

Partial sum of matrix entries of representations of the symmetric group and its asymptotics

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Abstract

Many aspects of the asymptotics of Plancherel distributed partitions have been studied in the past fifty years, in particular the limit shape, the distribution of the longest rows, characters of the associated representation matrices of the symmetric group, and connections with random matrix theory. The latter prompts us to study the representation matrices themselves. We consider thus partial sums of entries of the representation matrix. These partial sums have a natural decomposition as a main term and a remainder. In this paper we describe the fluctuations of the main term. Our main tool is the expansion of symmetric functions evaluated on Jucys-Murphy elements.

1 Introduction

Let λ be a partition of n , in short $\lambda \vdash n$, represented as a Young diagram in English notation. A filling of the boxes of λ with numbers from 1 to n , increasing towards right and downwards, is called a *standard Young tableau*. We call $\dim \lambda$ the number of standard Young tableaux of shape λ . We fix n and we associate to each λ the probability $\frac{(\dim \lambda)^2}{n!}$, which defines the *Plancherel measure*.

Let us recall briefly three motivations for the study of the asymptotics of Plancherel distributed random partitions. They relate algebraic combinatorics, representation theory of the symmetric group, combinatorics of permutations, and random matrix theory.

- The partitions of n index the irreducible representations of the symmetric group S_n . For each $\lambda \vdash n$ the dimension of the corresponding irreducible representation is $\dim \lambda$. A natural question concerns the asymptotics of the characters associated when λ is distributed with the Plancherel measure. A central limit theorem was proved with different techniques by Kerov, [Ker93a], [IO02], and Hora, [Hor98]:

$$n^{\frac{|\rho|-m_1(\rho)}{2}} \hat{\chi}_\rho^\lambda \xrightarrow{d} \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k). \quad (1)$$

Here ρ is a partition of n , $\hat{\chi}_\rho^\lambda$ is the renormalized character associated to λ calculated on a permutation of cycle type ρ , $m_k(\rho)$ is the number of parts of ρ which are equal to k , $\mathcal{H}_m(x)$ is the m -th Hermite polynomial, and $\{\xi_k\}_{k \geq 2}$ are i.i.d. standard gaussian variables.

- The Robinson-Schensted-Knuth algorithm allows us to interpret the longest increasing subsequence of a uniform random permutation as the first row of a Plancherel distributed partition. This motivates the study of the shape of a random partition. A limit shape result was proved independently by [KV77] and [LS77], then extended to a central limit theorem by Kerov, [IO02]. For an extensive introduction on the topic, see [Rom15].
- More recently it was proved that, after rescaling, the limiting distribution of the longest k rows of a Plancherel distributed partition λ coincides with the limit distribution of the properly rescaled k largest eigenvalues of a random Hermitian matrix taken from the Gaussian Unitary Ensemble. See for example [BOO00] and references therein. Such similarities also occur for fluctuations of linear statistics, see [IO02].

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Among the previous three connections, the last one is the least understood and motivates the study of the asymptotics of the representation matrix itself, rather than just the trace, to confront it with similar results on random matrix theory. We consider thus, for a real valued matrix A of dimension N and $u \in [0, 1]$, the partial trace and partial sum defined, respectively, as

$$PT_u(A) := \sum_{i \leq uN} \frac{A_{i,i}}{N}, \quad PS_u(A) := \sum_{i,j \leq uN} \frac{A_{i,j}}{N}.$$

We stress out that these two objects are not invariant by isomorphisms of representations. We will thus use an explicit construction of irreducible representations (Young seminormal representation) and a specific order on the set of standard Young tableaux (the last letter order), defined in Section 2.

The partial trace and sum have been considered in the context of random orthogonal matrices by D'Aristotile, Diaconis and Newman in [DDN03]. It is shown that, in this setting, both $PT_u(A)$ and $PS_u(A)$ converge, after an appropriate normalization, to Brownian motions. Our results for random representation matrices are quite different.

We define *subpartitions* μ_j of λ , denoted $\mu_j \nearrow \lambda$, as partitions of $n - 1$ obtained from λ by removing one box. Then we show in Proposition 2.8 that

$$PT_u^\lambda(\sigma) := PT_u(\pi^\lambda(\sigma)) = \sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) + \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PT_{\bar{u}}^{\mu_{\bar{j}}}(\sigma), \quad (2)$$

for some index \bar{j} and real number \bar{u} . We call the first term of the right hand side the *main term for the partial trace* $MT_u^\lambda(\sigma)$ and the second the *remainder for the partial trace* $RT_u^\lambda(\sigma)$. Our main result (Theorem 2.10) concerns the asymptotics for the main term for the partial trace:

$$n^{\frac{|\rho| - m_1(\rho)}{2}} MT_u^\lambda(\sigma) \xrightarrow{d} u \cdot \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k). \quad (3)$$

That is, the asymptotics of the main term is the same as the asymptotics of the trace multiplied by u . Notice that this result generalizes (1), although we use Kerov's result in the proof. Here lies the difference with the random matrix theory case: the partial trace of an orthogonal random matrix converges to a Brownian motion, and thus it has a high degree of randomness; on the contrary, if we condition the partial trace of a representation matrix by the total trace, the partial trace appears to be deterministic.

Informally, the main idea to prove 3 is to show that when n grows,

$$\sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) \sim \left(\sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} \right) \hat{\chi}^\lambda(\sigma).$$

To achieve this result we need to estimate the asymptotic of $\hat{\chi}^\lambda(\sigma) - \hat{\chi}^\mu(\sigma)$, for λ distributed with the Plancherel measure and $\mu \nearrow \lambda$. We consider power sums p_ν , where ν is the cycle type of σ in which we remove 1 to each part, calculated on the multiset of contents \mathcal{C}_λ . We prove in Section 3 that

$$p_\nu(\mathcal{C}_\lambda) - p_\nu(\mathcal{C}_\mu) \in o_P(n^{-\frac{|\sigma|}{2}}) \quad \text{implies} \quad \hat{\chi}^\lambda(\sigma) - \hat{\chi}^\mu(\sigma) \in o_P(n^{-\frac{|\sigma|}{2}}), \quad (4)$$

where $|\sigma|$ is the size of the support of the permutation σ . This will be accomplished by introducing modified power sums \tilde{p}_ν and by analyzing its highest terms with the appropriate filtration.

We cannot unfortunately prove asymptotic results on the remainder $RT_u^\lambda(\sigma)$, although we conjecture that

$$n^{\frac{|\rho| - m_1(\rho)}{2}} RT_u^\lambda(\sigma) \xrightarrow{P} 0 \quad (5)$$

for $u \in [0, 1]$. In Section 5 we describe a different conjecture, which would imply the one above, involving quotient of dimensions of irreducible representations. We give some numerical evidence. Our conjecture would imply

$$n^{\frac{|\rho| - m_1(\rho)}{2}} PT_u^\lambda(\sigma) \xrightarrow{d} u \cdot \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k).$$

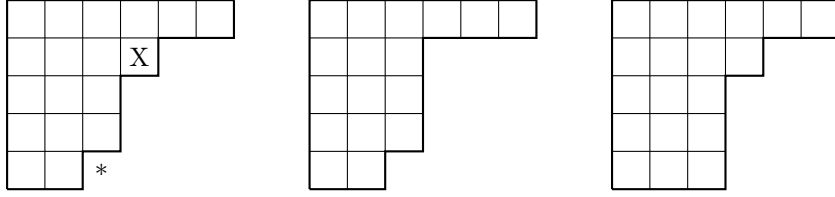


Figure 1: Example of Young diagrams. Starting from the left: $\lambda = (6, 4, 3, 3, 2)$, $\mu_3 = (6, 3, 3, 3, 2)$ and $\Lambda_2 = (6, 4, 3, 3, 3)$.

A similar decomposition as Proposition 2.8 can be shown for partial sums $PS_u^\lambda(\sigma) := PS_u(\pi^\lambda(\sigma))$, see Proposition 4.11: if we call *total sum*

$$TS^\lambda(\sigma) = PS_1^\lambda(\sigma) = \sum_{i \leq \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda},$$

then

$$PS_u^\lambda(\sigma) = \sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma) + \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma),$$

for some \bar{j} and \bar{u} .

Similarly to the partial trace case, we define the main term and remainder for the partial sum as, respectively,

$$MS_u^\lambda(\sigma) := \sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma), \quad RS_u^\lambda(\sigma) := \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma).$$

We show that, for $\sigma \in S_r$, the total sum $TS^\lambda(\sigma)$ can be written as linear combination of $\{\hat{\chi}^\lambda(\tau)\}_{\tau \in S_r}$. This implies a central limit theorem for the total sum (Theorem 4.2) and the main term of the partial sum (Corollary 4.12), and a law of large numbers for the partial sum itself (Theorem 4.13).

In section 2 we recall results on the co-transition measure, the Young seminormal representation, partial permutations, shifted symmetric functions and we introduce the partial trace. In section 3 we prove the result concerning the asymptotics of the main term. In section 4 we study the total and partial sum, while in section 5 we describe a conjecture which would imply a convergence result on the partial trace.

2 Preliminaries

2.1 Notation

Set $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ to be a partition of n , *i.e.* a nonincreasing sequence of positive integers whose sum is n . We associate partitions with *Young diagrams* represented in English notation, as in the example below. In this setting a *box* (in symbol \square) is an element $(a, b) \in \mathbb{N} \times \mathbb{N}$, and we write $\square = (a, b) \in \lambda$ if $1 \leq a \leq \lambda_b$. We say that a box $\square \in \lambda$ is a *corner* if there exists another Young diagram with the same shape as λ without that box. Likewise, a box $\square \notin \lambda$ is a *bay* if there exists a Young diagram with the same shape of λ together with \square . For a box $\square = (a, b)$ the *content* is defined as $c(\square) = c(a, b) = a - b$. Of particular interest are the contents of corners and bays of λ , and we call them respectively y_j and x_j , ordered in a way such that $x_1 < y_1 < x_2 < \dots < y_d < x_{d+1}$, where d is the number of corners. For a corner box \square such that $y_j = c(\square)$ for some j , the partition of $n - 1$ resulting from removing that box is called μ_j , and we write $\mu_j \nearrow \lambda$. We call such a μ_j a *subpartition* of λ . Similarly, Λ_j indicates the partition of $n + 1$ obtained from adding the bay box of content x_j .

Example 2.1. In Figure 1 we show three Young diagrams. The first is a partition $\lambda = (6, 4, 3, 3, 2) \vdash 18$ where we stress out a bay of content -2 and a corner of content 2 . The second is a subpartition corresponding to removing the box \boxed{X} from λ , while the third has the same shape of λ with an additional box corresponding to $\boxed{*}$.

We study the irreducible representation $\pi^\lambda(\sigma)$ associated to λ , and we call $\dim \lambda$ its dimension. It is a known fact that the set

$$\text{SYT}(\lambda) := \{T : T \text{ is a standard Young tableau of shape } \lambda\}$$

indexes a basis for π^λ , so that $\dim \lambda = |\text{SYT}(\lambda)|$.

Example 2.2. We present the set of standard Young tableaux of shape $\lambda = (3, 2)$.

$$T_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \quad T_3 = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array} \quad T_4 = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \quad T_5 = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}.$$

By setting $P_{PL}(\lambda) = (\dim \lambda)^2/n!$ we endow the set of partitions of n with a probability measure, called the *Plancherel measure* of size n . Throughout the paper we always assume that λ is distributed with the Plancherel measure.

2.2 Transition and co-transition measures

Fix $\lambda \vdash n$. Two identities follow from the branching rule (see for example [Sag13, section 2.8]):

$$\sum_{\mu \nearrow \lambda} \dim \mu = \dim \lambda, \quad \sum_{\lambda \nearrow \Lambda} \dim \Lambda = (n+1) \dim \lambda.$$

For $v \in \mathbb{R}$ we can define, respectively, the *normalized co-transition distribution* and the *normalized transition distribution*:

$$F_{ct}^\lambda(v) := \sum_{y_j \leq v\sqrt{n}} \frac{\dim \mu_j}{\dim \lambda}, \quad F_{tr}^\lambda(v) := \sum_{x_j \leq v\sqrt{n}} \frac{\dim \Lambda_j}{(n+1) \dim \lambda}.$$

In [Ker93b] Kerov proved that the normalized transition distribution converges almost surely to the semicircular distribution, when λ is equipped with the Plancherel measure. Building on Kerov's work, we prove the same result for the normalized co-transition distribution.

Lemma 2.3. *When the set of partitions of n is equipped with the Plancherel measure, then almost surely*

$$\lim_{n \rightarrow +\infty} F_{ct}^\lambda(v) = \frac{1}{2\pi} \int_{-2}^v \sqrt{4-t^2} dt$$

holds for all $|v| \leq 2$.

Proof. We prove the lemma by showing that the Stieltjes transform of the normalized co-transition measure associated to the random variable $\lambda^{(n)}$ converges to the Stieltjes transform of the semicircular distribution. It is showed in [GH03, corollary 1] that pointwise convergence of Stieltjes transforms implies convergence in distribution, which again implies convergence of distribution functions at continuity points (recall that the semicircular distribution function is continuous everywhere). We claim thus that

$$ST_{ct}^{\lambda^{(n)}}(u) := \sum_{\mu_j \nearrow \lambda} \frac{\dim \mu_j}{\dim \lambda^{(n)}} \frac{1}{u - y_j} \stackrel{a.s.}{\rightarrow} \frac{u - \sqrt{u^2 - 4}}{2} =: ST_{sc}(u). \quad (6)$$

In [Ker93b, sections 2 and 4], Kerov showed that the Stieltjes transform of the transition measure converges almost surely to the Stieltjes transform of the semicircular distribution. Moreover, he also showed that, if $Q^\lambda(u) := \prod_j (u - y_j)$ and $P^\lambda(u) := \prod_j (u - x_j)$, then

$$ST_{tr}^{\lambda^{(n)}}(u) = \frac{Q^\lambda(u)}{P^\lambda(u)}, \quad ST_{ct}^{\lambda^{(n)}}(u) = u - \frac{P^\lambda(u)}{Q^\lambda(u)}.$$

Hence

$$ST_{ct}^{\lambda^{(n)}}(u) = u - \frac{1}{ST_{tr}^{\lambda^{(n)}}(u)} \stackrel{a.s.}{\rightarrow} u - \frac{2}{u - \sqrt{u^2 - 4}} = \frac{u - \sqrt{u^2 - 4}}{2}.$$

This proves (6) and hence the lemma. \square

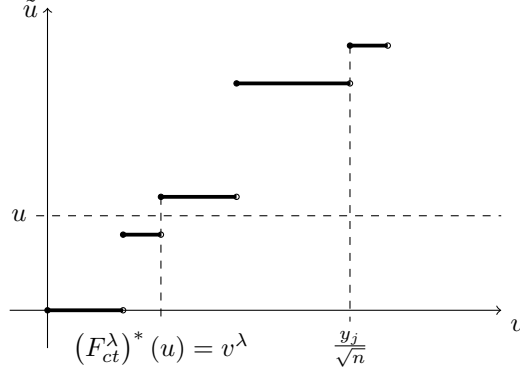


Figure 2: Example of graph of $\tilde{u} = F_{ct}^\lambda(v)$.

Set

$$(F_{ct}^\lambda)^*(u) := \sup \{ z \in \mathbb{R} \text{ s.t. } F_{ct}^\lambda(z) \leq u \}.$$

We consider $(F_{ct}^\lambda)^*$ as the inverse of the step function F_{ct}^λ . We want to show that $F_{ct}^\lambda(v^\lambda)$ converges to u :

Lemma 2.4. *For $v^\lambda = (F_{ct}^\lambda)^*(u)$ for a fixed u , then*

$$F_{ct}^\lambda(v^\lambda) = \sum_{y_j \leq v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \xrightarrow{p} u.$$

Proof. We show in Figure 2 an example of a normalized co-transition distribution. Since F_{ct}^λ is right continuous then $u \leq F_{ct}^\lambda(v^\lambda)$. Moreover, since the co-transition distribution is a step function, for each element of the image $\tilde{u} \in F_{ct}^\lambda(\mathbb{R})$ there exists j such that $F_{ct}^\lambda(\frac{y_j}{\sqrt{n}}) = \tilde{u}$. Call \bar{j} the index corresponding to $F_{ct}^\lambda(v^\lambda)$, that is, $F_{ct}^\lambda(\frac{y_{\bar{j}}}{\sqrt{n}}) = F_{ct}^\lambda(v^\lambda)$. Hence $\sum_{j < \bar{j}} \dim \mu_j / \dim \lambda \leq u$. Thus

- $v^\lambda \leq \frac{y_{\bar{j}}}{\sqrt{n}}$, since $F_{ct}^\lambda(\frac{y_{\bar{j}}}{\sqrt{n}}) \geq u$;
- $v^\lambda \geq \frac{y_{\bar{j}}}{\sqrt{n}}$, since for each $\epsilon > 0$, $F_{ct}^\lambda(\frac{y_{\bar{j}}}{\sqrt{n}} - \epsilon) \leq u$.

Thus $v^\lambda = \frac{y_{\bar{j}}}{\sqrt{n}}$. Therefore

$$\sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} = F_{ct}^\lambda(v^\lambda) - \frac{\dim \mu_{\bar{j}}}{\dim \lambda} \leq u \leq F_{ct}^\lambda(v^\lambda).$$

Equivalently

$$-\frac{\dim \mu_{\bar{j}}}{\dim \lambda} \leq u - F_{ct}^\lambda(v^\lambda) \leq 0,$$

and $\max \frac{\dim \mu_j}{\dim \lambda} \xrightarrow{p} 0$ because of the convergence of the normalized co-transition distribution towards an atom free distribution proved in the previous lemma. This proves the statement. \square

2.3 Young seminormal representation and last letter order

We recall the definition of Young seminormal representation, see [You77] for the original introduction, and [Gre92] for a more modern description. We need three preliminary definitions; set $\lambda \vdash n$, then

1. given a box $\square \in \lambda$ recall that the *content* $c(\square)$ is the difference between the row index and the column index of the box. For $k \leq n$ we also write $c_k(T)$ for the content of the box containing the number k in the tableau T . As instance, in the tableau T_1 of Example 2.2, $c_4(T_1) = -1$, $c_1(T_1) = c_5(T_1) = 0$, $c_2(T_1) = 1$ and $c_3(T_1) = 2$.
2. For $k \leq n-1$ the *signed distance* between k and $k+1$ in the tableau T is $d_k(T) := c_k(T) - c_{k+1}(T)$. For example $d_1(T_1) = d_2(T_1) = d_4(T_1) = -1$; $d_3(T_1) = 3$.

3. If k and $k + 1$ are in different columns and rows we define $(k, k + 1)T$ as the standard Young tableau equal to T but with the boxes containing k and $k + 1$ inverted. In the previous example $(3, 4)T_1 = T_2$, while $(2, 3)T_1$ is not defined.

Definition 2.5. *The Young seminormal representation defines the matrix associated to $\pi^\lambda((k, k + 1))$ entrywise in the following way: if T and S are standard Young tableaux of shape λ , then*

$$\pi^\lambda((k, k + 1))_{T,S} = \begin{cases} 1/d_k(T) & \text{if } T = S; \\ \sqrt{1 - \frac{1}{d_k(T)^2}} & \text{if } (k, k + 1)T = S; \\ 0 & \text{else.} \end{cases}$$

Notice that the adjacent transpositions $(k, k + 1)$, $k = 1, \dots, n - 1$, generate the group S_n , hence $\pi^\lambda(\sigma)$ is well defined for all $\sigma \in S_n$.

Example 2.6. Consider $\lambda = (3, 2)$, $\sigma = (2, 4, 3)$. We compute $\pi^\lambda((2, 4, 3))$:

$$\begin{aligned} \pi^\lambda((2, 4, 3)) &= \pi^\lambda((3, 4)(2, 3)) = \pi^\lambda((3, 4))\pi^\lambda((2, 3)) \\ &= \begin{bmatrix} -1/3 & \sqrt{8/9} & 0 & 0 & 0 \\ \sqrt{8/9} & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & \sqrt{3/4} & 0 & 0 \\ 0 & \sqrt{3/4} & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3/4} \\ 0 & 0 & 0 & \sqrt{3/4} & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -1/3 & -\sqrt{2/9} & \sqrt{2/3} & 0 & 0 \\ \sqrt{8/9} & -1/6 & \sqrt{1/12} & 0 & 0 \\ 0 & \sqrt{3/4} & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3/4} \\ 0 & 0 & 0 & -\sqrt{3/4} & -1/2 \end{bmatrix} \end{aligned}$$

We define also the *last letter order* in SYT(λ): let T, S be standard Young tableaux, then $T \leq S$ if the box containing n lies in a row in T which is lower than that in S ; if n lies in the same row in both tableaux, then we look at the rows containing $n - 1$, and so on. Notice that in the list of tableaux of shape $(3, 2)$ of Example 2.2 the tableaux are last letter ordered; also, the entries of the matrices in Example 2.6 are ordered accordingly. We write $\pi^\lambda(\sigma)_{i,j}$ instead of $\pi^\lambda(\sigma)_{T_i, T_j}$, where T_i is the i -th tableau of shape λ in the last letter order.

2.4 The partial trace and its main term

Let $\lambda \vdash n$ and $\sigma \in S_r$ a permutation of the set $\{1, \dots, r\}$ with $r \leq n$. We see S_r as a subgroup of S_n by adding fixed points, so that $\pi^\lambda(\sigma)$ is well defined. Similarly, if $\rho \vdash r \leq n$, then we define $\chi_\rho^\lambda = \chi_{\rho \cup 1^{n-r}}$ and $\hat{\chi}_\rho^\lambda = \hat{\chi}_{\rho \cup 1^{n-r}}$.

Definition 2.7. *Let $\lambda \vdash n$, $\sigma \in S_r$, $u \in [0, 1]$. Define the partial trace as*

$$PT_u^\lambda(\sigma) := \sum_{i \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda}.$$

When $u = 1$ then the partial trace correspond to the normalized trace

$$\hat{\chi}^\lambda(\sigma) = \frac{\chi^\lambda(\sigma)}{\dim \lambda} = \sum_{i \leq \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda}.$$

Our choice of order on the tableaux implies that, as long as $\sigma \in S_r$ and $r \leq n - 1$,

$$\pi^\lambda(\sigma) = \begin{bmatrix} \pi^{\mu_1}(\sigma) & 0 & 0 & \cdots \\ 0 & \pi^{\mu_2}(\sigma) & 0 & \\ 0 & 0 & \pi^{\mu_3}(\sigma) & \\ \vdots & & & \ddots \end{bmatrix}. \quad (7)$$

Hence $\hat{\chi}^\lambda(\sigma) = \sum_j \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma)$. This decomposition of the total trace can be easily generalized to the partial trace:

Proposition 2.8 (Decomposition of the partial trace). *Fix $u \in [0, 1]$, $\lambda \vdash n$ and $\sigma \in S_r$ with $r \leq n - 1$. Set $(F_{ct}^\lambda)^*(u) = \frac{y_{\bar{j}}}{\sqrt{n}} = v^\lambda$ for some \bar{j} as in Lemma 2.4. Define*

$$\bar{u} = \frac{\dim \lambda}{\dim \mu_{\bar{j}}} \left(u - \sum_{j < \bar{j}} \dim \mu_j \right) < 1,$$

then

$$PT_u^\lambda(\sigma) = \sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) + \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PT_{\bar{u}}^{\mu_{\bar{j}}}(\sigma). \quad (8)$$

Proof. We can decompose the partial trace as

$$PT_u^\lambda(\sigma) = \sum_{i \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} = \sum_{i \leq \sum_{j < \bar{j}} \dim \mu_j} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} + \sum_{\sum_{j < \bar{j}} \dim \mu_j < i \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda}.$$

The first sum in the RHS is

$$\sum_{i \leq \sum_{j < \bar{j}} \dim \mu_j} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} = \sum_{j < \bar{j}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma).$$

We consider now the second sum of the RHS: we have $u \leq \sum_{j \leq \bar{j}} \frac{\dim \mu_j}{\dim \lambda}$ by the definition of \bar{j} , thus

$$\sum_{j < \bar{j}} \dim \mu_j < i \leq u \dim \lambda \leq \sum_{j \leq \bar{j}} \dim \mu_j.$$

Hence $\pi^\lambda(\sigma)_{i,i} = \pi^{\mu_{\bar{j}}}(\sigma)_{\tilde{i},\tilde{i}}$, where $\tilde{i} = i - \sum_{j < \bar{j}} \dim \mu_j$, so that

$$0 < \tilde{i} \leq u \dim \lambda - \sum_{j < \bar{j}} \dim \mu_j = \bar{u} \dim \mu_{\bar{j}}.$$

Therefore

$$\sum_{\sum_{j < \bar{j}} \dim \mu_j < i \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} = \sum_{\tilde{i} \leq \bar{u} \dim \mu_{\bar{j}}} \frac{\pi^\lambda(\sigma)_{\tilde{i},\tilde{i}}}{\dim \lambda} = \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PT_{\bar{u}}^{\mu_{\bar{j}}}(\sigma),$$

and the proposition is proved. \square

We define $\sum_{y_j \leq v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma)$ to be the *main term for the partial trace* (denoted $MT_u^\lambda(\sigma)$), while $\frac{\dim \mu_{\bar{j}}}{\dim \lambda} PT_{\bar{u}}^{\mu_{\bar{j}}}(\sigma)$ is called the *remainder* (denoted $R_u^\lambda(\sigma)$). The main goal of this paper is to establish the asymptotic behavior of the main term for the partial trace. We settle some notation: if a permutation σ has cycle type ρ we write $m_k(\sigma) = m_k(\rho)$ for the number of parts of ρ which are equal to k or, equivalently, for the number of cycles of σ which are of length k . We define the *weights* of σ and ρ as $wt(\sigma) := wt(\rho) := |\text{Supp}(\sigma)| = |\rho| - m_1(\rho)$. We recall now a result due to [Ker93a], with a complete proof given by [IO02] and, independently, by [Hor98]:

Theorem 2.9. *Fix $\rho \vdash r \leq n$. The asymptotic behavior of the irreducible character $\hat{\chi}^\lambda$ is given by:*

$$n^{\frac{wt(\rho)}{2}} \hat{\chi}_\rho^\lambda \xrightarrow{d} \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k),$$

where $\{\xi_k\}_{k \geq 2}$ is a sequence of independent standard Gaussian random variables, and $\mathcal{H}_m(x)$, $m \geq 1$, is the Hermite polynomial of degree m defined by the recurrence relation $x\mathcal{H}_m(x) = \mathcal{H}_{m+1}(x) + m\mathcal{H}_{m-1}(x)$ and initial data $\mathcal{H}_0(x) = 1$ and $\mathcal{H}_1(x) = x$.

Our result is the following:

Theorem 2.10. *Let $v^\lambda = (F_{ct}^\lambda)^*(u)$, σ of cycle type ρ . Then*

$$n^{\frac{wt(\sigma)}{2}} MT_u^\lambda(\sigma) = n^{\frac{wt(\sigma)}{2}} \sum_{y_j \leq v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) \xrightarrow{d} u \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k).$$

Notice that the procedure of Proposition 2.8 can be iterated: consider $\sigma \in S_r$ and $s \in \mathbb{N}$ such that $n - r > s$. Set $\mu^{(s)} := \lambda$ and $\mu^{(s-1)} := \bar{\mu}$, where $\bar{\mu}$ is defined according to Proposition 2.8. Similarly set $u^{(s)} := u$ and $u^{(s-1)} := \bar{u}$, then we can rewrite (8) as

$$PT_u^\lambda(\sigma) = MT_{u^{(s)}}^{\mu^{(s)}}(\sigma) + \frac{\dim \mu^{(s-1)}}{\dim \lambda} PT_{u^{(s-1)}}^{\mu^{(s-1)}}(\sigma).$$

Applying again Proposition 2.8 to the second term of the right hand side, we see that there exists $\mu^{(s-2)} \nearrow \mu^{(s-1)}$ and $u^{(s-2)} \in [0, 1]$ such that

$$PT_u^\lambda(\sigma) = MT_{u^{(s)}}^{\mu^{(s)}}(\sigma) + \frac{\dim \mu^{(s-1)}}{\dim \lambda} MT_{u^{(s-1)}}^{\mu^{(s-1)}}(\sigma) + \frac{\dim \mu^{(s-2)}}{\dim \lambda} PT_{u^{(s-2)}}^{\mu^{(s-2)}}(\sigma).$$

By iterating we obtain the following proposition:

Proposition 2.11. *Let $\sigma \in S_r$. Set s such that $n - s > r$, then there exists a sequence of partitions $\mu^{(0)} \nearrow \mu^{(1)} \nearrow \dots \nearrow \mu^{(s)} = \lambda$ and a sequence of real numbers $0 \leq u_0, \dots, u_s < 1$ such that*

$$PT_u^\lambda(\sigma) = \sum_{i=1}^s \frac{\dim \mu^{(i)}}{\dim \lambda} MT_{u^{(i)}}^{\mu^{(i)}}(\sigma) + \frac{\dim \mu^{(0)}}{\dim \lambda} PT_{u^{(0)}}^{\mu^{(0)}}(\sigma).$$

2.5 Partial permutations

In this section we recall some results in the theory of partial permutations, introduced in [IK99]. All the definitions and results in this section can be found in [Fér12].

Definition 2.12. *A partial permutation is a pair (σ, d) , where $d \subset \mathbb{N}$ is finite and σ is a bijection $d \rightarrow d$.*

We call P_n the set of partial permutations (σ, d) such that $d \subseteq \{1, \dots, n\}$. The set P_n is endowed with the operation $(\sigma, d) \cdot (\sigma', d') = (\tilde{\sigma} \cdot \tilde{\sigma}', d \cup d')$, where $\tilde{\sigma}$ is the bijection from $d \cup d'$ to itself defined by $\tilde{\sigma}|_d = \sigma$ and $\tilde{\sigma}|_{d' \setminus d} = \text{Id}$; the same holds for $\tilde{\sigma}'$.

Define the algebra $\mathcal{B}_n = \mathbb{C}[P_n]$. There is an action of the symmetric group S_n on P_n defined by $\tau \cdot (\sigma, d) := (\tau \sigma \tau^{-1}, \tau(d))$, and we call \mathcal{A}_n the subalgebra of \mathcal{B}_n of the invariant elements under this action. Set

$$\alpha_{\rho;n} := \sum_{\substack{d \subseteq \{1, \dots, n\}, |d|=|\rho| \\ \sigma \in S_d \text{ of type } \rho}} (\sigma, d).$$

Proposition 2.13. *The family $(\alpha_{\rho;n})_{|\rho| \leq n}$ forms a linear basis for \mathcal{A}_n .*

There is a natural projection $\mathcal{B}_{n+1} \rightarrow \mathcal{B}_n$ which sends to 0 the partial permutations whose support contains $n + 1$. The projective limit through this projection is called $\mathcal{B}_\infty := \varprojlim \mathcal{B}_n$, and similarly $\mathcal{A}_\infty := \varprojlim \mathcal{A}_n$. The family $\alpha_\rho := (\alpha_{\rho;n})_{n \geq 1}$ forms a linear basis of \mathcal{A}_∞ .

We recall the definition of *Jucys-Murphy element*, described for example in [Juc66] and [Mur81], and its generalization as a partial permutation.

Definition 2.14. *The i -th Jucys-Murphy element is the element of the group algebra $\mathbb{C}[S_n]$ defined by $J_i := (1, i) + (2, i) + \dots + (i-1, i)$, $J_1 = 0$.*

The partial Jucys-Murphy element ξ_i is the element of \mathcal{B}_n defined by $\xi_i := \sum_{j < i} ((j, i), \{j, i\})$, $\xi_1 := 0$.

Proposition 2.15. *If f is a symmetric function, $f(\xi_1, \dots, \xi_n) \in \mathcal{A}_n$. The sequence $f_n = f(\xi_1, \dots, \xi_n)$ is an element of the projective limit \mathcal{A}_∞ , that we denote Γ_f ; moreover, $f \mapsto \Gamma_f$ is an algebra morphism. With the formalism of the virtual alphabet, we define Ξ as the projective limit of the sequence (ξ_1, \dots, ξ_n) for $n \rightarrow \infty$; equivalently, Ξ is the unique element of $\bigcup_{n=1}^\infty \mathbb{C}[S_n]$ such that $f(\Xi) := \Gamma_f$.*

For a partition $\nu = (\nu_1, \dots, \nu_q)$ and $n \geq |\nu|$ we consider $p_\nu(\xi_1, \dots, \xi_n) = \prod_{i=1}^q (\xi_1^{\nu_i} + \dots + \xi_n^{\nu_i})$. Then one can show that

$$p_\nu(\Xi) = \prod_i m_i(\nu)! \alpha_{\nu+\underline{1}} + \sum_{|\rho| < |\nu|+q} c_\rho \alpha_\rho,$$

with $\nu + \underline{1} = (\nu_1 + 1, \dots, \nu_q + 1)$ and c_ρ non-negative integers.

Let us fix some notation: we write $n^{\downarrow k}$ for the k -th falling factorial, that is, $n(n-1) \cdots (n-k+1)$, and $z_\rho = \prod_i \rho_i! \prod_i m_i(\rho)!$ for the order of the centralizer of a permutation of type ρ . We call

$$K_{\rho 1^{n-|\rho|}} = \frac{1}{\#\{\sigma \text{ of cycle type } \rho 1^{n-|\rho|}\}} \sum \sigma,$$

where the sum runs over the permutations $\sigma \in S_n$ of cycle type $\rho 1^{n-|\rho|}$. We consider a morphism of algebras $\varphi_n: \mathcal{A}_n \rightarrow Z(\mathbb{C}[S_n])$ which sends (σ, d) to σ . On the basis $(\alpha_{\rho;n})$ it acts as follows:

$$\varphi_n(\alpha_{\rho;n}) = \frac{n^{\downarrow |\rho|}}{z_\rho} K_{\rho 1^{n-|\rho|}}. \quad (9)$$

In particular $\hat{\chi}^\lambda(\varphi_n(\alpha_{\nu;n})) = \frac{1}{z_\nu} |\lambda|^{\downarrow |\nu|} \hat{\chi}_\nu^\lambda$. It follows from a result of Jucys [Juc74, Equation (12)] that

$$\hat{\chi}^\lambda(\varphi_n(f(\xi_1, \dots, \xi_n))) = f(\mathcal{C}_\lambda), \quad (10)$$

where f is a symmetric function and \mathcal{C}_λ is the multiset of contents of the diagram λ , $\mathcal{C}_\lambda = \{c(\square), \square \in \lambda\}$.

2.6 Shifted symmetric functions

We recall some results on the algebra of shifted symmetric functions Λ^* and their relations with \mathcal{A}_∞ . The results and definitions in this section follow [IK99].

Definition 2.16. *Let $\Lambda^*(n)$ be the algebra of polynomials with complex coefficients in x_1, \dots, x_n that become symmetric in the new variables $x'_i = x_i - i$. There is a natural projection $\Lambda^*(n+1) \rightarrow \Lambda^*(n)$ which sends x_{n+1} to 0. The algebra of shifted symmetric functions Λ^* is defined as the projective limit of $\Lambda^*(n)$ according to this projection.*

We can apply (shifted) symmetric functions to partitions in the following way: if f is a (shifted) symmetric function and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ a partition, then we set $f(\lambda) = f(\lambda_1, \lambda_2, \dots, \lambda_l)$.

As in the symmetric functions algebra, there are several interesting basis for the algebra of shifted symmetric functions. We present one of them:

Proposition 2.17. *There exists a family of shifted symmetric functions $\{p_\rho^\#\}_{\rho \vdash r}$ such that, for each $\lambda \vdash n$,*

$$p_\rho^\#(\lambda) = \begin{cases} n^{\downarrow r} \hat{\chi}_{\rho 1^{n-r}}^\lambda, & \text{if } n \geq r; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the family $\{p_\rho^\#\}$, where ρ runs over all partitions, forms a basis for Λ^* .

These functions are called *shifted power sums*; they were introduced in [OO98, Section 15], where it was proved that $p_\rho^\#(\lambda_1 - 1, \lambda_2 - 2, \dots, \lambda_l - l) = p_\rho(\lambda)$.

In order to simplify the notation we write $p_k^\#$, where k is a positive integer, instead of $p_{(k)}^\#$. Similarly, $p_{1^k}^\#$ replaces $p_{(1^k)}^\#$. As polynomials in the variables x_1, \dots, x_n, \dots the function $p_\rho^\#$ have degree $\deg p_\rho^\#(x_1, \dots, x_n, \dots) = |\rho|$. We are interested in two filtrations of the algebra Λ^* , which are described in [IK99, section 10]:

- $\deg(p_\rho^\#)_1 := |\rho|_1 := |\rho| + m_1(\rho)$. This filtration is usually referred as the *Kerov filtration*;
- $\deg(p_\rho^\#)_\mathbb{N} := |\rho|_\mathbb{N} = |\rho| + l(\rho)$, where $l(\rho) = \sum_{i \in \mathbb{N}} m_i(\rho)$ is the number of parts of ρ .

Lemma 2.18. *There is an isomorphism $F: \mathcal{A}_\infty \rightarrow \Lambda^*$ that sends α_ρ to $p_\rho^\#/z_\rho$.*

Our goal in this section is to develop [IO02, Proposition 4.12], which gives information about the top degree term of $p_\rho^\sharp \cdot p_\theta^\sharp$ for partitions ρ and θ , where top degree refers to the Kerov filtration. Consequently, we obtain results on the product $\alpha_\rho \cdot \alpha_\theta$ (Corollary 2.23).

We will write $V_k^<$ for the algebra whose basis is $\{p_\rho^\sharp \text{ such that } |\rho|_1 < k\}$. Notice that since $\text{deg}(\cdot)_1$ is a filtration,

$$p_\rho^\sharp \cdot V_k^< \subseteq V_{|\rho|_1+k}^<. \quad (11)$$

By abuse of notation we will often write $p_\rho^\sharp + V_k^<$ to indicate p_ρ^\sharp plus a linear combination of elements with $\text{deg}(\cdot)_1 < k$. We recall a result of Ivanov and Olshanski [IO02, Proposition 4.12]:

Lemma 2.19. *For any partition ρ and $k \geq 2$,*

$$p_\rho^\sharp p_k^\sharp = p_{\rho \cup k}^\sharp + \begin{cases} k \cdot m_k(\rho) \cdot p_{(\rho \setminus k) \cup 1^k}^\sharp + V_{|\rho|_1+k}^<, & \text{if } m_k(\rho) \geq 1 \\ V_{|\rho|_1+k}^<, & \text{if } m_k(\rho) = 0, \end{cases} \quad (12)$$

where the partition $\rho \setminus k$ is obtained by removing one part equal to k .

Lemma 2.20. *For a partition ρ ,*

$$p_\rho^\sharp p_{1^k}^\sharp = p_{\rho \cup 1^k}^\sharp + V_{|\rho|_1+2k}^<.$$

Proof. We prove the statement by induction on k , the case $k = 1$ being proved in [IO02, Proposition 4.11]. This implies that for any k

$$p_{1^k}^\sharp p_1^\sharp = p_{1^{k+1}}^\sharp + V_{2k+2}^<.$$

Suppose the lemma true for $k > 1$, then

$$\begin{aligned} p_\rho^\sharp p_{1^{k+1}}^\sharp &= p_\rho^\sharp p_{1^k}^\sharp p_1^\sharp + p_\rho^\sharp V_{2k+2}^< \\ &= p_{\rho \cup 1^k}^\sharp p_1^\sharp + V_{|\rho|_1+2k+2}^< \\ &= p_{\rho \cup 1^k \cup 1}^\sharp + V_{|\rho|_1+2k+2}^<, \end{aligned}$$

where we repetitively used property (11). □

Define $\tilde{V}_k^=$ as the algebra with basis $\{p_\theta^\sharp \text{ such that } |\theta|_1 = k \text{ and } m_1(\theta) > 0\}$, a slightly different statement as property (11) holds:

Lemma 2.21. *For a positive integer k and a partition ρ*

$$p_\rho^\sharp \cdot \tilde{V}_k^= \subseteq \tilde{V}_{|\rho|_1+k}^= + V_{|\rho|_1+k}^<. \quad (13)$$

Proof. It is enough to show that, for a partition θ such that $|\theta|_1 = k$ and $m_1(\theta) > 0$,

$$p_\rho^\sharp \cdot p_\theta^\sharp \subseteq \tilde{V}_{|\rho|_1+|\theta|_1}^= + V_{|\rho|_1+|\theta|_1}^<. \quad (14)$$

Set $\theta = \tilde{\theta} \cup 1$. By the previous lemma $p_\theta^\sharp = p_{\tilde{\theta}}^\sharp \cdot p_1^\sharp + V_{|\tilde{\theta}|_1+2}^<$ and

$$p_\rho^\sharp \cdot p_\theta^\sharp = p_\rho^\sharp \cdot p_{\tilde{\theta}}^\sharp \cdot p_1^\sharp + V_{|\rho|_1+|\tilde{\theta}|_1+2}^<.$$

Set $C_{\rho, \tilde{\theta}}^\tau$ to be the structure constants in the basis $\{p_\tau^\sharp\}$ of the product $p_\rho^\sharp \cdot p_{\tilde{\theta}}^\sharp$, that is,

$$p_\rho^\sharp \cdot p_{\tilde{\theta}}^\sharp = \sum_{|\tau|_1 \leq |\rho|_1 + |\tilde{\theta}|_1} C_{\rho, \tilde{\theta}}^\tau p_\tau^\sharp,$$

where the restriction $|\tau|_1 \leq |\rho|_1 + |\tilde{\theta}|_1$ is a consequence of (11). Thus

$$\begin{aligned} p_\rho^\sharp \cdot p_\theta^\sharp &= \sum_{|\tau|_1 \leq |\rho|_1 + |\tilde{\theta}|_1} C_{\rho, \tilde{\theta}}^\tau p_\tau^\sharp \cdot p_1^\sharp + V_{|\rho|_1 + |\tilde{\theta}|_1 + 2}^< \\ &= \sum_{|\tau|_1 \leq |\rho|_1 + |\tilde{\theta}|_1} C_{\rho, \tilde{\theta}}^\tau \left(p_{\tau \cup 1}^\sharp + V_{|\tau|_1+2}^< \right) + V_{|\rho|_1 + |\tilde{\theta}|_1 + 2}^< \\ &\subseteq \tilde{V}_{|\rho|_1 + |\tilde{\theta}|_1 + 2}^= + V_{|\rho|_1 + |\tilde{\theta}|_1 + 2}^< = \tilde{V}_{|\rho|_1 + |\theta|_1}^= + V_{|\rho|_1 + |\theta|_1}^<. \end{aligned}$$

This proves (14) and hence the lemma. □

Proposition 2.22. *Let ρ, θ be partitions. Then*

$$p_\rho^\sharp \cdot p_\theta^\sharp = p_{\rho \cup \theta}^\sharp + \tilde{V}_{|\rho|_1 + |\theta|_1}^= + V_{|\rho|_1 + |\theta|_1}^<.$$

Proof. We prove the statement by induction on the number $l(\theta) - m_1(\theta)$ of non-one parts of θ , with the base case being $\theta = 1^k$, shown in the previous lemma.

Consider now the claim true for $p_\rho^\sharp \cdot p_{\tilde{\theta}}^\sharp$ and set $\theta = \tilde{\theta} \cup k$ for $k \geq 2$. Then by Lemma 2.19

$$\begin{aligned} p_\theta^\sharp &= p_{\tilde{\theta} \cup k}^\sharp = p_{\tilde{\theta}}^\sharp \cdot p_k^\sharp + \begin{cases} k \cdot m_k(\tilde{\theta}) \cdot p_{(\tilde{\theta} \setminus k) \cup 1^k}^\sharp + V_{|\tilde{\theta}|_1}^<, & \text{if } m_k(\tilde{\theta}) \geq 1 \\ V_{|\tilde{\theta}|_1}^<, & \text{if } m_k(\tilde{\theta}) = 0, \end{cases} \\ &= p_{\tilde{\theta}}^\sharp \cdot p_k^\sharp + \tilde{V}_{|\tilde{\theta}|_1}^= + V_{|\tilde{\theta}|_1}^<. \end{aligned}$$

Because of Property (11) and Lemma (2.21),

$$p_\rho^\sharp \cdot p_\theta^\sharp = p_\rho^\sharp \cdot p_{\tilde{\theta}}^\sharp \cdot p_k^\sharp + \tilde{V}_{|\rho|_1 + |\tilde{\theta}|_1}^= + V_{|\rho|_1 + |\tilde{\theta}|_1}^<. \quad (15)$$

By the inductive step

$$p_{\tilde{\theta}}^\sharp \cdot p_k^\sharp = p_{\rho \cup \tilde{\theta}}^\sharp + \tilde{V}_{|\rho|_1 + |\tilde{\theta}|_1}^= + V_{|\rho|_1 + |\tilde{\theta}|_1}^<.$$

Hence, by applying again Lemma 2.19

$$\begin{aligned} p_\rho^\sharp \cdot p_{\tilde{\theta}}^\sharp \cdot p_k^\sharp &= p_{\rho \cup \tilde{\theta}}^\sharp \cdot p_k^\sharp + \tilde{V}_{|\rho|_1 + |\tilde{\theta}|_1 + k}^= + V_{|\rho|_1 + |\tilde{\theta}|_1 + k}^< \\ &= p_{\rho \cup \tilde{\theta} \cup k}^\sharp + \tilde{V}_{|\rho|_1 + |\tilde{\theta}|_1 + k}^= + V_{|\rho|_1 + |\tilde{\theta}|_1 + k}^<. \end{aligned}$$

We substitute the previous expression in (15)

$$p_\rho^\sharp \cdot p_\theta^\sharp = p_{\rho \cup \tilde{\theta} \cup k}^\sharp + \tilde{V}_{|\rho|_1 + |\tilde{\theta}|_1 + k}^= + V_{|\rho|_1 + |\tilde{\theta}|_1 + k}^<.$$

which concludes the proof. \square

We can obtain a similar result in the algebra \mathcal{A}_∞ by applying the isomorphism F^{-1} defined in 2.18. It is easy to see that $F^{-1}(\tilde{V}_k^=)$ is the space of linear combinations of α_ρ such that $|\rho|_1 = k$ and $m_1(\rho) > 0$. Similarly $F^{-1}(V_k^<)$ is the space of linear combinations of α_ρ such that $|\rho|_1 < k$.

Corollary 2.23. *Let ρ, θ be partitions. Then*

$$\alpha_\rho \cdot \alpha_\theta = \alpha_{\rho \cup \theta} + F^{-1}(\tilde{V}_k^=) + F^{-1}(V_k^<).$$

3 Asymptotic of the main term for the partial trace

In this section we prove Theorem 2.10. The main step is to prove that $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{wt(\rho)}{2}})$, where ρ is a fixed partition, $wt(\rho) = |\rho| - m_1(\rho)$, λ is a partition of n and $\mu \nearrow \lambda$.

Fro a partition $\nu = (\nu_1, \dots, \nu_q)$ we define slightly modified power sums \tilde{p}_ν ; we will show (Lemma 3.5 and Equation (10)) that

$$\hat{\chi}^\lambda(\varphi_n(\tilde{p}_\nu(\xi_1, \dots, \xi_n))) - \hat{\chi}^\mu(\varphi_{n-1}(\tilde{p}_\nu(\xi_1, \dots, \xi_{n-1}))) = \tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_\mu) \in o_P(n^{\frac{|\nu|+q}{2}}). \quad (16)$$

In order to translate this result on a bound on $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu$ we need to study the expansion of the modified power sums evaluated on Ξ , the infinite set of Jucys-Murphy elements in the theory of partial permutations:

$$\tilde{p}_\nu(\xi_1, \dots, \xi_n) = \sum_{\varsigma} c_\varsigma \alpha_{\varsigma; n}.$$

With the right choice of ν and the right filtration we will prove (Proposition 3.6) that

$$\tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_\mu) = \hat{\chi}^\lambda \circ \varphi_n \left(\sum_{\varsigma} c_\varsigma \alpha_{\varsigma; n} \right) - \hat{\chi}^\mu \circ \varphi_{n-1} \left(\sum_{\varsigma} c_\varsigma \alpha_{\varsigma; n-1} \right) = \hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu + o_P(n^{-\frac{wt(\rho)}{2}}). \quad (17)$$

Comparing (16) and (17), this gives

$$\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{wt(\rho)}{2}}).$$

Remark 3.1. It is easy to see that $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{|\rho|-l(\rho)}{2}})$, where $l(\rho)$ is the number of parts of ρ ; for example, considering balanced diagrams, in [FŚ11] the authors prove an explicit formula for the normalized character, which implies $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{|\rho|-l(\rho)}{2}})$. Alternatively one can look at the descriptions of normalized characters expressed as polynomials in terms of free cumulants, studied by Biane in [Bia03]. The action of removing a box from a random partition λ affects free cumulants in a sense described in [DFŚ10], which furnish the aforementioned bound. On the other hand we needed a stronger result, namely Proposition 3.6, and for this reason we introduce the modified power sums.

Through the section, σ will be a fixed permutation, ρ its cycle type, and we consider $\nu = (\nu_1, \dots, \nu_q)$, such that $\rho = \nu + \underline{1} = (\nu_1 + 1, \dots, \nu_q + 1, 1, \dots, 1)$.

3.1 Modified power sums

Definition 3.2. For a partition $\nu = (\nu_1, \dots, \nu_q)$ the modified power sum is

$$\tilde{p}_\nu(x_1, x_2, \dots) = \prod_{i=1}^q \left(p_{\nu_i}(x_1, x_2, \dots) - \text{Cat}\left(\frac{\nu_i}{2}\right) \cdot \left(\frac{\nu_i}{2}\right)! \alpha_{\left(\frac{\nu_i}{2}+1\right)} \right),$$

where $\text{Cat}(k) = \frac{\binom{2k}{k}}{k+1}$ is the k -th Catalan number if k is an integer, and 0 otherwise.

Lemma 3.3. Set $\nu = (\nu_1, \dots, \nu_q)$, then

$$\tilde{p}_\nu(\Xi) = \sum_{\varsigma} c_{\varsigma} \alpha_{\varsigma}, \quad (18)$$

for some non-negative integers c_{ς} . We have

1. the sum runs over the partitions ς such that $|\varsigma|_1 \leq |\nu|_{\mathbb{N}}$;
2. $c_{\nu+\underline{1}} = \prod_i (m_i(\nu)!)$;
3. if $c_{\varsigma} \neq 0$, $|\varsigma|_1 = |\nu|_{\mathbb{N}}$ and $\varsigma \neq \nu + \underline{1}$ then $m_1(\varsigma) > 0$.

Proof.

Part 1. Since $\deg(\alpha_{\varsigma})_1 = |\varsigma|_1$ is a filtration, it is enough to prove the statement for $q = 1$, that is, $\nu = (k)$ for some positive integer k . We consider

$$p_k(\Xi) = \sum_{1 \leq j_1, \dots, j_k < h} ((j_1, h), \{j_1, h\}) \cdot \dots \cdot ((j_k, h), \{j_k, h\}). \quad (19)$$

Let $(\sigma, d) = ((j_1, h) \cdot \dots \cdot (j_k, h), \{j_1, \dots, j_k, h\})$ be a term of the previous sum and let $\varsigma \vdash |d|$ be the cycle type of σ . This is the outline of the proof: firstly, we show an upper bound for $|\varsigma|_1$, that is, $|\varsigma|_1 \leq k + 2$. Then we check some conditions that σ must satisfy in order to have $|\varsigma|_1 = k + 2$ (if such a σ exists). We prove that $|\varsigma|_1 = k + 2$ if and only if $(\sigma, d) = (\text{Id}, d)$ with $|d| = k/2 + 1$ and k even. Finally, we calculate how many times the partial permutation (Id, d) , with $|d| = k/2 + 1$, appears in the modified power sum $p_k(\Xi)$. This number is the coefficient of $\alpha_{\left(\frac{k}{2}+1\right)}$ in $p_k(\Xi)$, and we show that it is equal to $\text{Cat}\left(\frac{k}{2}\right) \cdot \left(\frac{k}{2}\right)!$. Since we defined $\tilde{p}_k(\Xi) = p_k(\Xi) - \text{Cat}\left(\frac{k}{2}\right) \cdot \left(\frac{k}{2}\right)! \cdot \alpha_{\left(\frac{k}{2}+1\right)}$ this shows that all the terms in $\tilde{p}_k(\Xi)$ satisfy $|\varsigma|_1 \leq k + 1 = |\nu|_{\mathbb{N}}$ when $\nu = (k)$.

We first estimate

$$\begin{aligned} |\varsigma|_1 &= |\{j_1, \dots, j_k, h\}| + m_1(\varsigma) \\ &= 2\#\{j_i, \text{ s.t. } \sigma j_i = j_i\} + \#\{j_i, \text{ s.t. } \sigma j_i \neq j_i\} + 2\delta(h \text{ is fixed by } \sigma) + \delta(h \text{ is not fixed by } \sigma), \end{aligned}$$

where δ is the Kronecker delta. We stress out that, when counting the set cardinalities above, we do not count multiplicities; for example, if

$$\sigma = (1, 5)(2, 5)(2, 5)(1, 5)(3, 5) = (1)(2)(3, 5), \quad \text{then}$$

$$\#\{j_i, \text{ s.t. } \sigma j_i = j_i\} = \#\{1, 2\} = 2, \quad \#\{j_i, \text{ s.t. } \sigma j_i \neq j_i\} = \#\{3\} = 1, \quad \varsigma = (2, 1, 1).$$

Notice that in the sequence (j_1, \dots, j_k) all fixed points must appear at least twice, while non fixed points must appear at least once, hence

$$2\#\{j_i, \text{ s.t. } \sigma j_i = j_i\} + \#\{j_i, \text{ s.t. } \sigma j_i \neq j_i\} \leq k,$$

while obviously,

$$2\delta(h \text{ is fixed by } \sigma) + \delta(h \text{ is not fixed by } \sigma) \leq 2.$$

Thus $|\varsigma|_1 \leq k + 2$.

Suppose there exists σ such that $|\varsigma|_1 = k + 2$, then from the proof of the inequality $|\varsigma|_1 \leq k + 2$ we know that σ satisfies

$$\text{for each } i, j_i \text{ appears at most twice;} \quad (20)$$

$$j_i \text{ is fixed by } \sigma \text{ iff it appears exactly twice in the multiset } \{j_1, \dots, j_k\}; \quad (21)$$

$$h \text{ is a fixed point.} \quad (22)$$

We prove by induction on k that if $|\varsigma|_1 = k + 2$ and α_ς appears in the sum (19) then k is even and $\varsigma = (1^{\frac{k}{2}+1})$. If $k = 1$ then $\varsigma = (2)$ and $|\varsigma|_1 = 2 < 3 = k + 2$. If $k = 2$ then ς can be either $\varsigma = (3)$ or $\varsigma = (1, 1)$. By our request that $|\varsigma|_1 = 4 = k + 2$ we see that we must have $\varsigma = (1, 1)$. Consider now the statement true up to $k - 1$ and $\sigma = (j_1, h) \cdot \dots \cdot (j_k, h)$. By property (22) h is fixed, so that j_k must appear at least twice, and by (20) j_k appears exactly twice. Hence, by (21) j_k is fixed, thus it exists a unique $l < k - 1$ such that $j_{l+1} = j_k$ and

$$\sigma = (j_1, h) \cdot \dots \cdot (j_l, h) \cdot (j_k, h) \cdot (j_{l+2}, h) \cdot \dots \cdot (j_{k-1}, h)(j_k, h) = \tau_1(j_k, h)\tau_2(j_k, h) = \tau_1\tau_2,$$

where

$$\tau_1 = \begin{cases} \text{Id} & \text{if } l = 0 \\ (j_1, h) \cdot \dots \cdot (j_l, h) & \text{if } l > 0, \end{cases} \quad \tau_2 = \begin{cases} \text{Id} & \text{if } l = k - 2 \\ (j_{l+2}, j_k) \cdot \dots \cdot (j_{k-1}, j_k) & \text{if } l < k - 2. \end{cases}$$

If $l = 0$ or $l = k - 2$ we can apply induction on either τ_1 or τ_2 and the result follows, so consider $0 < l < k - 2$. We claim that the sets $\{j_1, \dots, j_l\}$ and $\{j_{l+2}, \dots, j_k\}$ are disjoint. Notice first that j_k does not appear in $\{j_1, \dots, j_l\}$. Suppose $j_a = j_b$ for some $a \leq l < b$, and choose b minimal with this property. Then j_b is not fixed by τ_2 , that is, $\tau_2(j_b) = j_{\tilde{b}}$ with either $\tilde{b} < b$ or $\tilde{b} = k$. In the first case ($\tilde{b} < b$), by minimality of b , $j_{\tilde{b}}$ does not appear in τ_1 ; in the second case ($\tilde{b} = k$) we know that j_k does not appear in $\{j_1, \dots, j_l\}$. Hence in either case $\sigma(j_b) = j_{\tilde{b}}$. This is a contradiction, since j_b appears twice in σ and therefore must be fixed (property (21)). This proves that $\{j_1, \dots, j_l\} \cap \{j_{l+2}, \dots, j_k\} = \emptyset$.

Therefore τ_1 and τ_2 respect properties (20),(21) and (22), and we can apply the inductive hypothesis to obtain that $\tau_1 = \text{Id} = \tau_2$ and both l and $k - l - 2$ are even. We conclude hence that if $|\varsigma|_1 = k + 2$ and α_ς appears in the sum (18) then $\varsigma = (1^{\frac{k}{2}+1})$ and k even.

We assume now that k is even and we calculate the coefficient of $\alpha_{(1^{\frac{k}{2}+1})}$ in $p_k(\Xi)$, which is equal to the coefficient of $(\text{Id}, d) = ((j_1, h) \cdot \dots \cdot (j_k, h), \{j_1, \dots, j_k, h\})$ in the sum (19), for a fixed d of cardinality $k/2 + 1$. In order to do this we count the number of lists $L = ((j_1, h), \dots, (j_k, h))$ such that $\{j_1, \dots, j_k, h\} = d$ and $(j_1, h) \cdot \dots \cdot (j_k, h) = \text{Id}$. We call \mathcal{L}_d the set of these lists.

Define a *pair set partition* of the set $[k] = \{1, \dots, k\}$ as a set partition $A = \{\{r_1, s_1\}, \dots, \{r_{\frac{k}{2}}, s_{\frac{k}{2}}\}\}$ of $[k]$ into pairs. Such a set partition is said to be *crossing* if $r_a < r_b < s_a < s_b$ for some $a, b \leq k/2$, otherwise the set partition is said to be *non crossing*. Calling \mathcal{S}_k the set of non crossing pair set partitions, it is known that $|\mathcal{S}_k| = \text{Cat}(k/2)$, see [Hor98]. We build a map $\psi_k: \mathcal{L}_d \rightarrow \mathcal{S}_k$ and we prove that this map is $(\frac{k}{2})!$ -to-one, which implies that $|\mathcal{L}_d| = c_{(1^{\frac{k}{2}+1})} = \text{Cat}(\frac{k}{2}) \cdot (\frac{k}{2})!$.

Let $L \in \mathcal{L}_d$, $L = (L_1, \dots, L_k) = ((j_1, h), \dots, (j_k, h))$; we have proven that, since $(j_1, h) \cdot \dots \cdot (j_k, h) = \text{Id}$ and $|d| = |\{j_1, \dots, j_k, h\}| = k/2 + 1$, then each element (j_i, h) must appear exactly twice in L . We construct a pair partition $\psi_k(L)$ such that a pair $\{r, s\} \in \psi_k(L)$ iff $j_r = j_s$. By [Hor98, Lemma 2] this set partition is non crossing. This map is clearly surjective, although not injective: every permutation γ acting on $d = \{j_1, \dots, j_k, h\}$ which fixes h acts also on \mathcal{L}_d : $\gamma(L) = ((\gamma(j_1), h), \dots, (\gamma(j_k), h))$. Notice that $h > j_1, \dots, j_k$ and this is why $\gamma(h) = h$ in order to have an action on \mathcal{L}_d . Moreover $\psi_k(L) = \psi_k(L')$ if and only if $L = \gamma(L')$ for some γ in $S_{d \setminus h}$. Thus ψ_k is a $(\frac{k}{2})!$ -to-one map, hence $c_{(1^{\frac{k}{2}+1})} = \text{Cat}(\frac{k}{2}) \cdot (\frac{k}{2})!$ and the first part of the proof is concluded.

Part 2. The second statement is shown in [Fér12, section 2].

Part 3. This claim is proven by induction on q , the number of parts of ν . The base case is $q = 1$ and

$$\tilde{p}_k(\Xi) = \sum_{|\zeta|_1 \leq k+1} c_\zeta \alpha_\zeta.$$

We consider the α_ζ 's appearing in the sum such that $|\zeta|_1 = k + 1$ and $m_1(\zeta) = 0$, so that $|\zeta| = k + 1$. Expanding the sum in a similar way as Equation (19), we see that $|\zeta| = k + 1 = \#\{j_1, \dots, j_k, h\}$, and all the elements in this set must be pairwise different. Hence $\sigma = (j_1, h) \cdot \dots \cdot (j_k, h) = (h, j_k, \dots, j_1)$ and $\varsigma = (k + 1)$. Thus the statement for $q = 1$ is proved.

Let $\nu = (\nu_1, \dots, \nu_q)$ with $q \geq 2$. Set $\tilde{\nu} = (\nu_1, \dots, \nu_{q-1})$ and suppose, by inductive hypothesis, that the assertion is true for $\tilde{p}_{\tilde{\nu}}(\Xi)$. Then

$$\begin{aligned} \tilde{p}_\nu(\Xi) &= \tilde{p}_{\tilde{\nu}}(\Xi) \tilde{p}_{\nu_q}(\Xi) \\ &= \left(\sum_{|\zeta|_1 \leq |\tilde{\nu}|_{\mathbb{N}}} c_\zeta \alpha_\zeta \right) \cdot \left(\sum_{|\theta|_1 \leq \nu_q + 1} c_\theta \alpha_\theta \right) \\ &= \sum_{\substack{|\zeta|_1 \leq |\tilde{\nu}|_{\mathbb{N}} \\ |\theta|_1 \leq \nu_q + 1}} c_\zeta c_\theta \cdot \alpha_\zeta \alpha_\theta. \end{aligned}$$

We apply now Corollary 2.23 and obtain

$$\tilde{p}_\nu(\Xi) = \sum_{\substack{|\zeta|_1 \leq |\tilde{\nu}|_{\mathbb{N}} \\ |\theta|_1 \leq \nu_q + 1}} c_\zeta c_\theta \alpha_{\zeta \cup \theta} + F^{-1}(\tilde{V}_{|\nu|_{\mathbb{N}}}^{\Xi}) + F^{-1}(V_{|\nu|_{\mathbb{N}}}^{\zeta}).$$

Hence there exists only one term in the previous sum such that $|\zeta|_1 = |\zeta \cup \theta|_1 = |\nu|_{\mathbb{N}}$ and $m_1(\zeta) = 0$, that is, $\zeta = \nu + \underline{1}$, and the proof is completed. \square

Let now $\lambda \vdash n$ be a random partition distributed with the Plancherel measure. We say that a function X on the set of partitions is $X \in o_P(n^\beta)$ if $n^{-\beta} X(\lambda) \xrightarrow{P} 0$. Similarly, $X \in O_P(n^\beta)$ is *stochastically bounded* by n^β if for any $\epsilon > 0$ there exists $M > 0$ such that $P_{PL}(|X(\lambda) n^{-\beta}| > M) \leq \epsilon$. For example, as a consequence of Kerov's result on the convergence of characters (Theorem 2.9), $\hat{\chi}_\rho^\lambda \in O_P(n^{-wt(\rho)/2})$. If a function $X(\lambda, \mu_j)$ depends also on a subpartition $\mu_j \nearrow \lambda$, by $X(\lambda, \mu_j) \in o_P(n^\beta)$ we mean that $\max_j X(\lambda, \mu_j) \in o_P(n^\beta)$, and similarly for the notion of stochastic boundedness.

Lemma 3.4. *For each partition ρ we have that $\hat{\chi}^\lambda(\varphi_n(\alpha_{\rho;n})) \in O_P(n^{\frac{|\rho|_1}{2}})$, where $|\rho|_1 = |\rho| + m_1(\rho)$.*

Proof. From Equation (9), $\hat{\chi}^\lambda(\varphi_n(\alpha_{\rho;n})) = \frac{c_\rho}{z_\rho} n^{|\rho|} \hat{\chi}_\rho^\lambda$. Since $\hat{\chi}_\rho^\lambda \in O_P(n^{-wt(\rho)/2})$ the lemma follows. \square

Lemma 3.5. *Let $\mu_j \nearrow \lambda$. In probability*

$$\hat{\chi}^\lambda(\varphi_n(\tilde{p}_\nu(\xi_1, \dots, \xi_n))) - \hat{\chi}^{\mu_j}(\varphi_{n-1}(\tilde{p}_\nu(\xi_1, \dots, \xi_{n-1}))) = \tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_{\mu_j}) \in o_P(n^{\frac{|\nu|_1 + q}{2}}).$$

Proof. The first equality come from the fact that $\hat{\chi}^\lambda \circ \varphi_n$ applied to a symmetric function of partial Jucys-Murphy elements equals the same symmetric function evaluated on the contents of λ . We prove now the asymptotic bound. Consider first the case $q = 1$, i.e., $\nu = (k)$ for some positive integer k , in which case

$$\begin{aligned} \tilde{p}_k(\mathcal{C}_\lambda) &= \tilde{p}_k(\mathcal{C}_{\mu_j} \cup y_j) = \tilde{p}_k(\mathcal{C}_{\mu_j}) + y_j^k + \frac{\text{Cat}(\frac{k}{2})}{\frac{k}{2} + 1} \left(n^{\downarrow(\frac{k}{2}+1)} - (n-1)^{\downarrow(\frac{k}{2}+1)} \right) \\ &= \tilde{p}_k(\mathcal{C}_{\mu_j}) + y_j^k + \text{Cat}\left(\frac{k}{2}\right) (n-1)^{\downarrow\frac{k}{2}}. \end{aligned}$$

Notice that $y_j < \max\{\lambda_1, \lambda'_1\}$ since y_j is the content of a box of λ . It is shown in [Rom15, Lemma 1.5] that, with probability that goes to 1, both λ_1 and λ'_1 are smaller than $3\sqrt{n}$, hence for each subpartition μ_j of λ , one has $y_j < 3\sqrt{n}$ with probability 1 for n big enough. Thus $\tilde{p}_k(\mathcal{C}_\lambda) - \tilde{p}_k(\mathcal{C}_{\mu_j}) \in O_P(n^{\frac{k}{2}}) \in o_P(n^{\frac{k+1}{2}})$.

In the general case $q > 1$ we expand the product, obtaining

$$\tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_{\mu_j}) = \sum_{A \subseteq \{1, \dots, q\}} \prod_{i \in A} \tilde{p}_{\nu_i}(\mathcal{C}_{\mu_j}) \prod_{i \notin A} \left(y_j^{\nu_i} + \text{Cat}\left(\frac{\nu_i}{2}\right) (n-1)^{\downarrow\frac{\nu_i}{2}} \right).$$

We use now Lemma 3.3 and 3.4, which show that the factor $\prod_{i \in A} \tilde{p}_{\nu_i}(\mathcal{C}_{\mu_j})$ is in $O_P(n^{\frac{1}{2}(\sum_{i \in A} \nu_i + |A|)})$ and

$$\prod_{i \notin A} \left(y_j^{\nu_i} + \text{Cat} \left(\frac{\nu_i}{2} \right) (n-1)^{\lfloor \frac{\nu_i}{2} \rfloor} \right) \in O_P(n^{\frac{1}{2} \sum_{i \notin A} \nu_i}).$$

Therefore $\tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_{\mu_j}) \in O_P(n^l)$, with

$$l = \max_{A \subsetneq \{1, \dots, q\}} \frac{1}{2} \sum_i \nu_i + \frac{|A|}{2} < \frac{|\nu| + q}{2}. \quad \square$$

Proposition 3.6. *Let $\lambda \vdash n$, $\mu \nearrow \lambda$. Let $\rho \vdash r < n$ then*

$$\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{wt(\rho)}{2}}).$$

Proof. Set $\nu = (\nu_1, \dots, \nu_q)$ such that $\rho = (\nu_1 + 1, \dots, \nu_q + 1, 1, \dots, 1)$. obviously, $\hat{\chi}_{\nu+1}^\lambda = \hat{\chi}_\rho^\lambda$ and $\hat{\chi}_{\nu+1}^\mu = \hat{\chi}_\rho^\mu$; moreover, $|\nu| + q = wt(\rho)$. We want to prove that $\hat{\chi}_{\nu+1}^\lambda - \hat{\chi}_{\nu+1}^\mu \in o_P(n^{-\frac{|\nu|+q}{2}})$. We prove the statement by induction on $|\nu + \underline{1}|_1 = |\nu| + q$. The base case is $\nu + \underline{1} = (1)$ and $\hat{\chi}_{(1)}^\lambda = 1 = \hat{\chi}_{(1)}^\mu$, so the proposition is trivially true.

Consider $n^{-\frac{|\nu|+q}{2}} (\tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_\mu)) \in o_P(1)$ because of the previous lemma. It can be rewritten as

$$n^{-\frac{|\nu|+q}{2}} \sum_{\substack{\varsigma \text{ s.t.} \\ |\varsigma|_1 \leq |\nu|+q}} (\hat{\chi}^\lambda \circ \varphi_n - \hat{\chi}^\mu \circ \varphi_{n-1}) (c_\varsigma \alpha_{\varsigma; n}) \xrightarrow{P} 0,$$

where the coefficients c_ς are described in Lemma 3.3.

Equivalently

$$\begin{aligned} n^{-\frac{|\nu|+q}{2}} (\tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_\mu)) &= n^{-\frac{|\nu|+q}{2}} \sum_{|\varsigma|_1 \leq |\nu|+q} \frac{c_\varsigma}{z_\varsigma} \left(n^{|\varsigma|} \hat{\chi}_\varsigma^\lambda - (n-1)^{|\varsigma|} \hat{\chi}_\varsigma^\mu \right) \\ &= n^{-\frac{|\nu|+q}{2}} \sum_{|\varsigma|_1 \leq |\nu|+q} \frac{c_\varsigma}{z_\varsigma} \left(\hat{\chi}_\varsigma^\lambda (n^{|\varsigma|} - (n-1)^{|\varsigma|}) + (n-1)^{|\varsigma|} (\hat{\chi}_\varsigma^\lambda - \hat{\chi}_\varsigma^\mu) \right). \end{aligned}$$

We split the previous sum and notice that $n^{-\frac{|\nu|+q}{2}} \sum_\varsigma \frac{c_\varsigma}{z_\varsigma} |\varsigma| (n-1)^{\downarrow(|\varsigma|-1)} \hat{\chi}_\varsigma^\lambda \xrightarrow{P} 0$. Indeed

$$(n-1)^{\downarrow(|\varsigma|-1)} \cdot n^{-\frac{|\nu|+q}{2}} \leq n^{\frac{|\nu|+q}{2}-1} \quad \text{and} \quad n^{\frac{|\nu|+q}{2}-1} \hat{\chi}_\varsigma^\lambda \xrightarrow{P} 0$$

since $|\varsigma|_1 \leq |\nu| + q$.

We deal with the sum $n^{-\frac{|\nu|+q}{2}} \sum_\varsigma \frac{c_\varsigma}{z_\varsigma} ((n-1)^{|\varsigma|} (\hat{\chi}_\varsigma^\lambda - \hat{\chi}_\varsigma^\mu))$. We separate in this sum the terms with $|\varsigma|_1 = |\nu| + q$ and $\varsigma \neq \nu + \underline{1}$, the terms with $|\varsigma|_1 < |\nu| + q$, and the term corresponding to $\varsigma = \nu + \underline{1}$.

- Case $|\varsigma|_1 = |\nu| + q$ and $\varsigma \neq \nu + \underline{1}$: we want to estimate

$$n^{-\frac{|\nu|+q}{2}} \sum_{\substack{|\varsigma|_1 = |\nu|+q \\ m_1(\varsigma) > 0}} \frac{c_\varsigma}{z_\varsigma} \left((n-1)^{|\varsigma|} (\hat{\chi}_\varsigma^\lambda - \hat{\chi}_\varsigma^\mu) \right),$$

where the restriction $m_1(\varsigma) > 0$ is a consequence of Lemma 3.3, part 3. We consider one term of the previous sum and we write $\tilde{\nu} := \varsigma - \underline{1}$, removed of the 0 parts. Notice as before that $\hat{\chi}_\varsigma^\lambda = \hat{\chi}_{\tilde{\nu}+1}^\lambda$, and $\hat{\chi}_\varsigma^\mu = \hat{\chi}_{\tilde{\nu}+1}^\mu$. Thus $|\tilde{\nu} + \underline{1}| < |\varsigma|_1 = |\nu| + q$ and we can apply the induction hypothesis. Therefore $\hat{\chi}_{\tilde{\nu}+1}^\lambda - \hat{\chi}_{\tilde{\nu}+1}^\mu \in o_P(n^{-\frac{|\tilde{\nu}+1|}{2}})$ and

$$n^{-\frac{|\varsigma|_1}{2}} \cdot (n-1)^{|\varsigma|} (\hat{\chi}_\varsigma^\lambda - \hat{\chi}_\varsigma^\mu) \in o_P \left(n^{\frac{|\varsigma|_1}{2} - \frac{m_1(\varsigma)}{2} - \frac{|\tilde{\nu}+1|}{2}} \right) = o_P(1).$$

- Case $|\varsigma|_1 < |\nu| + q$. We can apply induction again, to have $(\hat{\chi}_\varsigma^\lambda - \hat{\chi}_\varsigma^\mu) \in o_P(n^{-\frac{|\varsigma|_1 - m_1(\varsigma)}{2}})$. Therefore

$$n^{-\frac{|\nu|+q}{2}} \cdot (n-1)^{|\varsigma|} (\hat{\chi}_\varsigma^\lambda - \hat{\chi}_\varsigma^\mu) \in o_P \left(n^{-\frac{|\nu|+q}{2} + \frac{|\varsigma|_1}{2}} \right) \subseteq o_P(1).$$

We obtain thus that

$$n^{-\frac{|\nu|+q}{2}} \cdot (n-1)^{|\nu|+q} \frac{\prod_i m_i(\nu)!}{z_{\nu+1}} (\hat{\chi}_{\nu+1}^\lambda - \hat{\chi}_{\nu+1}^\mu) + o_P(1) \xrightarrow{P} 0,$$

which proves the statement. \square

Proof of Theorem 2.10. Fix $\sigma \in S_r$ and let ρ be its cycle type. We have

$$MT_u^\lambda(\sigma) = \sum_{y_j \leq v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}_\rho^{\mu_j} = \sum_{y_j \leq v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}_\rho^\lambda + o_P(n^{-\frac{|\nu|+q}{2}}),$$

since $\sum \dim \mu_j / \dim \lambda \leq 1$, and the previous proposition. Hence

$$n^{\frac{|\nu|+q}{2}} MT_u^\lambda(\sigma) = n^{\frac{|\nu|+q}{2}} \sum_{y_j \leq v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}_\rho^\lambda + o_P(1).$$

Finally, by Lemma 2.4 and Theorem 2.9, we obtain $n^{\frac{|\nu|+q}{2}} MT_u^\lambda(\sigma) \xrightarrow{d} u \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k)$. \square

4 Sum of the entries of an irreducible representation

In this chapter our goal is to describe the sum of the entries of the matrix associated to a Young seminormal representation, up to a certain index (depending on the dimension of the representation). We stress out that the objects we study really depend on the representation matrix, and change, for example, under isomorphisms of the representation. Some calculations are similar to those in the previous chapter: first we consider the sum of all the entries in the matrix (before this role was played by the trace), and then we study the sum of the entries whose indices (i, j) satisfy $i \leq u \dim \lambda$, $j \leq u \dim \lambda$, while before we were considering the partial trace.

4.1 Total sum

Definition 4.1. Define the normalized total sum associated to an irreducible representation and a permutation $\sigma \in S_r$ as

$$TS^\lambda(\sigma) := \sum_{i, j \leq \dim \lambda} \frac{\pi^\lambda(\sigma)_{i, j}}{\dim \lambda}$$

The following is the main result of the section:

Theorem 4.2. Fix $\sigma \in S_r$ and let $\lambda \vdash n$. Define the two real numbers

$$m := \mathbb{E}_{PL}^r [TS^\nu(\sigma)] \quad \text{and} \quad v := \binom{r}{2} \mathbb{E}_{PL}^r \left[\hat{\chi}_{(2, 1, \dots, 1)}^\nu TS^\nu(\sigma) \right],$$

Where $\mathbb{E}_{PL}^r[X]$ is the average of the random variable X considered with the Plancherel measure $(\dim \nu)^2 / r!$ for $\nu \vdash r$. Then

$$\sqrt{n} \cdot (TS^\lambda(\sigma) - m) \xrightarrow{d} \mathcal{N}(0, 2v^2),$$

a normal random variable of variance $2v^2$.

Note that the limit $\mathcal{N}(0, 2v^2)$ is degenerate in the case $v = 0$. It is nontrivial to see when v is nonzero. At the end of the section we study the values of v and m when σ is an adjacent transposition, and we write the explicit values of v and m for permutations of S_4 . In order to prove the Theorem, we need some preliminary results.

Proposition 4.3. Let $\lambda \vdash n$ and $r \leq n$. There exists a bijection ϕ_r between

$$\text{SYT}(\lambda) \stackrel{\phi_r}{\simeq} \bigsqcup_{\nu \vdash r} \text{SYT}(\nu) \times \text{SYT}(\lambda/\nu)$$

Proof. Let T be a standard Young tableau of shape λ , the image $\phi_r(T) = (U, V)$ of T is defined as follows: the boxes of T whose entries are smaller or equal than r identify a tableau U of shape $\nu \subseteq \lambda$. Define now V as a standard Young tableau of skew shape λ/ν , and in each box write $a - r$, where a is the value inside the corresponding box in T . \square

Example 4.4. Here is an example for $\lambda = (6, 4, 3, 3, 3, 1)$ and $r = 8$:

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 8 & 13 \\ \hline 3 & 7 & 9 & 16 & & \\ \hline 4 & 12 & 17 & & & \\ \hline 10 & 15 & 19 & & & \\ \hline 11 & 18 & 20 & & & \\ \hline 14 & & & & & \\ \hline \end{array} \leftrightarrow \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 8 \\ \hline 3 & 7 & & & \\ \hline 4 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 8 \\ \hline 4 & 9 \\ \hline 2 & 7 & 11 \\ \hline 3 & 10 & 12 \\ \hline 6 & & \\ \hline \end{array}, \boxed{5} \right)$$

Lemma 4.5. Let $\lambda \vdash n, \sigma \in S_r$ with $r \leq n$. For two standard Young tableaux $T, T' \in \text{SYT}(\lambda)$, set $\phi_r(T) = (U, V)$ and $\phi_r(T') = (U', V')$, then

$$\pi^\lambda(\sigma)_{T, T'} = \begin{cases} 0 & \text{if } V \neq V' \\ \pi^\nu(\sigma)_{U, U'} & \text{if } V = V', \end{cases}$$

where $\nu = \text{sh}(U) = \text{sh}(U')$ is the shape of the tableau U in the second case.

Proof. We will prove the lemma by induction on the number of factors in the (minimal) decomposition of σ into adjacent transpositions. Recall that if $\sigma = (k, k+1)$, $k < r$, by Definition 2.5

$$\pi^\lambda((k, k+1))_{T, T'} = \begin{cases} 1/d_k(T) & \text{if } T = T'; \\ \sqrt{1 - \frac{1}{d_k(T)^2}} & \text{if } (k, k+1)T = T'; \\ 0 & \text{else.} \end{cases}$$

Hence if $V \neq V'$ then $\pi^\lambda((k, k+1))_{T, T'} = 0$, otherwise $\pi^\lambda((k, k+1))_{T, T'} = \pi^\nu((k, k+1))_{U, U'}$ since $d_k(T) = d_k(U)$.

Consider a general $\sigma \in S_r$ and write it as $\sigma = (k, k+1)\tilde{\sigma}$. Then

$$\begin{aligned} \pi^\lambda(\sigma)_{T, T'} &= \sum_{S \in \text{SYT}(\lambda)} \pi^\lambda((k, k+1))_{T, S} \pi^\lambda(\tilde{\sigma})_{S, T'} \\ &= \frac{1}{d_k(T)} \pi^\lambda(\tilde{\sigma})_{T, T'} + \sqrt{1 - \frac{1}{d_k(T)^2}} \pi^\lambda(\tilde{\sigma})_{(k, k+1)T, T'} \delta_{\{(k, k+1)T \in \text{SYT}(\lambda)\}}. \end{aligned}$$

We apply the inductive hypothesis on $\tilde{\sigma}$, obtaining

$$\begin{aligned} \bullet \frac{1}{d_k(T)} \pi^\lambda(\tilde{\sigma})_{T, T'} &= \begin{cases} 0 & \text{if } V \neq V' \\ \frac{1}{d_k(U)} \pi^\nu(\tilde{\sigma})_{U, U'} & \text{if } V = V', \end{cases} \\ \bullet \sqrt{1 - \frac{1}{d_k(T)^2}} \pi^\lambda(\tilde{\sigma})_{(k, k+1)T, T'} &= \begin{cases} 0 & \text{if } V \neq V' \\ \sqrt{1 - \frac{1}{d_k(U)^2}} \pi^\lambda(\tilde{\sigma})_{(k, k+1)U, U'} & \text{if } V = V', \end{cases} \end{aligned}$$

since if $(k, k+1)T$ is a standard Young tableau, then $\phi_r((k, k+1)T) = ((k, k+1)U, V)$. Similarly, $\delta_{\{(k, k+1)T \in \text{SYT}(\lambda)\}} = \delta_{\{(k, k+1)U \in \text{SYT}(\nu)\}}$. To conclude

$$\frac{1}{d_k(U)} \pi^\lambda(\tilde{\sigma})_{U, U'} + \sqrt{1 - \frac{1}{d_k(U)^2}} \pi^\lambda(\tilde{\sigma})_{(k, k+1)U, U'} \delta_{\{(k, k+1)U \in \text{SYT}(\nu)\}} = \pi^\lambda(\sigma)_{U, U'} \quad \square$$

Corollary 4.6. Let λ and σ be as before, then

$$TS^\lambda(\sigma) = \sum_{\nu \vdash r} TS^\nu(\sigma) \cdot \dim \nu \cdot \frac{\dim \lambda / \nu}{\dim \lambda}.$$

Proof. By the previous lemma:

$$TS^\lambda(\sigma) := \sum_{T, T' \in \text{SYT}(\lambda)} \frac{\pi^\lambda(\sigma)_{i, j}}{\dim \lambda} = \sum_{\nu \vdash r} \sum_{U, U' \in \text{SYT}(\nu)} \pi^\nu(\sigma)_{U, U'} \cdot \frac{\dim \lambda / \nu}{\dim \lambda},$$

and the conclusion is immediate. \square

Lemma 4.7. *Let $|\nu| = r \leq n = |\lambda|$, then*

$$\dim \lambda/\nu = \frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^\lambda(\tau).$$

Proof. Fix ν , we prove the statement by induction on n . If $n = r$ then the LHS is equal to $\delta_{\lambda,\nu}$, the Kronecker delta, and same for the RHS by the character orthogonality relation of the first kind.

Suppose $r = n - 1$, then by (7) we have $\chi^\lambda(\tau) = \sum_{\mu_j \nearrow \lambda} \chi^{\mu_j}(\tau)$ for each $\tau \in S_{n-1}$. Hence the RHS of the statement can be written as

$$\frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^\lambda(\tau) = \sum_{\mu_j \nearrow \lambda} \left(\frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^{\mu_j}(\tau) \right) = \sum_{\mu_j \nearrow \lambda} \delta_{\mu_j = \nu} = \delta_{\nu \nearrow \lambda},$$

which is equal to the LHS.

By the inductive hypothesis we consider the statement true for each $\tilde{\lambda} \vdash n - 1$. Hence

$$\begin{aligned} \dim \lambda/\nu &= \sum_{\tilde{\lambda} \vdash n-1} \dim \lambda/\tilde{\lambda} \cdot \dim \tilde{\lambda}/\nu \\ &= \sum_{\tilde{\lambda} \nearrow \lambda} \left(\frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^{\tilde{\lambda}}(\tau) \right) \\ &= \frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^\lambda(\tau). \end{aligned} \quad \square$$

Proposition 4.8. *Let $\sigma \in S_r$, $\lambda \vdash n$, then*

$$TS^\lambda(\sigma) = \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] \hat{\chi}^\lambda(\tau).$$

Proof. We apply Corollary 4.6 and Lemma 4.7:

$$\begin{aligned} TS^\lambda(\sigma) &= \sum_{\nu \vdash r} TS^\nu(\sigma) \cdot \dim \nu \left(\frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \hat{\chi}^\lambda(\tau) \right) \\ &= \sum_{\tau \in S_r} \left(\sum_{\nu \vdash r} \frac{\dim \nu^2}{r!} \hat{\chi}^\nu(\tau) TS^\nu(\sigma) \right) \hat{\chi}^\lambda(\tau) \end{aligned}$$

and we recognize inside the parenthesis the average $\mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)]$ taken with the Plancherel measure of partitions of r . \square

Proof of Theorem 4.2. We write the previous proposition as

$$TS^\lambda(\sigma) = m + v \cdot \hat{\chi}^\lambda((1, 2)) + \sum_{\substack{\tau \in S_r \\ wt(\tau) > 2}} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] \hat{\chi}^\lambda(\tau),$$

where the first two terms correspond respectively to $\tau = \text{Id}$ and the sum of all transpositions. By Kerov's Theorem 2.9, we have that

$$\sqrt{n} \sum_{\substack{\tau \in S_r \\ wt(\tau) > 2}} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] \hat{\chi}^\lambda(\tau) \xrightarrow{P} 0;$$

on the other hand $\sqrt{n} \cdot v \cdot \hat{\chi}^\lambda((1, 2)) \xrightarrow{d} v\mathcal{N}(0, 2)$, therefore

$$\sqrt{n} \cdot (TS^\lambda(\sigma) - m) \xrightarrow{d} \mathcal{N}(0, 2v^2). \quad \square$$

Example 4.9. Let $\sigma = (r-1, r)$, $r > 2$, be an adjacent transposition. In this example we will show that $m = \mathbb{E}_{PL}^r [TS^\nu(\sigma)]$ is strictly positive and that $v := \binom{r}{2} \mathbb{E}_{PL}^r \left[\hat{\chi}_{(2,1,\dots,1)}^\nu TS^\nu(\sigma) \right] = 1$. First, notice that, for all $\nu \vdash r$ and for all T, S standard Young tableaux of shape ν , $d_{r-1}(T') = -d_{r-1}(T)$, where T' is a tableau of shape ν' conjugated to T , which implies that

$$\pi^\nu(\sigma)_{T',S'} = \begin{cases} -\pi^\nu(\sigma)_{T,S} & \text{if } T = S \\ \pi^\nu(\sigma)_{T,S} & \text{if } T \neq S, \end{cases}$$

and if $T \neq S$ then $\pi^\nu(\sigma)_{T,S} \geq 0$. Hence

$$m = \sum_{\nu \vdash r} \frac{(\dim \nu)^2}{r!} TS^\nu(\sigma) = \frac{1}{2} \sum_{\nu \vdash r} \frac{(\dim \nu)^2}{r!} (TS^\nu(\sigma) + TS^{\nu'}(\sigma)).$$

Since $TS^\lambda(\sigma) = \hat{\chi}^\lambda(\sigma) + \sum_{\substack{T,S \in \text{SYT}(\nu) \\ T \neq S}} \frac{\pi^\nu(\sigma)_{T,S}}{\dim \nu}$, then

$$m = \frac{1}{2} \sum_{\nu \vdash r} \frac{\dim \nu}{r!} \left(\sum_{\substack{T,S \in \text{SYT}(\nu) \\ T \neq S}} \pi^\nu(\sigma)_{T,S} + \pi^{\nu'}(\sigma)_{T',S'} \right) > 0.$$

For $r > 2$, at least one summand is nonzero.

Consider now v and decompose $TS^\nu(\sigma)$ as above:

$$v = \binom{r}{2} \left(\mathbb{E}_{PL}^r \left[\hat{\chi}_{(2,1,\dots,1)}^\nu \hat{\chi}^\nu(\sigma) \right] + \mathbb{E}_{PL}^r \left[\hat{\chi}_{(2,1,\dots,1)}^\nu \sum_{\substack{T,S \in \text{SYT}(\nu) \\ T \neq S}} \frac{\pi^\nu(\sigma)_{T,S}}{\dim \nu} \right] \right).$$

By the character relations of the second kind we get

$$\mathbb{E}_{PL}^r \left[\hat{\chi}_{(2,1,\dots,1)}^\nu \hat{\chi}^\nu(\sigma) \right] = \sum_{\nu \vdash r} \frac{(\dim \nu)^2}{r!} \hat{\chi}_{(2,1,\dots,1)}^\nu \hat{\chi}^\nu(\sigma) = \frac{2}{r(r-1)}.$$

On the other hand, using T' for the conjugate of T as above,

$$\begin{aligned} & \mathbb{E}_{PL}^r \left[\hat{\chi}_{(2,1,\dots,1)}^\nu \sum_{\substack{T,S \in \text{SYT}(\nu) \\ T \neq S}} \frac{\pi^\nu(\sigma)_{T,S}}{\dim \nu} \right] = \\ & \frac{1}{2} \sum_{\nu \vdash r} \frac{1}{r!} \sum_{\substack{T,S \in \text{SYT}(\nu) \\ T \neq S}} \left(\hat{\chi}_{(2,1,\dots,1)}^\nu \pi^\nu(\sigma)_{T,S} + \hat{\chi}_{(2,1,\dots,1)}^{\nu'} \pi^{\nu'}(\sigma)_{T',S'} \right) = 0 \end{aligned}$$

since $\pi^\nu(\sigma)_{T,S} = \pi^{\nu'}(\sigma)_{T',S'}$ if $T \neq S$ and $\hat{\chi}_{(2,1,\dots,1)}^\nu = -\hat{\chi}_{(2,1,\dots,1)}^{\nu'}$. Finally, $v = 1$.

Here we write the values of $m = \mathbb{E}_{PL}^r [TS^\nu(\sigma)]$ and $v = \binom{r}{2} \mathbb{E}_{PL}^r \left[\hat{\chi}_{(2,1,\dots,1)}^\nu TS^\nu(\sigma) \right]$ when $\sigma \in S_4$.

σ	Id	(3,4)	(2,3)	(2,3,4)	(2,4,3)	(2,4)	(1,2)	(1,2)(3,4)	(1,2,3)	(1,2,3,4)	(1,2,4,3)	(1,2,4)	(1,3,2)
m	1	1/2	2/3	5/12	1/6	-1/4	0	0	1/3	1/3	1/3	0	-1/3
v	0	1	1	1/2	4/3	13/6	1	1	0	0	2/3	1/3	0

σ	(1,3,4,2)	(1,3)	(1,3,4)	(1,3)(2,4)	(1,3,2,4)	(1,4,3,2)	(1,4,2)	(1,4,3)	(1,4)	(1,4,2,3)	(1,4)(2,3)
m	-1/12	-2/3	-1/6	7/12	1/6	-1/12	0	-5/12	-1/4	-2/3	-7/12
v	1/2	1	0	-7/6	-1/3	-7/6	-4/3	-5/6	-1/6	1/3	1/6

4.2 Partial sum of the entries of an irreducible representation

Definition 4.10. Let $\lambda \vdash n$, $\sigma \in S_n$, $u \in \mathbb{R}$. Then

$$PS_u^\lambda(\sigma) := \sum_{i,j \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda}.$$

We can now argue in a similar way as we did in Section 2.4: summing entries of $\pi^\lambda(\sigma)$ up to a certain index is equivalent to sum all the entries of the submatrices $\pi^{\mu_j}(\sigma)$ for $j < \bar{j}$ and the right choice of \bar{j} , plus a remainder which is again a partial sum. We present the analogous of Proposition 2.8; we omit the proof, since it follows the same argument of the partial trace version:

Proposition 4.11. *Fix $u \in [0, 1]$, $\lambda \vdash n$ and $\sigma \in S_r$ with $r \leq n - 1$. Set $(F_{ct}^\lambda)^*(u) = \frac{y_{\bar{j}}}{\sqrt{n}} = v^\lambda$. Define*

$$\bar{u} = \frac{\dim \lambda}{\dim \mu_{\bar{j}}} \left(u - \sum_{j < \bar{j}} \dim \mu_j \right) < 1,$$

then

$$PS_u^\lambda(\sigma) = \sum_{y_j < v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma) + \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma). \quad (23)$$

As before, we call the *main term for the partial sum* $MS_u^\lambda(\sigma) := \sum_{y_j < v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma)$ and *remainder for the partial sum* $RS_u^\lambda(\sigma) := \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma)$.

The connection between the main term for the partial trace and the main term for the partial sum is easily described by applying Proposition 4.8:

$$\begin{aligned} MS_u^\lambda(\sigma) &= \sum_{y_j < v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma) \\ &= \sum_{y_j < v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] \hat{\chi}^{\mu_j}(\tau) \\ &= \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] \left(\sum_{y_j < v^\lambda \sqrt{n}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\tau) \right) \\ &= \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] MT_u^\lambda(\tau), \end{aligned} \quad (24)$$

We can thus apply Theorem 2.10 on the convergence of the main term for the partial trace to describe the asymptotic of $MS_u^\lambda(\sigma)$; notice that the only term in the previous sum which does not disappear when λ increases is the one in which τ is the identity; moreover, the second highest degree term correspond to τ being a transposition, and thus of order $O_P(n^{-\frac{1}{2}})$. We get:

Corollary 4.12. *Set $\sigma \in S_r$ and let $\lambda \vdash n$. Consider as before*

$$m := \mathbb{E}_{PL}^r [TS^\nu(\sigma)] \quad \text{and} \quad v := \binom{r}{2} \mathbb{E}_{PL}^r [\hat{\chi}_{(2,1,\dots,1)}^\nu TS^\nu(\sigma)].$$

Then

$$\sqrt{n} \cdot (MS_u^\lambda(\sigma) - u \cdot m) \xrightarrow{d} u \cdot \mathcal{N}(0, 2v^2).$$

Proof. We rewrite (24) as:

$$MS_u^\lambda(\sigma) = m \cdot MT_u^\lambda(\text{Id}) + \sum_{\substack{\tau \in S_r \\ wt(\tau)=2}} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] MT_u^\lambda(\tau) + \sum_{\substack{\tau \in S_r \\ wt(\tau)>2}} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] MT_u^\lambda(\tau).$$

By Lemma 2.4, $MT_u^\lambda(\text{Id}) \xrightarrow{P} u$. By Theorem 2.10

$$n^{\frac{1}{2}} \sum_{\substack{\tau \in S_r \\ wt(\tau)=2}} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] MT_u^\lambda(\tau) \xrightarrow{d} v \cdot u \cdot \mathcal{N}(0, 2);$$

on the other hand

$$\sum_{\substack{\tau \in S_r \\ wt(\tau)>2}} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] MT_u^\lambda(\tau) \in o_P(n^{-\frac{1}{2}}).$$

The corollary is proved. \square

Although we cannot show a satisfying result on the convergence of the partial trace because we cannot prove that $n^{-\frac{wt(\rho)}{2}} \frac{\dim \mu_{\bar{j}}}{\dim \lambda} PT_u^{\mu_{\bar{j}}}(\sigma) \rightarrow 0$, we are more lucky with the partial sum:

Theorem 4.13. *Let $u \in \mathbb{R}$, $\lambda \vdash n$ and $\sigma \in S_r$, then*

$$PS_u^\lambda(\sigma) \xrightarrow{P} u \mathbb{E}_{PL}^r [TS^\nu(\sigma)].$$

We will prove the theorem after three lemmas.

Lemma 4.14. *Let as usual $\sigma \in S_r$ and $\lambda \vdash n$, with $n > r$. Then $\pi^\lambda(\sigma)_{i,j} = 0$ for all i, j such that $|i - j| > r!$*

Proof. We iteratively use the Formula (7):

$$\pi^\lambda(\sigma) = \bigoplus_{\mu^{(n-1)} \nearrow \lambda} \pi^{\mu^{(n-1)}}(\sigma) = \bigoplus_{\substack{\mu^{(n-1)} \nearrow \lambda \\ \mu^{(n-2)} \nearrow \mu^{(n-1)}}} \pi^{\mu^{(n-2)}}(\sigma) = \bigoplus_{\mu^{(r)} \nearrow \dots \nearrow \lambda} \pi^{\mu^{(r)}}(\sigma).$$

Therefore $\pi^\lambda(\sigma)$ is a block matrix such that the only nonzero blocks are those on the diagonal. To conclude, notice that $\dim \mu^{(r)} \leq r!$ \square

Lemma 4.15. *With the usual setting of λ , σ and u we have*

$$|PS_u^\lambda(\sigma)| \leq 2u \cdot r! \cdot 2^{l(\sigma)},$$

where $l(\sigma)$ is the length of the reduced word of σ , i.e. the minimal number of adjacent transpositions occurring in the decomposition of σ .

Proof. We first prove a bound on the absolute value of an entry of the matrix

$$|\pi^\lambda(\sigma)_{T,S}| \leq 2^{l(\sigma)} \text{ for each } T, S \quad (25)$$

by induction on $l(\sigma)$. The base case is $\sigma = (k, k+1)$, for which $|\pi^\lambda((k, k+1))_{T,S}| \leq 1$. Suppose $\sigma = (k, k+1)\tilde{\sigma}$, then

$$|\pi^\lambda(\sigma)_{T,S}| = \left| \frac{1}{d_k(T)} \pi^\lambda(\tilde{\sigma})_{T,S} + \sqrt{1 - \frac{1}{d_k(T)^2}} \pi^\lambda(\tilde{\sigma})_{(k,k+1)T,S} \delta_{(k,k+1)T \in SYT(\lambda)} \right| \leq 2^{l(\tilde{\sigma})} + 2^{l(\tilde{\sigma})} = 2^{l(\sigma)}.$$

We see that

$$|PS_u^\lambda(\sigma)| = \left| \sum_{i,j \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda} \right| \leq 2u \dim \lambda \cdot r! \max_{T,S \in SYT(\lambda)} \left| \frac{\pi^\lambda(\sigma)_{T,S}}{\dim \lambda} \right| \leq 2^{l(\sigma)+1} u \cdot r!,$$

which allows us to conclude. Notice that we used the previous lemma in the first inequality, which shows that the number of nonzero terms appearing in the sum is bounded by $2u \dim \lambda \cdot r!$ \square

Lemma 4.16. *Let $\sigma \neq \text{Id}$, then $PT_u^\lambda(\sigma) \xrightarrow{P} 0$.*

Proof. Recall the decomposition of the partial trace into main term and remainder (Proposition 2.8):

$$PT_u^\lambda(\sigma) = MT_u^\lambda(\sigma) + RT_u^\lambda(\sigma).$$

By Theorem 2.10, if $\sigma \neq \text{Id}$ then $MT_u^\lambda(\sigma) \in o_P(n^{-\frac{wt(\sigma)}{2}}) \subseteq o_P(1)$. On the other hand we can estimate the remainder through the bound on a singular entry calculated in (25):

$$|RT_u^\lambda(\sigma)| \leq 2^{l(\sigma)} \cdot \frac{\dim \mu_{\bar{j}}}{\dim \lambda} \xrightarrow{P} 0,$$

where $l(\sigma)$ is the length of the reduced word of σ , $\mu_{\bar{j}}$ is the subpartition corresponding to the renormalized content $\frac{y_{\bar{j}}}{\sqrt{n}} = (F_{ct}^\lambda)^*(u)$, and

$$\bar{u} = \frac{\dim \lambda}{\dim \mu_{\bar{j}}} \left(u - \sum_{j < \bar{j}} \dim \mu_j \right). \quad \square$$

Proof of Proposition 4.13. We claim that the partial sum $PS_u^\lambda(\sigma)$ and the following quantity are asymptotically close

$$\sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] PT_u^\lambda(\tau),$$

that is,

$$\left| PS_u^\lambda(\sigma) - \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] PT_u^\lambda(\tau) \right| \xrightarrow{p} 0.$$

We substitute in the previous expression the decomposition formulas for the partial sum and partial trace, respectively Propositions 4.11 and 2.8, and we simplify according to the equality of Equation (24), so that it remains:

$$\frac{\dim \mu_{\bar{j}}}{\dim \lambda} \left| PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma) - \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] PT_{\bar{u}}^{\mu_{\bar{j}}}(\tau) \right|.$$

We recall from (25) that $|\pi^\lambda(\tau)_{T,S}| \leq 2^{l(\tau)}$ for each T, S , which implies that

$$|PT_{\bar{u}}^{\mu_{\bar{j}}}(\tau)| \leq 2^{l(\tau)} \cdot \bar{u}$$

hence

$$\begin{aligned} & \frac{\dim \mu_{\bar{j}}}{\dim \lambda} \left| PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma) - \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] PT_{\bar{u}}^{\mu_{\bar{j}}}(\tau) \right| \\ & \leq \frac{\dim \mu_{\bar{j}}}{\dim \lambda} \left(2\bar{u} \cdot r! \cdot 2^{l(\sigma)} + \sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] 2^{l(\tau)} \cdot \bar{u} \right) \xrightarrow{p} 0 \end{aligned}$$

since the expression inside the parenthesis is bounded and $\dim \mu_{\bar{j}} / \dim \lambda \xrightarrow{p} 0$.

On the other hand

$$\sum_{\tau \in S_r} \mathbb{E}_{PL}^r [\hat{\chi}^\nu(\tau) TS^\nu(\sigma)] PT_u^\lambda(\tau) = \frac{[u \dim \lambda]}{\dim \lambda} \mathbb{E}_{PL}^r [TS^\nu(\tau)] + o(1) \xrightarrow{d} u \mathbb{E}_{PL}^r [TS^\nu(\tau)]. \quad \square$$

For the same reasons for which we cannot prove convergence of the partial trace, here we cannot show a second order asymptotic, indeed we know that the term $\frac{\dim \mu_{\bar{j}}}{\dim \lambda} PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma)$ disappear when λ grows, but we do not know how fast. This will be discussed more deeply in the next section.

We can now generalize the concept of partial sum, and Lemma 4.15 allows us to describe its asymptotics.

Definition 4.17. Let $\lambda \vdash n$, $\sigma \in S_n$, $u_1, u_2 \in \mathbb{R}$. Define the partial sum of the entries of the irreducible representation matrix associated to $\pi^\lambda(\sigma)$ stopped at (u_1, u_2) as

$$PS_{u_1, u_2}^\lambda(\sigma) := \sum_{\substack{i \leq u_1 \dim \lambda \\ j \leq u_2 \dim \lambda}} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda}.$$

Corollary 4.18. Let $u_1, u_2 \in [0, 1]$, then

$$PS_{u_1, u_2}^\lambda(\sigma) \xrightarrow{p} \min\{u_1, u_2\} \mathbb{E}_{PL}^r [TS^\nu(\sigma)]$$

Proof. Suppose $u_1 < u_2$, we only need to prove that

$$|PS_{u_1, u_2}^\lambda(\sigma) - PS_{u_1}^\lambda(\sigma)| = \left| \sum_{\substack{i \leq u_1 \dim \lambda \\ u_1 \dim \lambda < j \leq u_2 \dim \lambda}} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda} \right| \xrightarrow{p} 0.$$

The number of nonzero terms in the above sum is bounded by $(r!)^2$ by Lemma 4.14; moreover by Equation (25) we have $|\pi^\lambda(\sigma)_{i,j}| \leq 2^{l(\sigma)}$. Therefore

$$|PS_{u_1, u_2}^\lambda(\sigma) - PS_{u_1}^\lambda(\sigma)| \xrightarrow{p} 0,$$

and the corollary follows. \square

5 A conjecture

Conjecture 5.1. *Set as usual $\lambda \vdash n$. We say that $\mu \subseteq \lambda$ if μ is a partition obtained from λ by removing boxes.*

We conjecture that there exists $\alpha > 0$ such that, for all s ,

$$P_{PL}^n \left(\left\{ \lambda : \max_{\substack{\mu: \mu \subseteq \lambda \\ |\mu|=|\lambda|-s}} \frac{\dim \mu}{\dim \lambda} > n^{-\alpha s} \right\} \right) \rightarrow 0.$$

We run some tests which hint the conjecture to be true for $s = 1$; for example, if $\alpha = 0.2$ and $n \leq 70$, we have

$$P_{PL}^n \left(\left\{ \lambda : \max_{\substack{\mu: \mu \subseteq \lambda \\ |\mu|=|\lambda|-s}} \frac{\dim \mu}{\dim \lambda} > n^{-\alpha s} \right\} \right) \leq \begin{cases} 0.2 & \text{if } n \geq 7 \\ 0.1 & \text{if } n \geq 12 \\ 0.05 & \text{if } n \geq 37. \end{cases}$$

Notice that, for $s > 1$,

$$\max_{\substack{\mu: \mu \subseteq \lambda \\ |\mu|=|\lambda|-s}} \frac{\dim \mu}{\dim \lambda} \leq \max_{\mu^{(1)} \nearrow \mu^{(2)} \nearrow \dots \nearrow \mu^{(s-1)} \nearrow \lambda} \prod_i \left(\max_{\substack{\mu: \mu \subseteq \mu^{(i)} \\ |\mu|=|\mu^{(i)}|-1}} \frac{\dim \mu}{\dim \mu^{(i)}} \right),$$

so that it may seem that it is enough to prove the conjecture just for $s = 1$.

This is not true though, since the sequence $\mu^{(1)} \nearrow \dots \nearrow \mu^{(s-1)}$ in the RHS above is not Plancherel distributed. Hence we need the Conjecture in the more general form.

Proposition 5.2. *If Conjecture 5.1 is correct, then*

$$PT_u^\lambda(\sigma) \xrightarrow{d} u \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k).$$

Proof. Recall from Proposition 2.11 that for any $s < n - r$, there exists a sequence of partitions $\mu^{(0)} \nearrow \mu^{(1)} \nearrow \dots \nearrow \mu^{(s)} = \lambda$ and a sequence of real numbers $0 \leq c_0, \dots, c_s < 1$ such that

$$PT_u^\lambda(\sigma) = \sum_{i=1}^s \frac{\dim \mu^{(i)}}{\dim \lambda} MT_{u^{(i)}}^{\mu^{(i)}}(\sigma) + \frac{\dim \mu^{(0)}}{\dim \lambda} PT_{u^{(0)}}^{\mu^{(0)}}(\sigma).$$

We consider

$$n^{\frac{wt(\sigma)}{2}} PT_u^\lambda(\sigma) = n^{\frac{wt(\sigma)}{2}} \sum_{i=1}^s \frac{\dim \mu^{(i)}}{\dim \lambda} MT_{u^{(i)}}^{\mu^{(i)}}(\sigma) + n^{\frac{wt(\sigma)}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_{u^{(0)}}^{\mu^{(0)}}(\sigma).$$

In the first sum of the right hand side the term corresponding to $i = s$ is

$$n^{\frac{wt(\sigma)}{2}} MT_u^\lambda(\sigma) \xrightarrow{d} u \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k),$$

due to Theorem 2.10. On the other hand the other terms in the first sum of the RHS are of the form $n^{\frac{wt(\sigma)}{2}} \frac{\dim \mu^{(i)}}{\dim \lambda} MT_{u^{(i)}}^{\mu^{(i)}}(\sigma)$ for $i = 1, \dots, s-1$, and it is easy to see that $(n-s+i)^{\frac{wt(\sigma)}{2}} MT_{u^{(i)}}^{\mu^{(i)}}(\sigma)$ converge (because of Theorem 2.10), while $\frac{\dim \mu^{(i)}}{\dim \lambda} \in o_P(1)$ because of the convergence of the normalized co-transition distribution towards an atom free distribution (see Lemma 2.3). We study thus the term $n^{\frac{wt(\sigma)}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_{u^{(0)}}^{\mu^{(0)}}(\sigma)$, and Conjecture 5.1 implies that

$$\frac{\dim \mu^{(0)}}{\dim \lambda} \leq \max_{\substack{\mu: \mu \subseteq \lambda \\ |\mu|=|\lambda|-s}} \frac{\dim \mu}{\dim \lambda} \leq n^{-\alpha s},$$

with high probability. We choose s such that $wt(\sigma)/2 < \alpha s$. An easy application of Inequality (25) implies that $|PT_{u^{(0)}}^{\mu^{(0)}}(\sigma)| \leq u^{(0)} \cdot 2^{l(\sigma)}$, thus we have

$$n^{\frac{wt(\sigma)}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_{u^{(0)}}^{\mu^{(0)}}(\sigma) \leq n^{\frac{wt(\sigma)}{2}} n^{-\alpha s} u^{(0)} \cdot 2^{l(\sigma)} \rightarrow 0,$$

which implies

$$n^{\frac{wt(\sigma)}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_{u^{(0)}}^{\mu^{(0)}}(\sigma) \in o_P(1).$$

Therefore

$$PT_u^\lambda(\sigma) \xrightarrow{d} u \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k),$$

which concludes the proof. \square

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