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ASYMPTOTIC BEHAVIOUR OF OPERATORS SUM OF P-LAPLACIANS

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Dedicated to Professor Yihong Du
a bright mathematician and a pleasant colleague

Abstract. The goal of this paper is to study the asymptotic behaviour of a linear combination of operators of the p-Laplacian type in cylinder like domains having some directions going to infinity. The case of a combination of two such operators has been the topic of numerous studies in the recent years.

1. **Introduction and notation.** We will denote by Ω_ℓ the open subset of $\mathbb{R}^n$, $n \geq 2$ defined as

$$\Omega_\ell = \ell\omega_1 \times \omega_2 \tag{1}$$

where ω_1 is a bounded convex open subset of $\mathbb{R}^r$ containing 0, ω_2 being a bounded open subset of $\mathbb{R}^{n-r}$. For instance if $\omega_1 = (-1, 1)$ then Ω_ℓ is a cylinder, else Ω_ℓ is a kind of generalised cylinder contained in

$$\Omega_\infty = \mathbb{R}^r \times \omega_2. \tag{2}$$

The points in Ω_ℓ and Ω_∞ will be denoted by

$$x = (X_1, X_2)$$

where $X_1 = (x_1, \dots, x_r) \in \mathbb{R}^r$, and $X_2 = (x_{r+1}, \dots, x_n) \in \mathbb{R}^{n-r}$. We consider now $p_i, q_j \in \mathbb{R}$, $i = 1, \dots, k$, $j = 1, \dots, m$ such that

$$1 < p_1 < p_2 < \cdots < p_k < 2 \leq q_m < q_{m-1} < \cdots < q_1. \tag{3}$$

We denote by $W_0^{1,p_i}(\Omega)$, $W_0^{1,q_j}(\Omega)$ the usual Sobolev spaces constructed on $L^{p_i}(\Omega)$, $L^{q_j}(\Omega)$ respectively, vanishing on the boundary of Ω. We note that when Ω is a bounded open set of $\mathbb{R}^n$

$$L^{q_1}(\Omega) \subset L^{q_2}(\Omega) \subset \cdots \subset L^{q_m}(\Omega) \subseteq L^2(\Omega) \subset L^{p_k}(\Omega) \subset \cdots \subset L^{p_1}(\Omega), \tag{4}$$

$$W_0^{1,q_1}(\Omega) \subset W_0^{1,q_2}(\Omega) \subset \cdots \subset W_0^{1,q_m}(\Omega) \subseteq W_0^{1,2}(\Omega) \subset W_0^{1,p_k}(\Omega) \subset \cdots \subset W_0^{1,p_1}(\Omega). \tag{5}$$

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For any real $p > 1$ we denote by p' its conjugate i.e. the real number such that $\frac{1}{p} + \frac{1}{p'} = 1$. For $f \in L^{p_1'}(\omega_2)$ we would like to consider u_ℓ solution to

$$\begin{cases} u_\ell \in W_0^{1,q_1}(\Omega_\ell), \\ \nabla \cdot (\sum_{i=1}^k |\nabla u_\ell|^{p_i-2} + \sum_{j=1}^m |\nabla u_\ell|^{q_j-2})\nabla u_\ell = f(X_2) \ \text{ in } \Omega_\ell, \end{cases} \tag{6}$$

the second equation being understood in a weak sense. In particular we would like to see if when $\ell \to \infty$, u_ℓ converges toward u_∞ solution to

$$\begin{cases} u_\infty \in W_0^{1,q_1}(\omega_2), \\ \nabla_{X_2} \cdot (\sum_{i=1}^k |\nabla_{X_2} u_\infty|^{p_i-2} + \sum_{j=1}^m |\nabla_{X_2} u_\infty|^{q_j-2})\nabla_{X_2} u_\infty = f(X_2) \ \text{ in } \omega_2. \end{cases} \tag{7}$$

Note that when the p_i, q_j's reduce to two values such operator is sometimes referred as a (p,q)-Laplacian. Such problems have been popularised in the last decades as Euler equations of anisotropic energies (Cf. [7], [10], [8], [13], [14], [15], [12] and the references there).

It results from classical results (see for instance [9], [1]) that one has.

Theorem 1.1. *Assuming $f \in L^{p_1'}(\omega_2)$ the problems (6), (7) admit both a unique weak solution which can be obtained by minimising*

$$\int_{\Omega_\ell} \sum_{i=1}^k \frac{|\nabla v|^{p_i}}{p_i} + \sum_{j=1}^m \frac{|\nabla v|^{q_j}}{q_j} \ dx - \int_{\Omega_\ell} fv \ dx$$

$$(\textit{Resp.} \int_{\omega_2} \sum_{i=1}^k \frac{|\nabla_{X_2} v|^{p_i}}{p_i} + \sum_{j=1}^m \frac{|\nabla_{X_2} v|^{q_j}}{q_j} \ dX_2 - \int_{\omega_2} fv \ dX_2)$$

on $W_0^{1,q_1}(\Omega_\ell)$, (resp $W_0^{1,q_1}(\omega_2)$).

Note that in the above we used obvious notation for $\nabla_{X_2} = \partial_{x_{r+1}}, \dots, \partial_{x_n}$ and we will do the same later for $\nabla_{X_1} = \partial_{x_1}, \dots, \partial_{x_r}$, ∂_{x_i} denoting the partial derivative in the direction x_i. Let us first prove the following estimate that we will be using later.

Lemma 1.2. *Let u_ℓ be the solution to (6). One has for some constant C independent of ℓ*

$$\int_{\Omega_\ell} \sum_{i=1}^k |\nabla u_\ell|^{p_i} + \sum_{j=1}^m |\nabla u_\ell|^{q_j} dx \le C\ell^r \int_{\omega_2} |f(X_2)|^{p_1'} dX_2. \tag{8}$$

Proof. In the weak formulation of (6) let us take as test function u_ℓ. It comes

$$\int_{\Omega_\ell} \sum_{i=1}^k |\nabla u_\ell|^{p_i} + \sum_{j=1}^m |\nabla u_\ell|^{q_j} dx = \int_{\Omega_\ell} f(X_2) u_\ell \ dx.$$

Using the Hölder inequality we get

$$\int_{\Omega_\ell} \sum_{i=1}^k |\nabla u_\ell|^{p_i} + \sum_{j=1}^m |\nabla u_\ell|^{q_j} dx \le \big(\int_{\Omega_\ell} |f(X_2)|^{p_1'} \ dx \big)^{\frac{1}{p_1'}} \big(\int_{\Omega_\ell} |u_\ell|^{p_1} \ dx \big)^{\frac{1}{p_1}}. \tag{9}$$

Since $u_\ell \in W_0^{1,p_1}(\Omega_\ell)$ one has for almost every $X_1 \in \ell\omega_1$, $u_\ell(X_1, \cdot) \in W_0^{1,p_1}(\omega_2)$ and by the Poincaré inequality for some constant C_{p_1} and a.e. X_1

$$\int_{\omega_2} |u_\ell(X_1, X_2)|^{p_1} \ dX_2 \le C_{p_1} \int_{\omega_2} |\nabla_{X_2} u_\ell(X_1, X_2)|^{p_1} \ dX_2$$

$$\leq C_{p_1} \int_{\omega_2} |\nabla u_\ell(X_1, X_2)|^{p_1} \, dX_2.$$

Integrating this inequality in X_1 we obtain

$$\int_{\Omega_\ell} |u_\ell|^{p_1} \, dx \leq C_{p_1} \int_{\Omega_\ell} |\nabla u_\ell|^{p_1} \, dx.$$

Going back to (9) we deduce

$$\int_{\Omega_\ell} \sum_{i=1}^{k} |\nabla u_\ell|^{p_i} + \sum_{j=1}^{m} |\nabla u_\ell|^{q_j} dx \leq \big(\int_{\Omega_\ell} |f(X_2)|^{p_1'} \, dx \big)^{\frac{1}{p_1'}} \big(C_{p_1} \int_{\Omega_\ell} |\nabla u_\ell|^{p_1} \, dx \big)^{\frac{1}{p_1}},$$

and after an easy manipulation

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p_1} \, dx \leq (C_{p_1})^{\frac{p_1'}{p_1}} \int_{\Omega_\ell} |f(X_2)|^{p_1'} \, dx$$

and thus

$$\int_{\Omega_\ell} \sum_{i=1}^{k} |\nabla u_\ell|^{p_i} + \sum_{j=1}^{m} |\nabla u_\ell|^{q_j} dx \leq (C_{p_1})^{\frac{p_1'}{p_1}} \int_{\Omega_\ell} |f(X_2)|^{p_1'} \, dx.$$

The result follows by computing this last integral. Note that we will use subsequently again, as above, the Poincaré inequality in the section ω_2 leading to a Poincaré inequality in Ω_ℓ. □

In the next section we will introduce two lemmas which should help to simplify our presentation. Finally in the last part of the paper we will prove the results of convergence available at this time.

2. **Some lemmas.** Let us recall the following result (see [1], [4])

Lemma 2.1. *For any $q > 1$ there exist positive constants c_q, C_q such that*

$$||\xi|^{q-2}\xi - |\eta|^{q-2}\eta| \leq C_q |\xi - \eta| (|\xi| + |\eta|)^{q-2} \ \forall \xi, \eta \in \mathbb{R}^n, \tag{10}$$

$$(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta) \cdot (\xi - \eta) \geq c_q |\xi - \eta|^2 (|\xi| + |\eta|)^{q-2} \ \forall \xi, \eta \in \mathbb{R}^n. \tag{11}$$

We denote now by ρ a smooth function such that $0 \leq \rho \leq 1$. One has :

Lemma 2.2. *Suppose $p < 2$, $(\alpha - 2)\frac{2}{p} > \alpha \geq 1$ then one has for some constant C independent of ℓ*

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2} |\nabla_{X_1} u_\ell| |u_\ell - u_\infty| \rho^{\alpha - 1} \, dx \leq C \ I_p^{\frac{1}{2} + \frac{1}{p'}} \big(\int_{\Omega_\ell} \{ |\nabla u_\ell| + |\nabla u_\infty| \}^p \, dx \big)^{\frac{1}{p} - \frac{1}{2}} \tag{12}$$

with I_p given by

$$I_p = \int_{\Omega_\ell} \{ |\nabla u_\ell| + |\nabla u_\infty| \}^{p-2} |\nabla (u_\ell - u_\infty)|^2 \rho^\alpha \, dx. \tag{13}$$

.

Proof. One has, since u_∞ is independent of X_1

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2} |\nabla_{X_1} u_\ell| |u_\ell - u_\infty| \rho^{\alpha - 1} \, dx$$

$$= \int_{\Omega_\ell} ||\nabla u_\ell|^{p-2} \nabla_{X_1} u_\ell| |u_\ell - u_\infty| \rho^{\alpha - 1} \, dx$$

$$= \int_{\Omega_\ell} |\{ |\nabla u_\ell|^{p-2} \nabla_{X_1} u_\ell - |\nabla u_\infty|^{p-2} \nabla_{X_1} u_\infty \}| |u_\ell - u_\infty| \rho^{\alpha - 1} \, dx.$$

$\{|\nabla u_\ell|^{p-2}\nabla_{X_1}u_\ell - |\nabla u_\infty|^{p-2}\nabla_{X_1}u_\infty\}$ are the first r entries of $\{|\nabla u_\ell|^{p-2}\nabla u_\ell - |\nabla u_\infty|^{p-2}\nabla u_\infty\}$, thus, by the preceding lemma, we have for some constant $C = C_p$

$$\begin{aligned}
&\int_{\Omega_\ell} |\nabla u_\ell|^{p-2}|\nabla_{X_1}u_\ell||u_\ell - u_\infty|\rho^{\alpha-1}\ dx \\
&\leq \int_{\Omega_\ell} |\{|\nabla u_\ell|^{p-2}\nabla u_\ell - |\nabla u_\infty|^{p-2}\nabla u_\infty\}||u_\ell - u_\infty|\rho^{\alpha-1}\ dx \\
&\leq C\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{p-2}|\nabla(u_\ell - u_\infty)||u_\ell - u_\infty|\rho^{\frac{\alpha}{p'}}\rho^{\frac{\alpha}{p}-1}\ dx \\
&\leq C\Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^p\rho^{\alpha-p}\Big)^{\frac{1}{p}}\Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{(p-2)p'}|\nabla(u_\ell - u_\infty)|^{p'}\rho^{\alpha}\ dx\Big)^{\frac{1}{p'}}
\end{aligned}$$

by Hölder's inequality. Noting that $(p-2)p' = p - p'$, using the inequality

$$\begin{aligned}
&(|\nabla u_\ell| + |\nabla u_\infty|)^{(p-2)p'}|\nabla(u_\ell - u_\infty)|^{p'} \\
&\qquad = (|\nabla u_\ell| + |\nabla u_\infty|)^{p-p'}|\nabla(u_\ell - u_\infty)|^{p'-2}|\nabla(u_\ell - u_\infty)|^2 \\
&\qquad \leq (|\nabla u_\ell| + |\nabla u_\infty|)^{p-2}|\nabla(u_\ell - u_\infty)|^2
\end{aligned}$$

and the Poincaré inequality in the first integral above we obtain

$$\begin{aligned}
&\int_{\Omega_\ell} |\nabla u_\ell|^{p-2}|\nabla_{X_1}u_\ell||u_\ell - u_\infty|\rho^{\alpha-1}\ dx \\
&\leq C\Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^p\rho^{\alpha-p}\ dx\Big)^{\frac{1}{p}}\Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{p-2}|\nabla(u_\ell - u_\infty)|^2\rho^{\alpha}\ dx\Big)^{\frac{1}{p'}} \\
&\leq C\Big(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^p\rho^{\alpha-p}\ dx\Big)^{\frac{1}{p}} I_p^{\frac{1}{p'}}.
\end{aligned} \tag{14}$$

To estimate the integral above we use once more Hölder's inequality to get

$$\begin{aligned}
&\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^p\rho^{\alpha-p}\ dx \\
&\quad = \int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{(p-2)\frac{p}{2}}|\nabla(u_\ell - u_\infty)|^p\rho^{\alpha-p}(|\nabla u_\ell| + |\nabla u_\infty|)^{(2-p)\frac{p}{2}}\ dx \\
&\quad \leq \Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{(p-2)}|\nabla(u_\ell - u_\infty)|^2\rho^{(\alpha-p)\frac{2}{p}}\ dx\Big)^{\frac{p}{2}} \\
&\qquad\qquad\qquad \Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^p\ dx\Big)^{1-\frac{p}{2}}.
\end{aligned}$$

Since $(\alpha - p)\frac{2}{p} > \alpha$ we obtain

$$\begin{aligned}
&\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^p\rho^{\alpha-p}\ dx \\
&\quad \leq \Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{(p-2)}|\nabla(u_\ell - u_\infty)|^2\rho^{\alpha}\ dx\Big)^{\frac{p}{2}} \\
&\qquad\qquad\qquad \Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^p\ dx\Big)^{1-\frac{p}{2}} \\
&\quad = I_p^{\frac{p}{2}}\Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^p\ dx\Big)^{1-\frac{p}{2}}.
\end{aligned} \tag{15}$$

Collecting (14), (15) it comes

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2}|\nabla_{X_1} u_\ell||u_\ell - u_\infty|\rho^{\alpha-1}\ dx \le C\ I_p^{\frac{1}{2}+\frac{1}{p'}}\Big(\int_{\Omega_\ell}\{|\nabla u_\ell| + |\nabla u_\infty|\}^p\ dx\Big)^{\frac{1}{p}-\frac{1}{2}} \tag{16}$$

which completes the proof of the lemma. □

3. **Main results.** In the case where the sums in (6), (7) reduce to one term the exponential rate of convergence of u_ℓ towards u_∞ is only known when the operator at hand is the Laplace operator. When the only term in the sum is a q-Laplacian, convergence is known :

– when $f \ge 0$ but with no sharp rate of convergence (see [5], [6]),

– in some cases that we will recover below (see [4]).

For general results regarding asymptotic behaviour of this type we refer to [2], [3], [4].

Let us first show:

Theorem 3.1. *Suppose that $f \ge 0$. Then u_ℓ solution to (6) converges toward u_∞ solution to (7).*

Proof. The solution to (6) is the unique minimiser on $W_0^{1,q_1}(\Omega)$ of

$$E_{\Omega_\ell}(v) = \int_{\Omega_\ell} F(\nabla v) - fv\ dx$$

when $F(\xi)$ is given by

$$F(\xi) = \sum_{i=1}^{k} \frac{|\xi|^{p_i}}{p_i} + \sum_{j=1}^{m} \frac{|\xi|^{q_j}}{q_j}.$$

Since

$$\lambda|\xi|^{q_1} \le F(\xi) \le \Lambda|\xi|^{q_1} + \Lambda'$$

for some constant $\lambda, \Lambda, \Lambda'$, the result follows from the Theorem 10.5 of [4] applied to $F(\xi) - \Lambda'$. □

If $\big||\nabla v|\big|_{q,\Omega}$ denotes the $L^q(\Omega)$-norm of ∇v let us prove :

Theorem 3.2. *Let u_ℓ, u_∞ be the solutions to (6), (7).*

Suppose that the operator defined by (6) has only $q_j's \ge 2$, $q_1 > 2$. There exists a constant C independent of ℓ such that

$$\big||\nabla(u_\ell - u_\infty)|\big|_{q_1,\Omega_{\ell/2}}^{q_1} \le C\ell^{-(\frac{2q_1}{q_1-2}-r)}. \tag{17}$$

Suppose that the operator defined by (6) has only $p_i's < 2$. There exists a constant C independent of ℓ such that

$$\big||\nabla(u_\ell - u_\infty)|\big|_{p_1,\Omega_{\ell/2}}^{p_1} \le C\ell^{-(\frac{p_1^2}{2-p_1}-r)}. \tag{18}$$

Suppose that the operator defined by (6) has $p_i's$ and $q_j's$ satisfying (3), $q_1 > 2$. There exists a constant C independent of ℓ such that

$$\big||\nabla(u_\ell - u_\infty)|\big|_{q_1,\Omega_{\ell/2}}^{q_1} \le C\ell^{-\{(\frac{2p_1}{2-p_1}\wedge\frac{2q_1}{q_1-2})-r\}}. \tag{19}$$

($\wedge$ denotes the minimum of two numbers).

Suppose that the operator defined by (6) *has* $p_i's$ *satisfying* (3) *and* $q_1 = 2$. *Let*

$$k = [\frac{2p_1}{2-p_1}] + 1 \tag{20}$$

where [ ] *denotes the integer part of a number. There exists a constant* $C = C_k$ *independent of* ℓ *such that*

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|^2_{2,\Omega_{\ell/2^k}} \leq C_k \ell^{-(\frac{2p_1}{2-p_1} - r)}. \tag{21}$$

Remark 3.3. Note that the relevant couple in the estimates is $(p, q) = (p_1, q_1)$ formed by the extreme exponents.

Proof. Note that if $v \in W^{1,q_1}_0(\Omega_\ell)$ then $v(X_1, \cdot) \in W^{1,q_1}_0(\omega_2)$ for almost every $X_1 \in \ell\omega_1$. Thus from (7) one deduces that for almost every $X_1 \in \ell\omega_1$

$$\int_{\omega_2} (\sum_{i=1}^k |\nabla_{X_2} u_\infty|^{p_i-2} + \sum_{j=1}^m |\nabla_{X_2} u_\infty|^{q_j-2}) \nabla_{X_2} u_\infty \cdot \nabla_{X_2} v(X_1, \cdot) dX_2$$
$$= \int_{\omega_2} f v(X_1, \cdot) dX_2.$$

Integrating in X_1 on $\ell\omega_1$ and using the fact that u_∞ is independent of X_1 we get

$$\int_{\Omega_\ell} (\sum_{i=1}^k |\nabla u_\infty|^{p_i-2} + \sum_{j=1}^m |\nabla u_\infty|^{q_j-2}) \nabla u_\infty \cdot \nabla v dx = \int_{\Omega_\ell} f v dx.$$

Subtracting this equation from (6) we obtain for any $v \in W^{1,q_1}_0(\Omega_\ell)$

$$\int_{\Omega_\ell} (\sum_{i=1}^k |\nabla u_\ell|^{p_i-2} + \sum_{j=1}^m |\nabla u_\ell|^{q_j-2}) \nabla u_\ell -$$
$$(\sum_{i=1}^k |\nabla u_\infty|^{p_i-2} + \sum_{j=1}^m |\nabla u_\infty|^{q_j-2}) \nabla u_\infty \cdot \nabla v dx = 0. \tag{22}$$

Denote by $\rho_1 = \rho_1(X_1)$ a smooth function such that

$$0 \leq \rho_1 \leq 1, \;\; \rho_1 = 1 \text{ on } \frac{1}{2}\omega_1, \;\; \rho_1 = 0 \text{ outside } \omega_1, \;\; |\nabla_{X_1} \rho_1| \leq C \tag{23}$$

and set

$$\rho = \rho_\ell = \rho_1(\frac{X_1}{\ell}).$$

Proof of (17). It is clear that for any $\alpha \geq 1$ one has

$$(u_\ell - u_\infty)\rho^\alpha \in W^{1,q_1}_0(\Omega_\ell).$$

Thus we derive from (22) in this case that

$$\int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} \nabla u_\ell - |\nabla u_\infty|^{q_j-2} \nabla u_\infty \cdot \nabla\{(u_\ell - u_\infty)\rho^\alpha\} dx = 0.$$

This can also be written

$$\int_{\Omega_\ell} \sum_{j=1}^m \{|\nabla u_\ell|^{q_j-2} \nabla u_\ell - |\nabla u_\infty|^{q_j-2} \nabla u_\infty\} \cdot \{\nabla u_\ell - \nabla u_\infty\} \rho^\alpha dx$$
$$= - \int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} \nabla_{X_1} u_\ell \nabla_{X_1} \rho^\alpha (u_\ell - u_\infty) dx.$$

Thus using the Lemma 2.1 and the definition of ρ we derive easily for some constant C independent of ℓ

$$\int_{\Omega_\ell} \sum_{j=1}^m c_{q_j} \{|\nabla u_\ell| + |\nabla u_\infty|\}^{q_j-2} |\nabla(u_\ell - u_\infty)|^2 \rho^\alpha dx$$

$$\leq \frac{C}{\ell} \int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} |\nabla_{X_1}(u_\ell - u_\infty)||u_\ell - u_\infty|\rho^{\alpha-1} dx$$

$$\leq \frac{C}{\ell} \int_{\Omega_\ell} \{\sum_{j=1}^m |\nabla u_\ell|^{q_j-2}\}^{\frac{1}{2}} |\nabla(u_\ell - u_\infty)|\rho^{\frac{\alpha}{2}} \{\sum_{j=1}^m |\nabla u_\ell|^{q_j-2}\}^{\frac{1}{2}} |u_\ell - u_\infty|\rho^{\frac{\alpha}{2}-1} dx$$

$$\leq \frac{C}{\ell} \big(\int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} |\nabla(u_\ell - u_\infty)|^2 \rho^\alpha \big)^{\frac{1}{2}} \big(\int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} |u_\ell - u_\infty|^2 \rho^{\alpha-2} dx \big)^{\frac{1}{2}}$$

$$\leq \frac{C}{\ell} \big(\int_{\Omega_\ell} \sum_{j=1}^m \{|\nabla u_\ell| + |\nabla u_\infty|\}^{q_j-2} |\nabla(u_\ell - u_\infty)|^2 \rho^\alpha \big)^{\frac{1}{2}}$$

$$\big(\int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} |u_\ell - u_\infty|^2 \rho^{\alpha-2} dx \big)^{\frac{1}{2}}$$

$$\leq \frac{C}{(\min c_{q_j})\ell} \big(\int_{\Omega_\ell} \sum_{j=1}^m c_{q_j} \{|\nabla u_\ell| + |\nabla u_\infty|\}^{q_j-2} |\nabla(u_\ell - u_\infty)|^2 \rho^\alpha \big)^{\frac{1}{2}}$$

$$\big(\int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} |u_\ell - u_\infty|^2 \rho^{\alpha-2} dx \big)^{\frac{1}{2}}.$$

We used the fact that $q_j \geq 2$ and Cauchy-Schwarz inequality. From this we deduce that for some new constant C one has

$$\int_{\Omega_\ell} \sum_{j=1}^m c_{q_j} \{|\nabla u_\ell| + |\nabla u_\infty|\}^{q_j-2} |\nabla(u_\ell - u_\infty)|^2 \rho^\alpha dx$$

$$\leq \frac{C^2}{\ell^2} \int_{\Omega_\ell} \sum_{j=1}^m |\nabla u_\ell|^{q_j-2} |u_\ell - u_\infty|^2 \rho^{\alpha-2} dx.$$

Using Hölder's inequality in this last integral it comes

$$\int_{\Omega_\ell} c_{q_1} \{|\nabla u_\ell| + |\nabla u_\infty|\}^{q_1-2} |\nabla(u_\ell - u_\infty)|^2 \rho^\alpha dx$$

$$\leq \frac{C^2}{\ell^2} \big(\int_{\Omega_\ell} |u_\ell - u_\infty|^{q_1} \rho^{(\alpha-2)\frac{q_1}{2}} dx \big)^{\frac{2}{q_1}} \big(\int_{\Omega_\ell} (\sum_{j=1}^m |\nabla u_\ell|^{q_j-2})^{\frac{q_1}{q_1-2}} dx \big)^{1-\frac{2}{q_1}}.$$

Choosing now $(\alpha-2)\frac{q_1}{2} > \alpha$ and using the Poincaré inequality in the second integral we get for some new constant C

$$\int_{\Omega_\ell} c_{q_1} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^\alpha dx$$

$$\leq \frac{C^2}{\ell^2} \big(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^\alpha dx \big)^{\frac{2}{q_1}} \big(\int_{\Omega_\ell} (\sum_{j=1}^m |\nabla u_\ell|^{q_j-2})^{\frac{q_1}{q_1-2}} dx \big)^{1-\frac{2}{q_1}},$$

and thus for another constant

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^\alpha dx \le \frac{C}{\ell^{\frac{2q_1}{q_1-2}}} \int_{\Omega_\ell} (\sum_{j=1}^m |\nabla u_\ell|^{q_j-2})^{\frac{q_1}{q_1-2}} dx.$$

Since $q_1 > q_j \ \forall j$ one has

$$|\nabla u_\ell|^{q_j-2} \le (1 \vee |\nabla u_\ell|)^{q_1-2} \le (1+|\nabla u_\ell|)^{q_1-2},$$

and thus by (8)

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^\alpha dx \le \frac{C}{\ell^{\frac{2q_1}{q_1-2}}} \int_{\Omega_\ell} 1 + |\nabla u_\ell|^{q_1} dx \le \frac{C}{\ell^{\frac{2q_1}{q_1-2}}} \ell^r.$$

Since $\rho = 1$ on $\Omega_{\frac{\ell}{2}}$ we have obtained

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^{q_1} dx \le \frac{C}{\ell^{\frac{2q_1}{q_1-2}-r}}$$

which completes the proof of this part.
Proof of (18). It is clear that for any $\alpha \ge 1$ one has

$$(u_\ell - u_\infty)\rho^\alpha \in W_0^{1,p_k}(\Omega_\ell).$$

Thus we derive from (22) in this case that

$$\int_{\Omega_\ell} \sum_{i=1}^k |\nabla u_\ell|^{p_i-2}\nabla u_\ell - |\nabla u_\infty|^{p_i-2}\nabla u_\infty \cdot \nabla\{(u_\ell - u_\infty)\rho^\alpha\} dx = 0.$$

This can also be written

$$\int_{\Omega_\ell} \sum_{i=1}^k \{|\nabla u_\ell|^{p_i-2}\nabla u_\ell - |\nabla u_\infty|^{p_i-2}\nabla u_\infty\} \cdot \{\nabla u_\ell - \nabla u_\infty\}\rho^\alpha dx$$

$$= -\int_{\Omega_\ell} \sum_{i=1}^k |\nabla u_\ell|^{p_i-2} \nabla_{X_1} u_\ell \nabla_{X_1} \rho^\alpha (u_\ell - u_\infty) dx.$$

Thus using the lemma 2.1 and the definition of ρ we derive easily for some constant C

$$\int_{\Omega_\ell} \sum_{i=1}^k c_{p_i} \{|\nabla u_\ell| + |\nabla u_\infty|\}^{p_i-2} |\nabla(u_\ell - u_\infty)|^2 \rho^\alpha dx$$

$$\le \frac{C}{\ell} \int_{\Omega_\ell} \sum_{i=1}^k |\nabla u_\ell|^{p_i-2} |\nabla_{X_1} u_\ell||u_\ell - u_\infty| \rho^{\alpha-1} dx.$$

Using the lemma 2.2 we get for some constant C independent of ℓ

$$\sum_{i=1}^k c_{p_i} I_{p_i} \le \frac{C}{\ell} \sum_{i=1}^k I_{p_i}^{\frac{1}{2}+\frac{1}{p_i'}} \Big(\int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{p_i} dx \Big)^{\frac{1}{p_i}-\frac{1}{2}}.$$

When q, q' are conjugate one has for $a, b > 0$ the Young inequality

$$ab \le \frac{a^q}{q} + \frac{b^{q'}}{q'}$$

i.e. also for every $\epsilon > 0$

$$ab = (q\epsilon)^{\frac{1}{q}} a \frac{b}{(q\epsilon)^{\frac{1}{q}}} \le \epsilon a^q + C_\epsilon b^{q'}.$$

Applying this last inequality with $q = \frac{2p_i'}{2+p_i'}$ we get since then $q' = \frac{2p_i}{2-p_i}$

$$\begin{aligned}\sum_{i=1}^{k} c_{p_i} I_{p_i} &\leq \epsilon \sum_{i=1}^{k} I_{p_i} + C_\epsilon \sum_{i=1}^{k} \frac{1}{\ell^{\frac{2p_i}{2-p_i}}} \int_{\Omega_\ell} (|\nabla u_\ell| + |\nabla u_\infty|)^{p_i} dx \\ &\leq \epsilon \sum_{i=1}^{k} I_{p_i} + C_\epsilon \sum_{i=1}^{k} \frac{1}{\ell^{\frac{2p_i}{2-p_i}}} \int_{\Omega_\ell} 2^{p_i} (|\nabla u_\ell|^{p_i} + |\nabla u_\infty|^{p_i}) dx \\ &\leq \epsilon \sum_{i=1}^{k} I_{p_i} + C_\epsilon \sum_{i=1}^{k} \frac{1}{\ell^{\frac{2p_i}{2-p_i} - r}}\end{aligned}$$

by (8) and the fact that u_∞ is independent of X_1. From this, choosing ϵ small enough, we derive for some other constant

$$\sum_{i=1}^{k} I_{p_i} \leq \sum_{i=1}^{k} \frac{C}{\ell^{\frac{2p_i}{2-p_i} - r}},$$

and

$$I_{p_1} \leq \frac{C}{\ell^{\frac{2p_1}{2-p_1} - r}}.$$

Going back to (15) with $\alpha > p_1$ we obtain

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^{p_1} \rho^{\alpha - p_1} \; dx \leq \frac{C}{(\ell^{\frac{2p_1}{2-p_1} - r})^{\frac{p_1}{2}}} \ell^{r(1 - \frac{p_1}{2})}$$

which gives (18) since $\rho = 1$ on $\Omega_{\frac{\ell}{2}}$.

Proof of (19). Since for any $\alpha \geq 1$ one has

$$(u_\ell - u_\infty)\rho^\alpha \in W_0^{1,q_1}(\Omega_\ell).$$

One derives as above

$$\begin{aligned}\sum_{i=1}^{k} c_{p_i} I_{p_i} + \sum_{j=1}^{m} c_{q_j} I_{q_j} \leq& \frac{C}{\ell} \int_{\Omega_\ell} \sum_{i=1}^{k} |\nabla u_\ell|^{p_i - 2} |\nabla_{X_1} (u_\ell - u_\infty)| |u_\ell - u_\infty| \rho^{\alpha - 1} dx \\ &+ \frac{C}{\ell} \int_{\Omega_\ell} \sum_{j=1}^{m} |\nabla u_\ell|^{q_j - 2} |\nabla_{X_1} (u_\ell - u_\infty)| |u_\ell - u_\infty| \rho^{\alpha - 1} dx.\end{aligned}$$

The last term in this sum can be handled with similar techniques as before, that is

$$\begin{aligned}&\frac{C}{\ell} \int_{\Omega_\ell} \sum_{j=1}^{m} |\nabla u_\ell|^{q_j - 2} |\nabla_{X_1} (u_\ell - u_\infty)| |u_\ell - u_\infty| \rho^{\alpha - 1} dx \\ &= \frac{C}{\ell} \int_{\Omega_\ell} \sum_{j=1}^{m} |\nabla u_\ell|^{\frac{q_j - 2}{2}} |\nabla_{X_1} (u_\ell - u_\infty)| \rho^{\frac{\alpha}{2}} |\nabla u_\ell|^{\frac{q_j - 2}{2}} |u_\ell - u_\infty| \rho^{\frac{\alpha}{2} - 1} dx\end{aligned}$$

$$\le \epsilon \sum_{j=1}^{m} \int_{\Omega_\ell} |\nabla u_\ell|^{q_j-2} |\nabla_{X_1}(u_\ell - u_\infty)|^2 \rho^\alpha \, dx$$
$$+ \frac{C_\epsilon}{\ell^2} \sum_{j=1}^{m} \int_{\Omega_\ell} |\nabla u_\ell|^{q_j-2} |u_\ell - u_\infty|^2 \rho^{\alpha-2} \, dx$$
$$\le \epsilon \sum_{j=1}^{m} I_{q_j} + \frac{C_\epsilon}{\ell^2} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^{q_1} \rho^{(\alpha-2)\frac{q_1}{2}} \, dx \Big)^{\frac{2}{q_1}} \Big(\int_{\Omega_\ell} \big(\sum_{j=1}^{m} |\nabla u_\ell|^{q_j-2} \big)^{\frac{q_1}{q_1-2}} \Big)^{1-\frac{2}{q_1}}$$
$$\le \epsilon \sum_{j=1}^{m} I_{q_j} + \frac{C_\epsilon}{\ell^2} \Big(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^{(\alpha-2)\frac{q_1}{2}} \, dx \Big)^{\frac{2}{q_1}} \ell^{r(1-\frac{2}{q_1})}.$$

(In this last step we used the Poincaré inequality). Using the arguments we used above for the terms in p_i we derive

$$\sum_{i=1}^{k} c_{p_i} I_{p_i} + \sum_{j=1}^{m} c_{q_j} I_{q_j} \le \epsilon \sum_{i=1}^{k} I_{p_i} + \epsilon \sum_{j=1}^{m} I_{q_j} + \sum_{i=1}^{k} \frac{C_\epsilon}{\ell^{\frac{2p_i}{2-p_i}-r}}$$
$$+ \frac{C_\epsilon}{\ell^2} \Big(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^{(\alpha-2)\frac{q_1}{2}} \, dx \Big)^{\frac{2}{q_1}} \ell^{r(1-\frac{2}{q_1})}.$$

Let us fix ϵ small enough in such a way we have for some constant independent of ℓ

$$I_{q_1} \le \sum_{i=1}^{k} I_{p_i} + \sum_{j=1}^{m} I_{q_j} \le \frac{C}{\ell^{\frac{2p_1}{2-p_1}-r}}$$
$$+ \frac{C}{\ell^2} \Big(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^{(\alpha-2)\frac{q_1}{2}} \, dx \Big)^{\frac{2}{q_1}} \ell^{r(1-\frac{2}{q_1})}.$$

Using the Young inequality in this last term with δ in place of ϵ one gets

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^\alpha \, dx \le I_{q_1} \le \frac{C}{\ell^{\frac{2p_1}{2-p_1}-r}}$$
$$+ \delta \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^{(\alpha-2)\frac{q_1}{2}} \, dx + C_\delta \Big(\frac{\ell^{r(1-\frac{2}{q_1})}}{\ell^2} \Big)^{\frac{q_1}{q_1-2}}.$$

Choosing δ small enough and $(\alpha-2)\frac{q_1}{2} > \alpha$ we arrive easily to

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^{q_1} \rho^\alpha \, dx \le \frac{C}{\ell^{\frac{2p_1}{2-p_1}-r}} + \frac{C}{\ell^{\frac{2q_1}{2-q_1}-r}}$$

which leads to (19).
Proof of (21). For $\ell_1 \le \ell$ we set

$$\rho = \rho_1\Big(\frac{X_1}{\ell_1}\Big)$$

where ρ_1 is the function defined in (23). Then for $\alpha \ge 1$

$$(u_\ell - u_\infty)\rho^\alpha \in W_0^{1,2}(\Omega_\ell)$$

and from (22) one deduces

$$\int_{\Omega_\ell}\sum_{i=1}^{k}|\nabla u_\ell|^{p_i-2}\nabla u_\ell-|\nabla u_\infty|^{p_i-2}\nabla u_\infty\cdot\nabla\{(u_\ell-u_\infty)\rho^\alpha\}dx$$
$$+\int_{\Omega_\ell}\nabla(u_\ell-u_\infty)\cdot\nabla\{(u_\ell-u_\infty)\rho^\alpha\}dx=0.$$

This can also be written

$$\int_{\Omega_\ell}\sum_{i=1}^{k}|\nabla u_\ell|^{p_i-2}\nabla u_\ell-|\nabla u_\infty|^{p_i-2}\nabla u_\infty\cdot\nabla(u_\ell-u_\infty)\rho^\alpha dx+\int_{\Omega_\ell}|\nabla(u_\ell-u_\infty)|^2\rho^\alpha dx$$
$$=-\int_{\Omega_\ell}\sum_{i=1}^{k}|\nabla u_\ell|^{p_i-2}\nabla_{X_1}u_\ell\nabla_{X_1}\rho^\alpha(u_\ell-u_\infty)dx$$
$$-\int_{\Omega_\ell}\nabla_{X_1}(u_\ell-u_\infty)\cdot\nabla_{X_1}\rho^\alpha(u_\ell-u_\infty)dx.$$

This implies for some constant C independent of ℓ (see (11), (13))

$$\sum_{i=1}^{k}c_{p_i}I_{p_i}+\int_{\Omega_\ell}|\nabla(u_\ell-u_\infty)|^2\rho^\alpha dx\le\frac{C}{\ell_1}\int_{\Omega_\ell}\sum_{i=1}^{k}|\nabla u_\ell|^{p_i-2}|\nabla_{X_1}u_\ell||u_\ell-u_\infty|\rho^{\alpha-1}dx$$
$$+\frac{C}{\ell_1}\int_{\Omega_{\ell_1}\backslash\Omega_{\frac{\ell_1}{2}}}|\nabla_{X_1}(u_\ell-u_\infty)||u_\ell-u_\infty|\rho^{\alpha-1}dx.$$

(We used the fact that $\nabla_{X_1}\rho=0$ on $\Omega_{\frac{\ell_1}{2}}$). Using the lemma 2.2 we get easily (recall that $\rho=1$ on $\frac{\ell_1}{2}\omega_1$)

$$\sum_{i=1}^{k}c_{p_i}I_{p_i}+\int_{\Omega_{\frac{\ell_1}{2}}}|\nabla(u_\ell-u_\infty)|^2dx\le\frac{C}{\ell_1}\sum_{i=1}^{k}I_{p_i}^{\frac12+\frac{1}{p_i'}}\Big(\int_{\Omega_\ell}(|\nabla u_\ell|+|\nabla u_\infty|)^{p_i}dx\Big)^{\frac{1}{p_i}-\frac12}$$
$$+\frac{C}{\ell_1}\int_{\Omega_{\ell_1}\backslash\Omega_{\frac{\ell_1}{2}}}|\nabla_{X_1}(u_\ell-u_\infty)|^2+|u_\ell-u_\infty|^2dx.$$

Using the Poincaré inequality in ω_2 one derives easily for some constant independent of ℓ or ℓ_1

$$\int_{\Omega_{\ell_1}\backslash\Omega_{\frac{\ell_1}{2}}}|u_\ell-u_\infty|^2dx\le C\int_{\Omega_{\ell_1}\backslash\Omega_{\frac{\ell_1}{2}}}|\nabla_{X_2}(u_\ell-u_\infty)|^2dx$$

and the inequality above becomes

$$\sum_{i=1}^{k}c_{p_i}I_{p_i}+\int_{\Omega_{\frac{\ell_1}{2}}}|\nabla(u_\ell-u_\infty)|^2dx\le\frac{C}{\ell_1}\sum_{i=1}^{k}I_{p_i}^{\frac12+\frac{1}{p_i'}}\Big(\int_{\Omega_\ell}(|\nabla u_\ell|+|\nabla u_\infty|)^{p_i}dx\Big)^{\frac{1}{p_i}-\frac12}$$
$$+\frac{C}{\ell_1}\int_{\Omega_{\ell_1}\backslash\Omega_{\frac{\ell_1}{2}}}|\nabla(u_\ell-u_\infty)|^2dx.$$

Arguing as in the proof of (18) we get

$$\sum_{i=1}^k c_{p_i} I_{p_i} + \int_{\Omega_{\frac{\ell_1}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \leq \epsilon \sum_{i=1}^k I_{p_i} + C_\epsilon \sum_{i=1}^k \frac{1}{\ell_1^{\frac{2p_i}{2-p_i}-r}} + \frac{C}{\ell_1} \int_{\Omega_{\ell_1} \setminus \Omega_{\frac{\ell_1}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx.$$

Chosing ϵ smaller than all the c_{p_i}'s we derive

$$\begin{aligned}\int_{\Omega_{\frac{\ell_1}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx &\leq C_\epsilon \sum_{i=1}^k \frac{1}{\ell_1^{\frac{2p_i}{2-p_i}-r}} + \frac{C}{\ell_1} \int_{\Omega_{\ell_1} \setminus \Omega_{\frac{\ell_1}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \\ &\leq \frac{C}{\ell_1^{\frac{2p_1}{2-p_1}-r}} + \frac{C}{\ell_1} \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx \\ &\quad - \frac{C}{\ell_1} \int_{\Omega_{\frac{\ell_1}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx.\end{aligned}$$

Thus we obtain

$$\int_{\Omega_{\frac{\ell_1}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \leq \frac{C}{\ell_1^{\frac{2p_1}{2-p_1}-r}} + \frac{C}{\ell_1} \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx.$$

Starting from $\ell_1 = \frac{\ell}{2^k}$ and iterating k times this inequality leads (for different constants C_k) to

$$\begin{aligned}\int_{\Omega_{\frac{\ell}{2^k}}} |\nabla(u_\ell - u_\infty)|^2 dx &\leq \frac{C_k}{\ell^{\frac{2p_1}{2-p_1}-r}} + \frac{C_k}{\ell} \int_{\Omega_{\frac{\ell}{2^{k-1}}}} |\nabla(u_\ell - u_\infty)|^2 dx \\ &\leq \frac{C_k}{\ell^{\frac{2p_1}{2-p_1}-r}} + \frac{C_k}{\ell^{\frac{2p_1}{2-p_1}-r+1}} + \frac{C_k}{\ell^2} \int_{\Omega_{\frac{\ell}{2^{k-2}}}} |\nabla(u_\ell - u_\infty)|^2 dx \\ &\leq \frac{C_k}{\ell^{\frac{2p_1}{2-p_1}-r}} + \frac{C_k}{\ell^k} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \\ &\leq \frac{C_k}{\ell^{\frac{2p_1}{2-p_1}-r}} + \frac{C_k}{\ell^k} \int_{\Omega_\ell} |\nabla u_\ell|^2 + |\nabla u_\infty|^2 dx \\ &\leq \frac{C_k}{\ell^{\frac{2p_1}{2-p_1}-r}}\end{aligned}$$

by definition of $k > \frac{2p_1}{2-p_1}$ (see (20)) and due to the lemma 1.1.

This completes the proof of the theorem. □

Remark 3.4. Note that our results extend trivially in the case of operators of the type

$$\nabla \cdot (\sum_\alpha a_\alpha |\nabla u_\ell|^{\alpha-2})$$

or

$$\nabla \cdot (\sum_\alpha A_\alpha(\nabla u_\ell))$$

where a_α are for instance positive constants and A_α are nonlinear functions with suitable structures. Unfortunately the case when a_α is depending on x or even X_2 only required some additional work.

Remark 3.5. In the case $r = 1$ the theorem above insures convergence of u_ℓ toward u_∞. In any other case one should notice that when q_1 or p_1 get closer to 2 then one gets convergence for larger r. It is somehow consistent with the fact that if the operator reduces to the usual Laplacian then $u_\ell \to u_\infty$ with an exponential rate of convergence (Cf. [4]).

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