i=2}^{n} \sum_{k=1}^{\dim\lambda} c_{T_k}(\boxed{i}) = \dim\lambda \sum_{\square\in\lambda} c(\square)$$

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$$c_\rho n^{\downarrow(wt(\rho))}\hat{\chi}^\lambda_\rho = \prod_{i=1}^{l}\left(\sum_{\square\in\lambda} c(\square)^{\nu_i}\right) - \sum_{wt(\tilde{\rho})<wt(\rho)} c_{\tilde{\rho}} n^{\downarrow(|\tilde{\rho}|-m_1(\tilde{\rho}))}\hat{\chi}^\lambda_{\tilde{\rho}}$$

where $\rho_i = \nu_i + 1$

Considering $\chi^\lambda(J_2 + \ldots + J_n)$ we get

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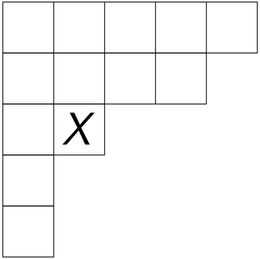
Considering $\chi^\lambda\left(\prod_{i=1}^{l}\left(J_2^{\nu_i} + \ldots + J_n^{\nu_i}\right)\right)$ we get

$$\hat{\chi}^\lambda(\sigma) n^{\frac{wt(\rho)}{2}} \sim \prod_{i=1}^{l}\left(\sum_{\square\in\lambda} c(\square)^{\nu_i}\right)$$

$\mu \nearrow \lambda =$ 

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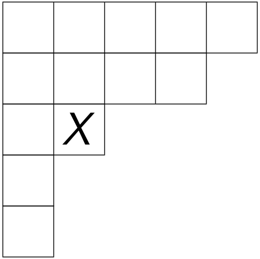

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$$\| $$

$$\prod_{i=1}^{l} \left(\sum_{\square \in \lambda} c(\square)^{\nu_i} - c(\boxed{X})^{\nu_i} \right)$$

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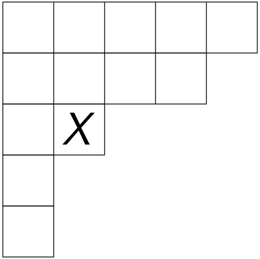

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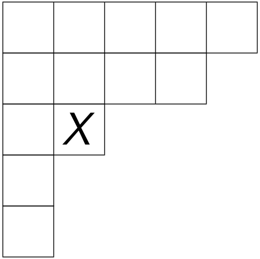

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$$\parallel$$

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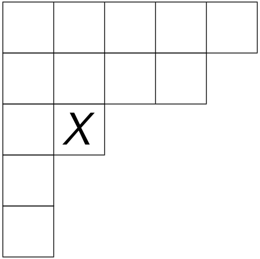

$$n^{\frac{wt(\sigma)}{2}} \sum_{j<\bar{k}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma)$$

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$$\downarrow$$

$$A \cdot B$$

Telescopic sum

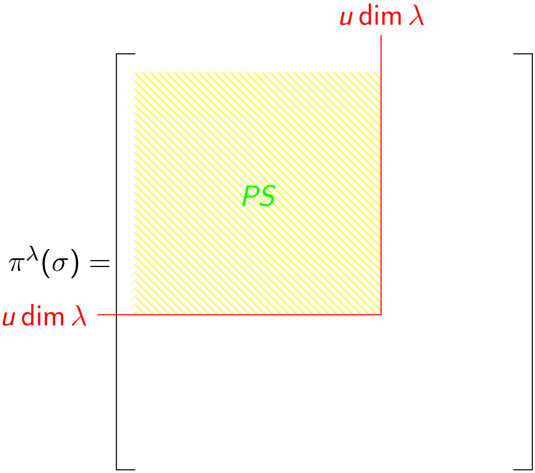
$$PT_u^\lambda(\sigma) = \sum_{j<\bar{k}_1} \frac{\dim \mu_j^{(1)}}{\dim \lambda} \hat{\chi}^{\mu_j^{(1)}}(\sigma) + \sum_{j<\bar{k}_2} \frac{\dim \mu_j^{(2)}}{\dim \lambda} \hat{\chi}^{\mu_j^{(2)}}(\sigma) + \dots$$

Unfortunately, I cannot prove convergence...

Partial sum

$$PS_u^\lambda(\sigma) := \sum_{i,j \le u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda}$$

Visually



Decomposition of PS

$$\sigma \in S_r$$

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$$= \sum_{\tau \in S_r} \mathbb{E}_{PL}^r \left[\hat{\chi}^{\cdot}(\tau) PS_1^{\cdot}(\sigma)\right] PT_u^\lambda(\tau)$$

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$$= \sum_{\tau \in S_r} \mathbb{E}_{PL}^r \left[\hat{\chi}^{\cdot}(\tau) PS_1^{\cdot}(\sigma)\right] PT_u^{\lambda}(\tau)$$

And we have convergence $PS_u^{\lambda}(\sigma) \to u \mathbb{E}_{PL}^r \left[PS_1^{\cdot}(\sigma)\right]$

THANK YOU